Non-dictatorial Social Choice Rules Are Safely Manipulable

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Abstract

When a number of like-minded voters vote strategically and have limited abilities to communicate the under and overshooting may occur when too few or too many of them vote insincerely. In this paper we discuss this phenomenon and define the concept of a safe strategic vote. We prove that for any onto and non-dictatorial social choice rule there exist a profile at which a voter can make a safe strategic vote. This means that on occasion a voter will have an incentive to make a strategic vote and know that he will not be worse off regardless of how other voters with similar preferences would vote, sincerely or not. We also extend the Gibbard-Satterthwaite theorem. We prove that an onto, non-dictatorial social choice rule which is employed to choose one of at least three alternatives is safely manipulable by a single voter. We discuss new problems related to computational complexity that appear in this new framework.

1 Introduction

The classical Gibbard-Satterthwaite theorem (1973–75) claims that for any non-dictatorial social choice rule which has at least three alternatives in its range, a voter, on occasion, will have an opportunity to vote strategically. This opportunity allows her to change the result and get a better outcome than the one that she would get if voted sincerely, ceteris paribus. However such occasions become rare when the number of voters is large. This statement, known as Pattanaik's conjecture ([13], p.102) was made rigorous — see, for example, [15, 16, 17, 18], — where, in particular, it was proved that under the Impartial Culture (IC) assumption, the probability of obtaining a manipulable profile is bounded from above by a scalar multiple of $1/\sqrt{n}$, where n is the number of voters. So, in large scale elections individual manipulability is not really an issue, as Pattanaik suspected.

However the issue of group manipulability, which is sometimes called coalitional manipulability, remains an issue since the probability of having a group of voters that can successfully manipulate does not go to zero when n grows [11]. It was shown [17] that, under the IC, to be able to manipulate with non-zero probability, a group of voters must include a scalar multiple of $\sqrt{n}$ voters. Moreover, in [12] it was shown that the average size of minimal manipulating coalition also contain a scalar multiple of $\sqrt{n}$ voters. Thus any group that wants to have a chance to manipulate must be large.

In practice there are several significant barriers for group manipulation happening. Firstly, this manipulating coalition must be somehow formed. If it has to be formed endogenously its formation, given its size, must be complex with a lot of private communication.

Secondly, even if it is exogenously defined, this group must include a coordinator who calculates who should submit which linear order and then privately communicates those to coalition members. Thirdly, all the coalition members must obey the instructions of the coordinator and there does not seem to be obvious ways to reinforce the discipline. In any case it is extremely unlikely that such a coalition can be formed in the absence of means of private voter-to-voter communication.

Here we would like to suggest a new model for a group formation of a manipulating coalition. We assume that it is possible for a voter to send a single message to the whole electorate (say through the media) but it is not possible to send a large number of "individualised" messages. The example of such a communication would be an important public figure calling upon her supporters to vote in a certain way. Obviously that only voters who share their views with this public figure might respond and change their vote accordingly. However, not all of them will respond; some of them might consider voting strategically unethical.

We make the classical social choice assumption that voters know sincere preferences of others but do not know their voting intentions. Therefore issuing a call to supporters the public figure will not know exactly how many supporters will follow her example and vote as she recommends. If the value of the social choice function may not drop below the status quo, then we say that such call is safe.

Due to the uncertainty in the number of like-minded voters attempting to vote strategically, sometimes an overshooting may occur when too many like-minded voters act strategically, and as a result the value of the social choice function drops below of what it would be if everybody voted sincerely. The same thing may happen if too few like-minded voters act on the incentive to manipulate; in such a case we talk about undershooting. If one of these situations (or both) is possible we classify such a strategic vote as unsafe. Making a safe strategic vote is safe in the sense that the value of the social choice function would not drop below the status quo no matter how many like-minded voter would respond and join in the manipulation attempt.

The main result of this paper states that if there are at least three alternatives, then for any non-dictatorial and onto social choice function a profile can be found at which one (or more) voter can make a safe strategic vote.

The issue of overshooting can also appear in the context of the Gibbard-Satterthwaithe theorem. Suppose that we have a profile which is manipulable by a single voter. Then it will be most likely manipulable by many voters (at least if the function is anonymous, then any voter with the same preferences will have the same incentive to manipulate). It might be the case that the value of the social choice function gets initially better, when one voter votes insincerely but then gradually drops even below the status quo as more and more like-minded voters join in a manipulating attempt. We call such individual manipulation unsafe and possible overshooting will be a significant deterrent to such manipulation.

