How Large a Coalitional Should Be to Manipulate an Election?

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To the memory of Murat Sertel

Abstract: Assuming the IC conjecture, we show that, for any faithful scoring rule, when the number of participating agents n tends to infinity, the probability that a random profile will be manipulable for a coalition of size Cn^α, with $0 \leq \alpha < 1/2$ and C constant, is of order $O\left(1/n^{1/2-\alpha}\right)$.

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The well-known impossibility theorem of Gibbard (1973) and Satterthwaite (1975) states that every non-dictatorial social choice function at certain profiles is manipulable by a single individual. Further research nevertheless showed that when the number of voting agents become large, the probability of individual manipulation tends to zero. This asymptotic behavior was observed for both of the two major models for the distribution of voters' preferences, i.e. the IC (Impartial Culture conjecture) and the IAC (Impartial Anonymous Culture conjecture) although the speed of convergence of the probability of individual manipulation to zero is different. Peleg (1979), Baharad and Neeman (2002) and Slinko (2002a, 2002b) showed that it is of order $O\left(1/n^{1/2}\right)$ for the IC conjecture. In a contrust to that, Slinko (2002c) showed that this probability is only of order $O\left(1/n\right)$ for the IAC conjecture[1].

We show that, for the IC conjecture, for all faithful weighted scoring rules, including Borda, when the number of participating agents n tends to infinity, the probability that a random profile will be manipulable by a coalition of size n^{α}, with $0 \leq \alpha < 1/2$, is being of order $O\left(1/n^{1/2-\alpha}\right)$. For the IAC conjecture, Slinko (2002c) showed that the probability that a random voting situation will be manipulable for a coalition of size n^{α}, with $0 \leq \alpha < 1$, is of order $O\left(1/n^{1-\alpha}\right)$.

Let A and $\mathcal{N}$ be two finite sets of cardinality m and n respectively. The elements of A will be called alternatives, the elements of $\mathcal{N}$ agents. We assume that the agents have preferences over the set of alternatives. By $\mathcal{L} = \mathcal{L}(A)$ we denote the set of all linear orders on A; they represent the preferences of agents over A. The elements of the Cartesian product

$$\mathcal{L}(A)^n = \mathcal{L}(A) \times \ldots \times \mathcal{L}(A) \quad (n \text{ times})$$

are called profiles. They represent the collection of preferences of an n-element society of agents $\mathcal{N}$. If a linear order $R_i \in \mathcal{L}(A)$ represents the preferences of the i-th agent, then by aR_ib, where $a, b \in A$, we denote that this agent prefers a to b. A family of mappings $F_n \colon \mathcal{L}(A)^n \to A$, $n \in \mathbb{N}$, is called a social choice function (SCF).

Definition 1 *Let $R = (R_1, \ldots, R_n)$ be a profile. We say that a profile $R^{'}$ occurred as a result of strategic behaviour of k voters, if some k voters who previously submitted linear orders $R_{i_1}, \ldots, R_{i_k}$ now submit linear orders $R'_{i_1}, \ldots, R'_{i_k}$ while the remaining voters submit their original linear orders.*

[1] We write $g(n) = O(f(n))$ in case there is a positive constant C such that $|g(n)| \leq Cf(n)$ for all sufficiently large values of n

Definition 2 *Let F be an SCF and let R be a profile. We say that R is k-manipulable for F if there is a profile $R^{'}$, which occurred as a result of strategic behaviour of k voters, with the linear orders $R_{i_1}, \ldots, R_{i_k}$ being replaced by the linear orders $R'_{i_1}, \ldots, R'_{i_k}$, such that $F(R')R_{i_s}F(R)$ for all $s = 1, 2, \ldots, k$. We also say that a profile R is k-unstable if there exists a profile $R^{'}$, which occurred as a result of strategic behaviour of k voters, such that $F(R^{'}) \neq F(R)$.*

It should be noted that the existing concept of coalitional manipulability (see e.g. Lepelley et al., 1987 and Favardin et al., 2002) does not restrict the size of the coalition.

