Comparative Probability Orders and the Flip Relation

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Abstract

In this paper we study the flip relation on the set of comparative probability orders on n atoms introduced by Maclagan (1999). With this relation the set of all comparative probability orders becomes a graph $\mathcal{G}_n$. Firstly, we prove that any comparative probability order with an underlying probability measure is uniquely determined by the set of its neighbours in $\mathcal{G}_n$. This theorem generalises the theorem of Fishburn, Pekeč and Reeds (2002). We show that the existence of the underlying probability measure is essential for the validity of this result. Secondly, we obtain the numerical characteristics of the flip relation in $\mathcal{G}_6$. Thirdly, we prove that a comparative probability order on n atoms can have in $\mathcal{G}_n$ up to ϕ_{n+1} neighbours, where ϕ_n is the nth Fibonacci number. We conjecture that this number is maximal possible. This partly answers a question posed by Maclagan.

Keywords. comparative probability, flippable pair, probability elicitation, subset comparisons, simple game, weighted majority game, desirability relation

1 Introduction

Considering comparative probability orders from the combinatorial viewpoint, Maclagan [13] introduced the concept of a flippable pair of subsets. We show that the concept of flippable pair is important for several other reasons and adds richness to the whole theory of comparative probability orders. In particular, we show that comparisons of subsets in flippable pairs correspond to irreducible vectors in the discrete cone of a comparative probability order. Fishburn at al [9] showed that in any minimal set of comparisons that define a representable comparative probability order all pairs of subsets in those comparisons are critical. We strengthen this theorem by showing that they must be not only critical but also flippable.

We show that there is an important distinction in algebraic properties of discrete cones for representable and non-representable comparative probability orders. In the former case the cone has a basis of irreducible vectors and in the latter irreducible vectors may not generate the cone.

Maclagan formulated a number of very interesting questions (see [13, p. 295]) which we partly answer here. In particular, she asked how many flippable pairs a comparative probability order may have. In this paper we show that a representable comparative probability order may have up to ϕ_{n+1} flippable pairs, which is the $(n+1)$th Fibonacci number. We conjecture that this lower bound on maximal number of flippable pairs is sharp. The latter results was obtained by Dominic Searles in his summer scholarship project (2006) under supervision of the other two authors.

Section 2 contains preliminary results and formulates Maclagan's problem. Section 3 discusses the concept of a flippable pair and proves the aforementioned generalisation of Fishburn-Pekeč-Reeds theorem. Section 4 numerically characterises the flip relation on six atoms. In Section 5 we discuss Searles' conjecture in relation to Maclagan's problem and prove the aforementioned lower bound. Section 6 introduces a class of simple games related to comparative probability orders and Section 7 concludes with stating sveral open problems.

2 Preliminaries

2.1 Comparative Probability Orders and Probability Measures

Given a (weak) order[1] $\preceq$ on a set A, the symbols $\prec$ and $\sim$ will, as usual, denote the corresponding (strict) linear order and indifference, respectively.

Definition 1. *Let X be a finite set. A linear order $\preceq$ on 2^X is called a* comparative probability order *on*

[1] reflexive, complete and transitive binary relation

X if $\emptyset \prec A$ for every non-empty subset A of X, and $\preceq$ satisfies de Finetti's axiom, namely

$$A \preceq B \iff A \cup C \preceq B \cup C, \tag{1}$$

for all $A, B, C \in 2^X$ such that $(A \cup B) \cap C = \emptyset$.

As in [7, 8] at this stage of investigation we preclude indifferences between sets. For convenience, we will further suppose that $X = [n] = \{1, 2, \ldots, n\}$ and denote the set of all comparative probability orders on $2^{[n]}$ by $\mathcal{P}_n$.

If we have a probability measure $\mathbf{p} = (p_1, \ldots, p_n)$ on X, where p_i is the probability of i, then we know the probability of every event A by the rule $p(A) = \sum_{i \in A} p_i$. We may now define an order $\preceq_{\mathbf{p}}$ on 2^X by

$$A \preceq_{\mathbf{p}} B \quad \text{if and only if} \quad p(A) \leq p(B).$$

If probabilities of all events are different, then $\preceq_{\mathbf{p}}$ is a comparative probability order on X. Any such order is called *(additively) representable.* The set of representable orders is denoted by $\mathcal{L}_n$. It is known [10] that $\mathcal{L}_n$ is strictly contained in $\mathcal{P}_n$ for all $n \geq 5$.

Since a representable comparative probability order does not have a unique probability measure representing it but a class of them, any comparative probability order can be viewed as a credal set [12] of a very special type. We will return to this interpretation slightly later.

As in [7, 8], it is often convenient to assume that $1 \prec 2 \prec \ldots \prec n$, This reduces the number of possible orders under consideration by a factor of $n!$. The set of all comparative probability orders on $[n]$ that satisfy this condition, will be denoted by $\mathcal{P}_n^*$ and the set of all representable comparative probability orders on $[n]$ will be denoted by $\mathcal{L}_n^*$.