Unfortunately such deterrent does not always exist. As a corollary to our main result we extend the Gibbard-Satterthwaithe theorem by showing that any onto non-dictatorial social choice rule is, on occasion, safely manipulable if more than three alternatives are involved.

There is an extensive research into the computational complexity of strategic voting. Computational complexity of manipulating classical voting procedures has been studied, for example, in Bartholdi *et al* [2], Bartholdi and Orlin [3] and, more recently, Conitzer and Sandholm [4, 6], Faliszewski [9] and Conitzer [7]. Some artificial protocols for which manipulation is hard (in worst case scenario) have been also designed by Conitzer and Sandholm [5] and Elkind and Lipmaa [8]. At the same time it has been shown that no voting protocol is hard to manipulate in the average case scenario [6]. It would be interesting to extend the complexity analysis to the new type of manipulation introduced in this paper.

To the best of our knowledge, the distinction between safe and unsafe strategic votes, as we define them, was first made (albeit in the context of parliament choosing rules) in Slinko and White [19].

2 Strategic overshooting and undershooting

For the remainder of this section let us fix the set of alternatives $\mathcal{A}$, a set of voters $[n] = \{1, 2, \ldots, n\}$, and a social choice rule F. Preferences of each voter i are represented by a linear order R_i on $\mathcal{A}$ and the sequence $R = (R_1, \ldots, R_n)$ is called a *profile.* We will say that two voters i and j are of the *same type* if they have identical preferences, i.e. $R_i = R_j$. The type of the voter i is denoted as $<i>$. It is identified with R_i. Voters of the same type will be also called *like-minded*.

As usual, if $V \subseteq [n]$, by $R_{-V}(L)$ we will denote the profile obtained from R when all voters from V vote L and all other voters retain their original linear orders. For $V \subseteq [n]$ we will write $a \succ_V b$ if all voters from V strictly prefer a to b.

Definition 1 (An incentive to vote strategically)**.** *Fix a voter i, and define V to be the set of all voters with preferences identical to those of i at R. If there exists a linear order $L \neq R_i$ over $\mathcal{A}$, and a subset $V_1 \subseteq V$ containing i such that*

$$F(R_{-V_1}(L)) \succ_V F(R)$$

then we will say that, at R, voter i has an incentive to vote strategically *L.*

This is the key concept of the paper. We note that to have incentive to vote strategically does not mean that the voter is pivotal. What this voter can hope for is that there will be a sufficient number of like-minded voters with preferences identical to hers who will make a strategic move. She can call upon them to do so.

In some circumstances (we will present an example in the next section) a voter may hesitate to act on an incentive to vote strategically or to issue a call. One reason for hesitation would be this: in attempting to vote strategically, the voter could realise a gain or could realise a loss depending on which other voters with the same preferences also vote strategically. We now describe such circumstances formally.

Definition 2 (Strategic overshooting)**.** *Fix a voter i, and define V to be the set of all voters of type $<i>$ at R. Suppose that there exist two sets V_1 and V_2 such that $i \in V_1 \subset V_2 \subseteq V$,and a linear order $L \neq R_i$ such that:*

- *every voter in V_2 has an incentive to strategically vote L, and*
- *$F(R_{-V_1}(L)) \succ_V F(R) \succ_V F(R_{-V_2}(L))$.*

Then voter i (together with other voters of type $<i>$) can strategically overshoot *at R voting L.*

If we reverse the roles of the sets V_1 and V_2 we will obtain a definition of *strategic undershooting*. These two concepts are not mutually exclusive. Theoretically it is possible that a voter can both strategically overshoot and strategically undershoot at R with a vote of L.