Let us define the following two indices. Given the rule F, the index of k-manipulability

$$K_F(n, m, k) = \frac{d_F(n, m, k)}{(m!)^n}, \tag{1}$$

where $d_F(n, m, k)$ is the total number of all k-manipulable profiles, and the index of k-instability

$$L_F(n, m, k) = \frac{e_F(n, m, k)}{(m!)^n}, \tag{2}$$

where $e_F(n, m, k)$ is the total number of all k-unstable profiles. We note that under the IC conjecture $\mathcal{L}(A)^n$ is a discrete probability space with the uniform distribution, hence the indices $K_F(m, n, k)$ and $L_F(m, n, k)$ become the probabilities of drawing a k-manipulable profile, or an k-unstable profile, respectively.

Every k-manipulable profile is k-unstable, hence $K_F(m, n, k) \leq L_F(m, n, k)$. Therefore any upper bound for $L_F(n, m, k)$ is an upper bound for $K_F(m, n, k)$.

Every scoring rule F is characterized by the sequence of weights $w_1 \geq w_2 \geq \ldots \geq w_m = 0$, and we may consider them to be integers. It is required that $w_1 > 0$. The scoring rule is faithful if $w_i \neq w_{i+1}$ for all $i = 1, 2, \ldots, m{-}1$. For each profile $R \in \mathcal{L}(A)^n$ and for every alternative $a \in A$, we can define the score of a, denoted $Sc_F(R, a)$, which can be computed as $Sc_F(R, a) = \sum_{\ell=1}^{m} w_\ell i_\ell$, where the number i_k shows how many times the alternative a was ranked kth. The most commonly used score is the Borda score $Sc_B(R, a)$, where B is the Borda rule defined by the weights $(m{-}1, m{-}2, \ldots, 1, 0)$.

The following obvious lemma explains how the scores can be changed during a manipulation attempt undertaken by k agents.

Lemma 1 *Let F be any scoring rule and let R be a profile. Let $R^{'}$ be another profile which occurred as a result of strategic behaviour of k voters. Then*

$$|Sc_F(R,a) - Sc_F(R^{'},a)| \leq kw_1, \tag{3}$$

where w_1 is the largest weight.

Let us consider a multiset $A_{n,q} = \{1^q, 2^q, \ldots, n^q\}$, which contains q identical copies of each of the elements $1, 2, \ldots, n$, with its subsets partially ordered by inclusion. A collection of subsets of a given multiset is called an antichain if for any two subsets from the collection none of them is a subset of the other. It is well-known that the collection of all subsets of $A_{n,q}$ of middle size, $\lfloor qn/2 \rfloor$, is a maximal antichain[2] in $A_{n,q}$; if n and q are both odd, then the subsets of size $\lceil qn/2 \rceil$ also form a maximal antichain. Anderson (1969) proved that the length $s(n)$ of a maximal antichain in $A_{n,q-1}$ satisfies the inequality

$$c_q \frac{q^{n-1}}{\sqrt{n}} \leq s(n) \leq C_q \frac{q^{n-1}}{\sqrt{n}}, \tag{4}$$

for some constants c_q, C_q, which depend on q but not on n. This result is crucial for the proof of our main theorem.

Given a linear order R and an alternative a, let us recall that the lower contour set of a relative to R is the set $L(R,a) = \{x \in A \mid aRx\}$.

Let $R = (R_1, \ldots, R_n)$ be a profile on A and $A' = A \setminus \{a_m\}$. Then we define the restrictions $Q_i = R_i|_{A'}$ of the linear orders R_i on A' and the restricted profile $Q = (Q_1, \ldots, Q_n)$ on A'. We will also say that R is an extension of Q.

Let us now consider a profile Q on A' and let $E(Q)$ be the set of all possible extensions of Q.

Definition 3 *Let R and S belong to $E(Q)$. We say that $R \leq S$ if the inclusion $L(R_i, a_m) \subseteq L(S_i, a_m)$ for the lower contour sets holds for all $i = 1, 2, \ldots, n$.*

Lemma 2 *The poset $(E(Q), \leq)$ is isomorphic to the poset $(\mathcal{P}(A_{n,m-1}), \subseteq)$ of all subsets of $A_{n,m-1}$.*

Proof: Suppose that R is an extension of Q and $card(L(R_i, a_m)) = t_i$. Then our isomorphism should assign to this particular extension the subset $\{1^{t_1}, 2^{t_2}, \ldots, n^{t_n}\}$ of $A_{n,m-1}$. The proof of this isomorphism is obvious.