We can also define a representable comparative probability order by any vector of positive utilities $\mathbf{u} = (u_1, \ldots, u_n)$ by

$$A \preceq_{\mathbf{u}} B \quad \text{if and only if} \quad \sum_{i \in A} u_i \leq \sum_{i \in B} u_i.$$

We do not get anything new since this will be the order $\preceq_{\mathbf{p}}$ for the measure $\mathbf{p} = \frac{1}{S}\mathbf{u}$, where $S = \sum_{i=1}^n u_i$. However, sometimes it is convenient to have the coordinates of $\mathbf{u}$ integers. We will call $u(A) = \sum_{i \in A} u_i$ the *utility* of A.

2.2 Discrete Cones

To every linear order $\preceq \in \mathcal{P}_n^*$, there corresponds a *discrete cone* $C(\preceq)$ in T^n, where $T = \{-1, 0, 1\}$ (as defined in [11, 7]).

Definition 2. *A subset $\mathcal{C} \subseteq T^n$ is said to be a discrete cone if the following properties hold:*

D1. $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} \subseteq \mathcal{C}$, *where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$,*

D2. *for every $\mathbf{x} \in T^n$, exactly one vector of the set $\{-\mathbf{x}, \mathbf{x}\}$ belongs to $\mathcal{C}$,*

D3. $\mathbf{x} + \mathbf{y} \in C$ *whenever $\mathbf{x}, \mathbf{y} \in \mathcal{C}$ and $\mathbf{x} + \mathbf{y} \in T^n$.*

We note that in [7] Fishburn requires $\mathbf{0} \notin \mathcal{C}$ because his orders are anti-reflexive. In our case, condition D2 implies $\mathbf{0} \in \mathcal{C}$.

For each subset $A \subseteq X$ we define the indicator vector χ_A of this subset by setting $\chi_A(i) = 1$, if $i \in A$, and $\chi_A(i) = 0$, if $i \notin A$. Given a comparative probability order $\preceq$ on X, we define the indicator vector $\chi(A, B) = \chi_B - \chi_A \in T^n$ for every possible comparison $A \preceq B$. The set of all indicator vectors $\chi(A, B)$, for $A, B \in 2^X$ such that $A \preceq B$, is denoted by $C(\preceq)$. The two axioms of comparative probability guarantee that $C(\preceq)$ is a discrete cone (see [7, Lemma 2.1]).

Definition 3. *A comparative probability order $\preceq$ satisfies the mth cancellation condition C_m if and only if there is no set $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ of non-zero vectors in $C(\preceq)$ for which there exist positive integers $a_1, \ldots, a_m$ such that*

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_m\mathbf{x}_m = 0. \tag{2}$$

It is known [10, 7, 4] that a comparative probability order $\preceq$ is representable if and only if all cancellation conditions for $C(\preceq)$ are satisfied.

There is an interpretation of discrete cones in terms of gambles. Any vector of T^n represents a gamble. The gamble

$$\mathbf{x} = (x_1, \ldots, x_n) \in T^n$$

pays $x_i \in T$ if the state i materialises. On appearance of $\mathbf{0} \neq \mathbf{x} \in T^n$ a participating agent must be ready to accept either $\mathbf{x}$ or $-\mathbf{x}$. The basic rationality assumption requires that the set of acceptable gambles form a discrete cone.

One may measure rationality of an agent looking at how consistent she was in accepting and rejecting various gambles. We need the following concept.

Definition 4. *Let $\mathcal{C}$ be a discrete cone corresponding to a personal comparative probability of an agent. A multiset*

$$P = \{\mathbf{x}_1^{a_1}, \mathbf{x}_2^{a_2}, \ldots, \mathbf{x}_m^{a_m}\},$$

where $\mathbf{x}_i \in \mathcal{C}$ and $a_i \in \mathbb{N}$, is called a portfolio of acceptable gambles.

Gambles are like risky securities. You may own a different number of shares of the same company. Similarly, a portfolio can contain several identical gambles. If the personal comparative probability of an agent is representable by a measure, then all portfolios of acceptable gambles are (in the long run) profitable.

Definition 5. *The portfolio P is said to be* neutral *if (2) is satisfied.*

The criterion of representability given in [10] can be reformulated in terms of portfolios as follows

Theorem 1 ([10])**.** *Suppose $\preceq$ be the agent's comparative probability order on 2^Ω and $\mathcal{C}$ be the corresponding discrete cone. Then $\preceq$ is representable iff $\mathcal{C}$ has no neutral portfolios of acceptable gambles.*

One can measure the degree of rationality of the agent by the minimal size of the portfolio of gambles which she cannot handle correctly.

2.3 Generation of Cones and Preference Elicitation

Let us define a restricted sum for vectors in a discrete cone $\mathcal{C}$. Let $\mathbf{u}, \mathbf{v} \in \mathcal{C}$. Then

$$\mathbf{u} \oplus \mathbf{v} = \begin{cases} \mathbf{u}+\mathbf{v} & \text{if } \mathbf{u}+\mathbf{v} \in T^n, \\ \text{undefined} & \text{if } \mathbf{u}+\mathbf{v} \notin T^n. \end{cases}$$

This makes a discrete cone an algebraic object, first studied by Kumar [11].

Definition 6. *We say that the cone $\mathcal{C}$ is weakly generated by vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ if every non-zero vector $\mathbf{c} \in \mathcal{C}$ can be expressed as a restricted sum of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, in which each generating vector can be used as many times as needed. We denote this by $\mathcal{C} = <\mathbf{v}_1, \ldots, \mathbf{v}_k>_w$.*

For the cone of a representable comparative probability order there is a much stronger tool to produce new vectors of the cone from a set of given ones. The following condition is a reformulation of Axiom 3 in [9] in terms of discrete cones associated with $\preceq$. See also [3].