Definition 3 (Safe and unsafe strategic votes)**.** *Fix a voter i, and a profile R. Suppose that there exists a linear order $L \neq R_i$ such that*

- *at R, voter i has an incentive to strategically vote L; and*
- *voter i cannot strategically overshoot or strategically undershoot at R with a vote of L.*

Then voter i can make a safe strategic vote *at R. If the first condition is satisfied but the second is not, we will say that voter i can make an* unsafe strategic vote *at R.*

It is important to note that the fact that the fact that a voter can make a safe strategic vote does not mean that she is pivotal. But, even if she is not pivotal, she has a strong incentive to vote strategically and try to rally supporters to vote likewise. If a strategic vote is unsafe, a voter has an incentive to vote strategically but she also has a disincentive, namely the prospect of making the outcome worse rather than better. In a similar circumstances, with respect to parliament choosing rules, Slinko and White [19] suggested that in this case voters will act in accord with their attitude towards uncertainty.

The main theorem of this paper is:

Theorem 1 (Slinko & White, 2008)**.** *Suppose an onto, non-dictatorial social choice rule is employed to choose one of at least three alternatives. Then there exist a profile at which a voter can make a safe strategic vote.*

Proof. The proof is rather technical and can be found in a preprint [20]. □

It has proved useful to classify a certain kind of strategic moves as an escape. Suppose preferences are such that voter i ranks no element of $\mathcal{A}$ lower than X. Further suppose that R is the profile of sincere preferences (over $\mathcal{A}$), and $F(R) = X$. If, at R, voter i has an incentive to vote strategically then voter i (together with other voters of type $<i>$) will be said to be able to *escape* at R. Notice that an escape is more than an ordinary safe strategic vote. Indeed, voter i cannot get worse no matter how all other voters vote (not only voters of type $<i>$). The concept of escaping appears often during the proofs.

3 Examples and Their Geometric Interpretation

Given any scoring social choice rule other than plurality, and a set of voters which is sufficiently large (more than 50, say), it is easy to create examples of strategic overshooting. In this section we will deal only with anonymous rules so we will use the so-called "succinct input" [9] which in Social Choice is known as voting situation [1]. Passing from a profile to a voting situation we forget the order of linear orders in the profile which makes votes anonymous. This is especially convenient when we have only three alternatives A, B, and C. In particular, the table

Preference order	Number of voters
ABC	n_1
ACB	n_2
BAC	n_3
BCA	n_4
CAB	n_5
CBA	n_6

denotes the voting situation when n_1 voters prefer A to B to C, n_2 voters prefer A to C to B, etc. This voting situation is denoted as $(n_1, n_2, n_3, n_4, n_5, n_6)$.

Example 1 (Strategic overshooting, escaping under the Borda rule)**.** *Suppose 94 voters are using the Borda rule to choose one of three alternatives and that the corresponding voting situation of sincere preferences is* $(17, 15, 18, 16, 14, 14)$*. If all voters vote sincerely then A would score 96, B 99, and C 87; B would win. If between four and eight ABC types vote* ACB*, ceteris paribus, then A would win. If 10 or more ABC types vote* ACB*, ceteris paribus, then C would win. So the given voting situation of sincere preferences is prone to unsafe manipulation. This voting situation is also prone to safe manipulation: if 13 or more* ACB *voters vote* CAB*, ceteris paribus, then C will win, and the manipulators will have made an escape.*

To explain Example 1 geometrically we need a graphical representation of the scores and strategic moves. Firstly we normalise the scores. Given weights $w_1 \geq w_2 \geq \ldots \geq w_m = 0$ we define the corresponding score function $sc\colon A \to \mathbb{R}$. For a profile $R = (R_1, \ldots, R_n)$ we set

$$sc(a) = \sum_{i=1}^{n} w_{pos(a,R_i)},$$

where $pos(a, R_i)$ is the position of a in R_i, i.e. the number of alternatives which are no worse than a relative to R_i. Then the *normalised positional score* of a candidate a is given by:

$$scn(a) = \frac{sc(a)}{sc(a_1) + \ldots + sc(a_m)}.$$

After this normalisation we have

$$scn(a_1) + scn(a_2) + \ldots + scn(a_m) = 1.$$

A normalised vector of scores $scn(a)$ can be represented as a point X of the m-dimensional simplex S^{m-1}:

$$\mathbf{x} = (x_1, \ldots, x_m), \qquad x_1 + \ldots + x_m = 1,$$

where $x_i = scn(a_i)$ is the normalised score of the ith alternative. We treat $x_1, \ldots, x_n$ as the homogeneous barycentric coordinates of X.