[2]i.e. maximal by the number of subsets in it

Lemma 3 *Let F be a faithful scoring rule. Let Q be a profile on A'. Let us denote $a_m = a$ and let $b \in A'$. Then any two extensions $R, S \in E(Q)$ such that*

$$Sc_F(R, a) - Sc_F(R, b) = Sc_F(S, a) - Sc_F(S, b) \tag{5}$$

are not comparable relative to $\leq$.

Proof: Suppose $R \leq S$ and $R \neq S$. Then $Sc_F(R, a) < Sc_F(S, a)$ because the rule is faithful. This means that $L(R_i, a) \subseteq L(S_i, a)$ for all $i = 1.2. \ldots, n$ and this inclusion is strict at least for one i. Hence to obtain S from R we move a up in several linear orders of the profile. At the same time for every i there must be $L(R_i, b) \supseteq L(S_i, b)$ because S_i is obtained from R_i by moving a up and it is possible that a is in $L(R_i, b)$ but not in $L(S_i, b)$. This can happen only with a. Hence $Sc_F(R, b) \geq Sc_F(S, b)$ and the equation (5) cannot hold. This proves the lemma.

Lemma 4 *Let F be a faithful scoring rule, $a, b \in A$, and K be an integer. Then, for some $C_m > 0$ depending on m but not on n, there exist no more than $C_m(m!)^n/\sqrt{n}$ profiles $R \in \mathcal{L}(A)^n$ such that*

$$Sc_F(R, a) - Sc_F(R, b) = K. \tag{6}$$

Proof: We will show that the same constant C_m as in Anderson's theorem works. Let $A' = A \setminus \{a\}$. There are $((m-1)!)^n$ profiles on A' and we have m^n extensions for each of them to a profile on A. Let us take an arbitrary profile Q on A'. Then by Lemma 3 any two extensions with the property (6) will not be comparable. Thus by Lemma 2 and by the aforementioned result of Anderson we have no more than $C_m m^n/\sqrt{n}$ such extensions. In total we cannot have more than $\frac{C_m m^n}{\sqrt{n}} \cdot ((m-1)!)^n = \frac{C_m (m!)^n}{\sqrt{n}}$ profiles for which (6) is satisfied. The lemma is proved.

Theorem 1 *For any faithful scoring rule F there exists a constant D_m depending on m but not on n and k such that*

$$L_F(n, m, k) \leq D_m \frac{k}{\sqrt{n}}. \tag{7}$$

Proof: Let $a, b \in A$. Then by Lemma 4 for each K satisfying $-w_1 k \leq K \leq w_1 k$ the total number of profiles satisfying

$$Sc_F(R, a) - Sc_F(R, b) = K \tag{8}$$

will be not greater than $\frac{C_m(m!)^n}{\sqrt{n}}$. Thus the total number of profiles where the difference between the scores of some two alternatives is smaller than or equal to $2kw_1$ will be not greater than $4kw_1\binom{m}{2}\frac{C_m(m!)^n}{\sqrt{n}}$. Since by Lemma 1 every manipulable and even every unstable profile is among those counted, we see that (8) holds with $D_m = 4w_1\binom{m}{2}C_m$. This proves the theorem.

The main result follows immediately from this theorem.

Our results show that to be able to manipulate with a nonzero probability the manipulating coalition should have $O\left(\sqrt{n}\right)$ members. We conclude with an interesting hypothesis. Chamberlin (1985) introduced a very interesting characteristic of a rule, namely, the average minimum size of a coalition capable of manipulation. Since then there were no theoretical results related to it. Our results suggest that it is very likely that this characteristic of a scoring rule F will be of the form $K_F(m)\sqrt{n}$, where $K_F(m)$ is the constant which depend only on the number of alternatives m (and, of course, on the rule F) but not on n. This will open a new way of comparing the rules by comparing their constants $K_F(m)$ for some fixed m.

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