Lemma 1. *Let $\prec \in \mathcal{L}_n^*$ be a representable comparative probability order and $C(\prec)$ the corresponding discrete cone. Suppose $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\} \subseteq C(\prec)$ and suppose that for some positive rational numbers $a_1, \ldots, a_m$ and $\mathbf{x} \in T^n$,*

$$\mathbf{x} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_m\mathbf{x}_m. \tag{3}$$

Then $\mathbf{x} \in C(\prec)$.

Definition 7. *Let $\preceq$ be a representable comparative probability order. We say that the cone $\mathcal{C} = C(\preceq)$ is* strongly generated *by vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ if every non-zero vector $\mathbf{c} \in \mathcal{C}$ can be obtained from $\mathbf{v}_1, \ldots, \mathbf{v}_k$ by taking linear combinations with positive rational coefficients. We denote this by $\mathcal{C} = <\mathbf{v}_1, \ldots, \mathbf{v}_k>$.*

These two latter concepts are important in the light of probability elicitation problem that Fishburn et al [9] considered. When we elicit a comparative probability without knowing that an underlying probability measure exists, then queries

$$A_1?B_1, \ldots, A_k?B_k, \tag{4}$$

resulting in comparisons $A_1 \prec B_1, \ldots, A_k \prec B_k$, determine the order $\preceq$ if and only if the vectors $\mathbf{v}_1 = \chi(A_1, B_1), \ldots, \mathbf{v}_k = \chi(A_k, B_k)$ weakly generate $C(\preceq)$. If it is already known that a representable order is being elicited, then (4) defines $\preceq$ if and only if the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ strongly generate $C(\preceq)$. Conder et al [2] give an example where the set of strong generators of the cone does not generate the cone weakly.

2.4 Geometric Representation of Representable Orders

Let $A, B \subseteq [n]$ be disjoint subsets, of which at least one is non-empty. Let $H(A,B)$ be a hyperplane consisting of all points $\mathbf{x} \in \mathbb{R}^n$ satisfying the equation

$$\sum_{a\in A} x_a - \sum_{b \in B} x_b = 0.$$

We denote the corresponding hyperplane arrangement by $\mathcal{A}_n$. Also let J be the hyperplane

$$x_1 + x_2 + \ldots + x_n = 1,$$

and let $\mathcal{H}_n = \mathcal{A}_n^J$ be the induced hyperplane arrangement.

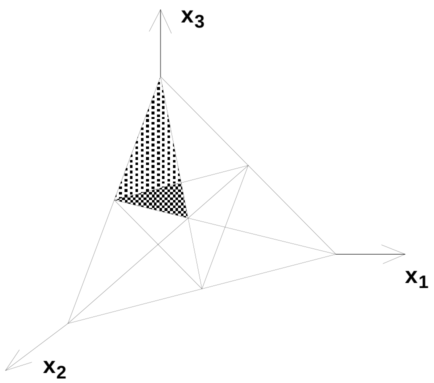

Figure 1

Fine and Gill [5] showed that the regions of $\mathcal{H}_n$ in the positive orthant $\mathbb{R}_+^n$ of $\mathbb{R}^n$ correspond to representable orders from $\mathcal{P}_n$.

Example 1. *The 12 regions of $\mathcal{H}_3$ on Figure 1 represent all 12 comparative probability orders on $\{1,2,3\}$. The two shaded triangular regions correspond to the two orders for which $1 \prec 2 \prec 3$, namely*

$$1 \prec 2 \prec 12 \prec 3 \prec 13 \prec 23 \prec 123, \tag{5}$$

$$1 \prec 2 \prec 3 \prec 12 \prec 13 \prec 23 \prec 123, \tag{6}$$

with the lighter one corresponding to the first order (the lexicographic order).

Now we can see what is special in the credal sets that correspond to comparative probability orders. They are not only convex, as credal sets must be, but they are in fact polytopes.

Problem 1 (Maclagan [13]). *How many facets do regions of $\mathcal{H}_n$ have?*

The minimal number of facets of a region in $\mathcal{H}_n$ is n [4, 2]. The maximal number of facets is not known. Searles' conjecture which we discuss in Section 5 states that the maximal number of facets is ϕ_{n+1}, the $(n+1)$th Fibonacci number.

3 Critical and flippable pairs

Definition 8. *Let A and B be disjoint subsets of $[n]$. The pair (A,B) is said to be* critical[2] *for $\preceq$ if $A \prec B$ and there is no $C \subseteq [n]$ for which $A \prec C \prec B$.*

Definition 9. *Let A and B be disjoint subsets of $[n]$. The pair (A,B) is said to be* flippable *for $\preceq$ if for every $D \subseteq [n]$, disjoint from $A \cup B$, the pair $(A \cup D, B \cup D)$ is critical.*

Since in the latter definition we allow the possibility that $D = \emptyset$, every flippable pair is critical.

We note that the set of flippable pairs is not empty, since the central pair of any comparative probability order is flippable [10]. Indeed, this consists of a certain set A and its complement $A^c = X \setminus A$, and there is no D which has empty intersection with both of these sets. It is not known whether this can be the *only* flippable pair of the order.