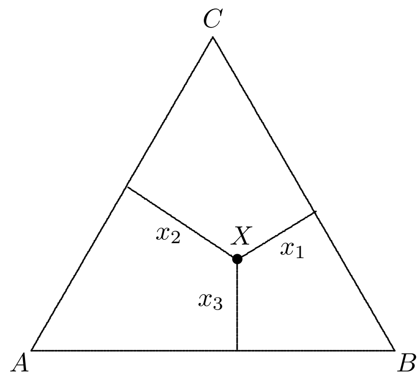

Figure 1: Geometric representation of a normalised score

Irrespective of the positional scoring rule, by voting insincerely a voter who prefers A to B to C cannot improve the score of A, nor can she worsen the score of C. If she votes insincerely, she will expect the vector of scores to fall in the shaded area.

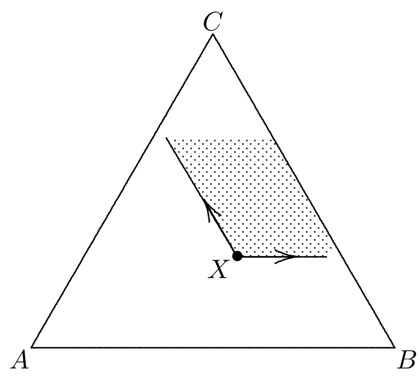

Figure 2: Possible directions of change under a manipulation attempt

By insincerely reporting her preferences to be BAC, she will move the vector of scores horizontally east. This she can do so long as the score function is not antiplurality. By insincerely reporting ACB, she moves the vector of scores north west, parallel to BC, and this is possible except in the event the score function is plurality.

Now the geometry of Example 1 can be represented as follows: The simplex S^2 is split into four quadrilaterals $AKOM$, $BMOL$, $CLOK$, where the winners will be A, B and C, respectively. For the sincere profile we are in the quadrilateral $BMOL$ where B wins. When ABC types report ACB they move the vector of normalised scores in north-western direction parallel to BC. Overshooting occurs, when the normalised vector of scores crosses the quadrilateral $AKOM$, where A wins and penetrates $CLOK$, where the winner is C. If ACB voters vote CAB they move the vector of normalised scores from $BMOL$ to $CLOK$ without crossing $AKOM$. This precludes over and undershooting.

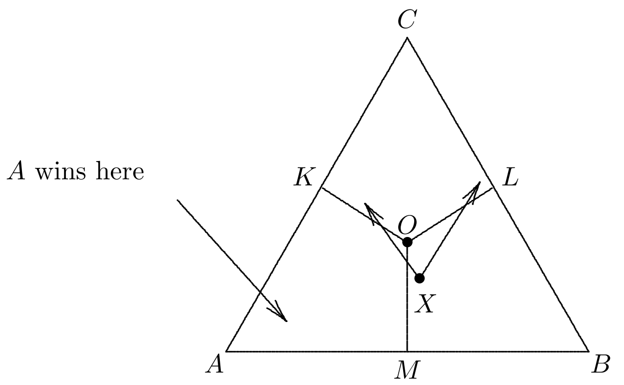

Figure 3: Geometric representation of the two moves from Example 1

The arrow parallel to BC graphically represents the unsafe manipulation that overshoots; it crosses the area where A wins and ends up in the area where the winner is C. The arrow parallel to AC represents the safe strategic move that is successful.

Here is yet another example for a multistage elimination rule..

Example 2 (Strategic overshooting under plurality with a run-off)**.** *Suppose 23 voters are using a plurality with a run-off social choice rule to choose one of three alternatives and suppose the corresponding voting situation is* $(4, 6, 7, 0, 0, 6)$*. If all voters vote sincerely then B beats A 13-10 in the run-off. If 2 voters of type ABC vote for C in the first round, ceteris paribus, then A beats C 17-6 in the run-off. If 4 or more voters of type ABC vote for C in the first round, ceteris paribus, then C beats B 12-11 in the run-off. Thus the profile of sincere preferences described is unsafely manipulable.*

We continue our series of examples with an example of a profile where only unsafe strategic vote can be made.