Suppose now that a pair (A,B) is flippable for a comparative probability order $\preceq$, and $A \neq \emptyset$. Then reversing each comparison $A \cup D \prec B \cup D$ (to $B \cup D \prec A \cup D$), we will obtain a new comparative probability order $\preceq'$, since the de Finetti axiom will still be satisfied. We say that $\preceq'$ is obtained from $\preceq$ by *flipping over* $A \prec B$. The orders $\preceq$ and $\preceq'$ are called *flip-related.* This flip relation turns $\mathcal{P}_n$ into a graph which we will denote $\mathcal{G}_n$.

[2] We follow Fishburn [9] in this definition, while Maclagan [13] calls such pairs *primitive.*

A pair (A,B) with $A = \emptyset$ can be flippable with no possibility of flipping over. Below we mark with an asterisk the three flippable pairs of the comparative probability order (5):

$$\emptyset \prec_* 1 \prec_* 2 \prec 12 \prec_* 3 \prec 13 \prec 23 \prec 123.$$

The first comparison $\emptyset \prec_* 1$ cannot be flipped over while the other two can be. For example, if we flip this order over $(12,3)$ we will obtain the order (6) which geometrically means passing from the lightly shaded triangle to the darkly shaded one. Or else we can say that flipping over takes us from one credal set to the adjacent one.

Definition 10. *An element $\mathbf{w}$ of the cone $\mathcal{C}$ is said to be* reducible *if there exist two other vectors $\mathbf{u}, \mathbf{v} \in \mathcal{C}$ such that $\mathbf{w} = \mathbf{u} \oplus \mathbf{v}$, and* irreducible *otherwise. The set of all irreducible elements of $\mathcal{C}$ will be denoted as $Irr(\mathcal{C})$.*

Theorem 2. *A pair (A,B) of disjoint subsets is flippable for $\preceq$ if and only if the corresponding indicator vector $\chi(A,B)$ is irreducible in $\mathcal{C}(\preceq)$.*

Proof. Suppose (A,B) is flippable but $\mathbf{w} = \chi(A,B)$ is reducible. Then $\mathbf{w} = \mathbf{u} \oplus \mathbf{v}$, where $\mathbf{u} = \chi(C,D)$ and $\mathbf{v} = \chi(E,F)$ for some C,D,E,F such that $C \prec D$ and $E \prec F$. We may assume without loss of generality that $C \cap D = E \cap F = \emptyset$. Since $\mathbf{u} + \mathbf{v} \in \mathcal{C}(\preceq) \subset T^n$ and $C \cap D = E \cap F = \emptyset$, we have $C \cap E = D \cap F = \emptyset$. Also since $\chi(A,B) = \chi(C,D) + \chi(E,F)$, it is easy to see that

$$A = (C \setminus F) \cup (E \setminus D) \quad \text{and} \quad B = (D \setminus E) \cup (F \setminus C).$$

Let $X = C \cap F$. Then $X \cap (A \cup B) = \emptyset$, and since $(C \cup D) \cap (E \setminus D) = (E \cup F) \cap (D \setminus E) = \emptyset$ we have

$$A \cup X = C \cup (E \setminus D) \prec D \cup (E \setminus D) =$$

$$(D \setminus E) \cup E \prec (D \setminus E) \cup F = B \cup X.$$

In particular, $A \cup X$ and $B \cup X$ are not neighbours in $\preceq$, so (A,B) is not flippable — contradiction.

Suppose now that $A \prec B$ but (A,B) is not flippable. Then there exist subsets C and D such that $(A \cup B) \cap C = \emptyset$ and

$$A \cup C \prec D \prec B \cup C.$$

We may assume that C is minimal for which such D exists. In this case we must have $C \cap D = \emptyset$, for otherwise the common elements in C and D can be removed and a contradiction with minimality of C follows. Now if $\mathbf{u} = \chi(A \cup C, D)$ and $\mathbf{v} = \chi(D, B \cup C)$, then

$$\mathbf{u} \oplus \mathbf{v} = \chi(A \cup C, B \cup C) = \chi(A,B) = \mathbf{w},$$

and so $\mathbf{w}$ is reducible. ∎

Fishburn et al [9, Theorem 3.7] proved that any smallest set of comparisons that determines a representable comparative probability order in $\mathcal{L}_n$ must consist of critical pairs. Here we prove a stronger result.

Theorem 3. *Let $\preceq$ be a representable comparative probability order. Then the set of irreducible elements of $\mathcal{C} = C(\preceq)$ is the smallest set that weakly generates $\mathcal{C}$.*

Proof. It is clear that the set of *all* irreducible elements $\mathrm{Irr}(\mathcal{C})$ of $\mathcal{C} = C(\preceq)$ is contained in any set of weak generators. Let $\mathbf{x} \in \mathcal{C}$. We will prove that either $\mathbf{x}$ belongs to $\mathrm{Irr}(\mathcal{C})$ or $\mathbf{x}$ can be represented as a restricted sum of elements of $\mathrm{Irr}(\mathcal{C})$. Suppose $\mathbf{x} \notin \mathrm{Irr}(\mathcal{C})$. Then $\mathbf{x} = \mathbf{x}_1 \oplus \mathbf{x}_2$ for some $\mathbf{x}_i \in \mathcal{C}$. If both of them belong to $\mathrm{Irr}(\mathcal{C})$, we are done. If at least one of them does not, then we continue representing both as restricted sums of vectors of $\mathcal{C}$. In this way, we obtain a binary tree of elements of $\mathcal{C}$. We claim that not a single branch of this tree can be longer than the cardinality of $\mathcal{C}$. If one of the branches were longer, then there would be two equal elements in it. Hence it would be possible to start a tree with some element and find the same element deep inside the tree. Without loss of generality, we can assume that $\mathbf{x}$ itself can be found in a tree generated by $\mathbf{x}$. If we stop when $\mathbf{x}$ has appeared for the second time, then we will have

$$\mathbf{x} = G(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_m),$$

where G is some term in the algebra $\langle \mathcal{C}, \oplus \rangle$. Then if we express restricted addition through the ordinary one, the term $\mathbf{x}$ will cancel on both sides, and we will obtain an expression

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \ldots + a_m\mathbf{x}_m = \mathbf{0}$$

with all coefficients a_i positive integers. This will violate the mth cancellation condition. $\square$

This theorem strengthens the aforementioned result of Fishburn, Pekec and Reeds in two directions. Firstly we prove a stronger property for pairs, secondly we prove this for a larger set of pairs.

We see that a minimal set of queries (4) that define a representable comparative probability order in $\mathcal{P}_n$ is unique. In contrast, a minimal set of queries (4) that define a representable comparative probability order in $\mathcal{L}_n$ is not unique. This can be seen, for example, from Example 2 of [2].

Theorem 3 does not hold for non-representable orderings as the following example shows.

Example 2. *In the following non-representable comparative probability order we mark all flippable pairs with an asterisk:*

$$\emptyset \prec 1 \prec 2 \prec 3 \prec 12 \prec 13 \prec_* 4 \prec 14 \prec_* 23 \prec 5$$

$$\prec_* 123 \prec 24 \prec 34 \prec_* 15 \prec 124 \prec 25 \prec_* 134 \ldots\ .$$

There are five such pairs. Let $\mathbf{f}_1 = \chi(13,4)$, $\mathbf{f}_2 = \chi(14,23)$, $\mathbf{f}_3 = \chi(5,123)$, $\mathbf{f}_4 = \chi(34,15)$, $\mathbf{f}_5 = \chi(25,134)$, and also let $\mathbf{x} = \chi(23,5)$. Then it is easy to check that

$$\mathbf{x} = \mathbf{f}_1 \oplus ((\mathbf{f}_5 \oplus (\mathbf{f}_2 \oplus \mathbf{x})) \oplus \mathbf{f}_4). \tag{7}$$

But on the other hand, $\mathbf{x}$ cannot be represented as a restricted sum of $\mathbf{f}_1, \ldots, \mathbf{f}_5$ since it is not in the subspace spanned by $\mathbf{f}_1, \ldots, \mathbf{f}_5$. The reason for (7) is of course the equation $\mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_4 + \mathbf{f}_5 = \mathbf{0}$, which is a violation of the fourth cancellation condition C_4.

There is a marked difference in algebraic properties of representable cones (Theorem 3) and the cone of the non-representable comparative probability order in the previous example. We wonder if this can be made a criterion of representability.

Problem 2. *Is it true that a discrete cone is representable if and only if it is generated by its irreducible vectors?*

4 Characteristics of the flip relation and Maclagan's problem

It is clear that it is sufficient to solve Maclagan's problem (Problem 1) for comparative probability orders in $\mathcal{L}_n^*$. For $n = 5$ and $n = 6$ we can find a solution computationally, using the following fact:

Proposition 1 ([2])**.** *Let $\preceq$ be a representable comparative probability order in $\mathcal{L}_n$, and let P be the corresponding convex polytope, which is a region of the hyperplane arrangement $\mathcal{H}_n$. Then the number of facets of P equals the number of representable comparative probability orders that are flip-related to $\preceq$ (plus one if the pair $\emptyset \prec 1$ is flippable).*

As we know, the flip relation turns $\mathcal{P}_n$ into a graph. Let $\preceq$ and $\preceq'$ be two comparative probability orders which are connected by an edge in this graph (and so are flip-related). We say that $\preceq$ and $\preceq'$ are in *friendly* relation if they are either both representable or both non-representable.

In the following tables, by the number of flips of the order $\preceq$ we mean the number of flippable pairs of $\preceq$. Let $A \prec B$ be a flippable pair of $\preceq$ such that $A \neq \emptyset$. We say that the flip of the pair $A \prec B$ is *friendly* if the given order $\preceq$ and the order $\preceq'$ resulting from this flip are in friendly relation.

Let $\preceq \in \mathcal{P}_n^*$ be a representable comparative probability order. There are two situations when a flip of $\preceq$ fails to be friendly: either the corresponding flippable pair is $\emptyset \prec 1$, or the order $\preceq'$ resulting from this flip is of a type different to $\preceq$.