Example 3 (A profile that is unsafely but not safely manipulable)**.** *Suppose 80 voters are using a scoring social choice rule to choose one of three alternatives. Suppose that a first* *place ranking on a ballot is worth three points, while a second place ranking is worth one point, i.e.* $w_1 = 3$*,* $w_2 = 1$*,* $w_3 = 0$*. Let the sincere voting situation be* $(30, 0, 20, 0, 0, 30)$*. If all voters are honest then A scores 110, B 120, and C 90; B wins. Those that rank B or C highest have no incentive to act strategically. Consider the voters of type ABC. If between 11 and 19 of them state they are of type ACB, ceteris paribus, then A will win. If more than 21 of them state they are of type ACB, ceteris paribus, then C will win. Voters of type ABC have no other way to manipulate the vote. Thus the profile of sincere preferences described is 'completely' unsafe to manipulate.*

From geometric consideration it is easy to understand that for three alternatives no strategic undershooting for scoring rules is possible. Example below presents an example of strategic undershooting for Borda with five alternatives.

Example 4 (Strategic undershooting under the Borda rule)**.** *Suppose 41 voters are using the Borda rule to select one of five alternatives. Let the number of different voter types present at the profile of sincere preferences be given by the following table:*

Preference order	*Number of voters*
BCADE	*15*
CABED	*14*
CEDBA	*2*
DABEC	*10*

When all voters vote honestly, A scores 102, B 110, C 109, D 59, and E 30; B wins. If between two and six DABEC types vote ADEBC, ceteris paribus, then C wins. If eight or more DABEC types vote ADEBC, ceteris paribus, then A wins. So DABEC voters can strategically undershoot at the profile of sincere preferences.

Strategic undershooting and overshooting are also possible under plurality. The examples that we have do, however, rely upon the tie-breaking procedure adopted.

4 Implications for the Gibbard and Satterthwaite theorem.

Proposition 1. *Suppose, at a profile* R*, a voter* i *can make a safe strategic vote* L*. Then there exists another profile* Q*, where the voter can make a safe strategic vote* L *and* $F(Q_{-i}(L) \succ_i F(Q)$*.*

Proof. Let i be a voter, and define V to be the set of all voters with preferences identical to those of i at R. Suppose that there exists a subset $U \subseteq V$ such that $i \in U$, and a linear order $L \neq R_i$ such that:

- every voter in U has an incentive to strategically vote L, and

- $F(R_{-U}(L)) \succ_V F(R)$,
- for every subset $W \subseteq V$ such that $W \supset U$ we have $F(R_{-W}(L)) \succeq_V F(R)$.

Without loss of generality we may assume that U is the smallest subset withe above properties relative to the set-theoretic inclusion. Then $F(R) = F(R_{U'}(L))$ for every subset $U' \subset U$. Let $U_1 = U - \{i\}$. Then voter i can make a safe strategic vote and is pivotal at $Q = R_{-U_1}(L)$. Indeed,

$$F(Q_{-i}(L) = F(R_{-U}(L)) \succ_V F(R) = F(Q).$$

since $U_1 \subset U$. □

From This and Theorem 1 we immediately deduce:

Theorem 2 (Extention of the GS Theorem)**.** *Suppose that the number of alternatives is at least three. Then any onto and non-dictatorial social choice rule is safely manipulable by a single voter.*

5 Conclusion and Further Research

This paper has formally distinguished between safe and unsafe manipulation of voting rules. Examples of unsafe manipulations were presented. The Gibbard-Satterthwaite theorem was extended to show that all onto, non-dictatorial social choice rules are safely manipulable.

The main two questions that stem from this research.

1. Under a reasonable probability distribution of votes (say, IC or IAC) what is the probability that someone can make a safe strategic vote?

Conitzer and Sandholm [4] observed that, for every election system for which the winner problem is in P, if the voters are unweighted and there are a fixed number m of candidates then the manipulation problem is in P. This result holds because a manipulator can easily evaluate all possible $m!$ manipulations. It is more difficult for her to decide whether or not she can make a safe strategic vote since she has to examine an exponential number (in the number of voters n) of subsets of like-minded voters who may join her.

2. Let the number of alternatives be fixed. For various social choice rules is the safe strategic vote problem in P?

We focused on social choice rules for two reasons. Firstly, it allowed us to formally introduce and illustrate the concepts of over and undershooting relatively simply. Secondly, our main result applies only to social choice rules. But the difference between safe and unsafe strategic votes is applicable to a much wider class of choice rules. For example, Slinko and White (2006) identified that strategic over and undershooting can occur under systems of proportional representation. Future research might consider the over and undershooting phenomena in other settings.

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