The characteristics of the flip relation for $n = 5$ are given in the following table

Representable orders in $\mathcal{P}_5^*$		
# flips	# friendly flips	# of orders
5	5	169
	4	11 (11)
6	6	159
	5	82 (3)
7	7	65
	6	15
	5	6
8	8	9
Non-representable orders in $\mathcal{P}_5^*$		
# flips	# friendly flips	# of orders
5	3	6
	2	2
	1	16
6	2	6

Note that the numbers in parentheses are the numbers of orders for which the pair $\emptyset \prec 1$ is flippable. The total number of comparative probability orders in $\mathcal{P}_5$ in each category can be obtained by multiplying by $5! = 120$. The corresponding indicators of the flip relation for $n = 6$ are given in [2].

The number of facets of the regions of $\mathcal{H}_5$ corresponding to orders of $\mathcal{L}_5^*$ are given here:

# facets	5	6	7	8	all
# regions	265	177	65	9	512

The number of facets of the regions of $\mathcal{H}_6$ corresponding to $\mathcal{L}_6^*$ is given in [2]. Here will only notice that the smallest number of facets is 6 with 38,025 regions and the maximal is 13 with 20 regions.

It is worth paying attention to the fact that for $n = 5$ and $n = 6$, all comparative probability orders with the largest possible number of flips (namely 8 for $n = 5$, and 13 for $n = 6$) are representable, and all of their flips are friendly. This does not always happen, when an order has the smallest possible number of flips. Nevertheless, this is true for any representable order with smallest possible number of flips: all its flips are friendly [4, 2].

Maclagan [13] gave an example of a non-representable comparative probability order in $\mathcal{P}_6$ whose set of flippable pairs was a subset of the set of all flippable pairs of a representable comparative probability order. She concluded that for $n \geq 6$, an order might not be determined by the set of its flippable pairs.

Strictly speaking, we have to talk about the sequence of flippable pairs of an order $\preceq$, since these pairs may occur in $\preceq$ in different order. Strengthening the result of Maclagan, we have found eight sequences of comparisons with the property that each is the sequence of flippable pairs for two different non-representable comparative probability orders in $\mathcal{P}_6$ [2]. We list one such sequence below: $14 \prec 5$, $15 \prec 24$, $125 \prec 34$, $45 \prec 16$, $26 \prec 145$, $1245 \prec 36$. These eight sequences were found with the help of the Magma [1] system, which we used to determine and analyse several examples of orderings on sets of small order.

5 Searles' Conjecture

Let us summarise what we know about the cardinality of $|\mathrm{Irr}(\mathcal{C})|$ in the following

Theorem 4 ([4, 2])**.** *Let $\preceq$ be a comparative probability order on 2^X with $|X| = n$, and $\mathcal{C}$ be the corresponding discrete cone. Then*

- *if $\preceq$ is representable, then the set of all irreducible elements $\mathrm{Irr}(\mathcal{C})$ generates $\mathcal{C}$ and $|\mathrm{Irr}(\mathcal{C})| \geq n$, while*
- *if $\preceq$ is non-representable, then the set of all irreducible elements $\mathrm{Irr}(\mathcal{C})$ may not generate $\mathcal{C}$ and it may be that $|\mathrm{Irr}(\mathcal{C})| < n$.*

As we mentioned, Magma computations show that

- in $\mathcal{G}_5$: $5 \leq |\mathrm{Irr}(\mathcal{C})| \leq 8$, and
- in $\mathcal{G}_6$: $5 \leq |\mathrm{Irr}(\mathcal{C})| \leq 13$,

and all intermediate values are attainable.

Searles noticed that $8 = \phi_6$ and $13 = \phi_7$, where ϕ_n is the nth Fibonacci number, that is the nth member of the sequence defined by $\phi_1 = \phi_2 = 1$ and $\phi_{n+2} = \phi_{n+1} + \phi_n$. Its initial values are: $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ He conjectured that

Conjecture 1. *The maximal number of facets of regions of $\mathcal{H}_n$ is equal to the maximal cardinality of $Irr(\mathcal{C}(\preceq))$ for $\preceq \in \mathcal{L}_n^*$, and equal to the Fibonacci number ϕ_{n+1}.*

The first part of this conjecture will be proved if we show that for some representative comparative probability order $\preceq$, for which $|\mathrm{Irr}(\mathcal{C}(\preceq))|$ is maximal, all flips of $\preceq$ are friendly. The existence of such order was checked for all $n \leq 12$.

Searles made the following advance towards proving the second part of this conjecture.

Theorem 5. *In $\mathcal{P}_n$ there exists a comparative probability order with a discrete cone $\mathcal{C}$ for which $|\mathrm{Irr}(C)| = \phi_{n+1}$, where ϕ_n is the nth Fibonacci number.*

The proof will be split into several observations. Let us introduce the following notation first. Let $\mathbf{u} = (u_1, \ldots, u_n)$ be a vector such that $0 < u_1 < \ldots < u_n$ and $q > 0$ be a number such that $u_j < q < u_{j+1}$ for some j (we assume that $u_{n+1} = \infty$). In this case we set $(\mathbf{u}, q)$ to be the vector of $\mathbb{R}^{n+1}$ such that

$$(\mathbf{u}, q) = (u_1, \ldots, u_j, q, u_{j+1}, \ldots, u_n).$$

We also denote $\ell_n = (1, 2, 4, \ldots, 2^{n-1})$ and $2\ell_n = (2, 4, 8, \ldots, 2^n)$. An easy observation is this:

Proposition 2. *$\preceq_{\ell_n}$ is the lexicographic order, and the utilities of any two consecutive terms in it differ by 1. These utilities cover the whole range between 0 and $2^n - 1$.*

Proof. We leave the verification to the reader. $\square$

Proposition 3. *Let q be an odd number such that $q < 2^{n+1}$ and $\mathbf{m} = (2\ell_n, q)$. Then the difference between the utilities of any two consecutive terms of $\preceq_{\mathbf{m}}$ is not greater than 2.*

Proof. Suppose $2^j < q < 2^{j+1}$, that is, q is the utility of j in $\preceq_{\mathbf{m}}$. Suppose now that (A, B) is a critical pair for $\preceq_{\mathbf{m}}$. If $j \notin B$, then the statement follows from Proposition 2. If $j \in B$ and $j \in A$, the statement follows from the same proposition. Assume now that $j \in B$ but $j \notin A$. Then $B = \{j\} \cup B'$, where B' does not contain j. If $B' \neq \emptyset$, then by Proposition 2 there exists A', not containing j, such that $0 \leq u(B') - u(A') \leq 2$. Then A must be $\{j\} \cup A'$ and the proposition is true. Finally, if $B = \{j\}$, then since $u(j) \leq 2^{n+1} - 1$ by Proposition 2 there will be an A, not containing j, such that $u(B) - u(A) = 1$. $\square$

Let us denote by $\mathcal{S}_{n+1}$ the class of orderings on $X = \{1, 2, \ldots, n+1\}$ of the type $\preceq_{\mathbf{m}}$, where $\mathbf{m} = (2\ell_n, q)$ for some odd $q < 2^n$. And let j denote the number such that $2^j < q < 2^{j+1}$. Obviously, $j < n + 1$.

Proposition 4. *From the position at which the subset $\{j\}$ appears in the order $\preceq_{\mathbf{m}}$ until the position at which all subsets contain j, subsets not containing j alternate with those containing j, with the difference in utilities for any two consecutive terms being* 1.

Proof. All subsets not containing j have even utility and all those containing j have odd utilities. If we consider these two sequences separately, by Proposition 2 the difference of utilities of neighboring terms in each sequence will be equal to 2. Hence they have to alternate in $\preceq_{\mathbf{m}}$. $\square$

Lemma 2. *Let $\preceq_{\mathbf{m}}$ be an order from the class $\mathcal{S}_{n+1}$ and let (A, B) be a critical pair for $\preceq_{\mathbf{m}}$. Then the following conditions are equivalent:*

(a) *(A, B) is flippable;*

(b) *either A or B contains j;*

(c) *$u(B) - u(A) = 1$.*

Proof. (a) $\Longrightarrow$ (b): Suppose (A, B) is flippable. As (A, B) is critical, it is impossible for A and B each to contain j. We only have to prove that it is impossible for both of them not to contain j. If $j \notin A$ and $j \notin B$, then $u(A) + 2 = u(B) < u(j)$. Then $u(A) < u(B) < u(n+1) = 2^n$, hence neither A nor B contains $n+1$. But then for $A' = A \cup \{n+1\}$ and $B' = B \cup \{n+1\}$ we have $u(j) < u(A') < u(B')$. Both A' and B' do not contain j, hence they are in the alternating part of the ordering, and since $u(B') - u(A') = 2$, they cannot be consecutive terms. As (A, B) is flippable, this is impossible, which proves that either A or B contain j.

(b) $\Longrightarrow$ (c): This follows from Proposition 4.

(c) $\Longrightarrow$ (a): This is true not only for orders from our class, but also for all orders defined by integer utility vectors. Indeed, if $u(B) - u(A) = 1$, then for any $C \cap (A \cup B) = \emptyset$ we have $u(B \cup C) - u(A \cup C) = 1$, and $B \cup C$ and $A \cup C$ are consecutive. $\square$

Up to now, the utility of q and j did not matter. Now we will try to maximise the number of flippable pairs in $\preceq_{\mathbf{m}}$, so we will need to choose q carefully. It should come as no surprise that the optimal choice of q will depend on n, so we will talk about q_n now. For the rest of the proof we will set

$$q_n = \frac{(-1)^{n+1} + 2^n}{3}. \tag{8}$$

An equivalent way of defining q_n would be by the recurrence relation

$$q_n = q_{n-1} + 2q_{n-2} \tag{9}$$

with the initial values $q_3 = 3$, $q_4 = 5$. We also note:

Proposition 5. $q_n \equiv 2 + (-1)^{n+1} \pmod 4$.

Proof. Easy induction using (9). $\square$

Let us now consider a flippable pair (A, B) for $\preceq_{\mathbf{m}}$. Since $j = n-2$, we have either $A = A' \cup \{n-2\}$ or $B = B' \cup \{n-2\}$. In the first case, (A', B) is a pair of nonintersecting subsets of the lexicographic order on $[n+1] \setminus \{n-2\}$ with $u(B) - u(A') = q+1$. In the second, the pair will be (B', A) with $u(A) - u(B') = q-1$.

Let g_n be the number of pairs $A \prec B$ with $u(B) - u(A') = q+1$ in the lexicographic order $\preceq_{2\ell_n}$, and let h_n be the number of pairs $A \prec B$ with $u(B) - u(A') = q-1$ in the same order. We have proved the following:

Lemma 3. *The number of flippable pairs in $\preceq_{\mathbf{m}}$ is $g_n + h_n$.*

This reduces our calculations to a rather understandable lexicographic order $\preceq_{2\ell_n}$.

For convenience we will denote $q_n^+ = q_n + 1$ and $q_n^- = q_n - 1$. We note that Proposition 5 implies

Proposition 6. *$q_n^- \equiv 1 + (-1)^{n+1} \pmod 4$, and $q_n^+ \equiv 3 + (-1)^{n+1} \pmod 4$.*

A direct calculation also shows that the following equations hold:

Proposition 7.

$$\begin{aligned} q_{n+1}^- &= 2q_n^- && \text{for all odd } n \geq 3, && (10)\\ q_{n+1}^- &= 2q_n^- + 2 && \text{for all even } n \geq 4, && (11)\\ q_{n+1}^+ &= 2q_n^+ - 2 && \text{for all odd } n \geq 3, && (12)\\ q_{n+1}^+ &= 2q_n^+ && \text{for all even } n \geq 4. && (13) \end{aligned}$$

Lemma 4. *For any odd $n \geq 3$ the following recurrence relations hold:*

$$\begin{aligned} g_{n+1} &= g_n + h_n, && (14)\\ h_{n+1} &= h_n, && (15) \end{aligned}$$

and for any even $n \geq 4$

$$\begin{aligned} g_{n+1} &= g_n, && (16)\\ h_{n+1} &= g_n + h_n. && (17) \end{aligned}$$

Proof. Firstly we assume that n is odd. Then $n+1$ is even. We know from (10) that $q_{n+1}^- = 2q_n^-$. Given any nonintersecting pair $A < B$ in $\preceq_{2\ell_n}$, we may shift it to the right, replacing each element i with the element $i+1$, to obtain a nonintersecting pair $\bar{A} < \bar{B}$ of $\preceq_{2\ell_{n+1}}$. This procedure of shifting doubles the difference in utilities, so $u(\bar{B}) - u(\bar{A}) = 2q_n^- = q_{n+1}^-$. This proves $h_{n+1} \geq h_n$. Moreover, by Proposition 6, $q_{n+1}^- \equiv 0 \pmod 4$ hence no nonintersecting pair $C < D$ of $\preceq_{2\ell_{n+1}}$ with difference q_{n+1}^- can involve 1, either in C or in D. Therefore $C = \bar{A}$ and $D = \bar{B}$ for some nonintersecting pair $A < B$, and so $h_{n+1} = h_n$.

We can also use h_n nonintersecting pairs of $\preceq_{2\ell_{n+1}}$ as described above to construct the same number of nonintersecting pairs of $\preceq_{2\ell_{n+1}}$ with utility difference $q_{n+1}^+ = q_{n+1}^- + 2$. If $A < B$ is such a pair, we notice that 1 belongs neither to A nor to B. Adding 1 to B will create a pair $A < B \cup \{1\}$ with the utility difference q_{n+1}^+. We can also use (12) and a shifting technique to create another g_n nonintersecting pairs with utility difference q_{n+1}^+. Indeed, if $A < B$ is a nonintersecting pair in $\preceq_{2\ell_n}$ with utility difference q_n^+, then the pair $\{1\} \cup \bar{A} < \bar{B}$ will be nonintersecting in $\preceq_{2\ell_{n+1}}$ with utility difference $2q_n^+ - 2 = q_{n+1}^+$. Thus $g_{n+1} \geq g_n + h_n$.

We have now two ways of obtaining nonintersecting pairs from $\preceq_{2\ell_{n+1}}$ with utility difference q_{n+1}^+. The first method gives us pairs $C < D$ with $1 \in D$, while the second method gives us pairs $C < D$ with $1 \in C$. Now, let $C < D$ be a nonintersecting pair in $\preceq_{2\ell_{n+1}}$ with utility difference q_{n+1}^+. As $n+1$ is even, Proposition 6 gives $q_{n+1}^+ \equiv 2 \pmod 4$. This implies that either $1 \in C$ or $1 \in D$. Now as above, we can show that $C < D$ can be obtained by the second or the first method, respectively. Thus $g_{n+1} = g_n + h_n$.

For even n, the statement can be proved similarly, using the other two equations in Proposition 7. □

Proof of Theorem 5. Let us consider the case $n = 3$. We have $q_3 = 3$, so $q_3^- = 2$ and $q_3^+ = 4$. We have three nonintersecting pairs in $\preceq_{2\ell_3}$ with utility difference two, namely $\emptyset < 1$, $1 < 2$, and $12 < 3$, and two nonintersecting pairs with utility difference four, namely, $\emptyset < 2$ and $2 < 3$. Thus $g_3 = 2$ and $h_3 = 3$. Alternatively, we may say that $(g_3, h_3) = (\phi_3, \phi_4)$. It is also easy to check that $(g_4, h_4) = (5, 3) = (\phi_5, \phi_4)$. A simple induction argument now shows that $(g_n, h_n) = (\phi_n, \phi_{n+1})$ for odd n and $(g_n, h_n) = (\phi_{n+1}, \phi_n)$ for even n. By Lemma 3 we find that the number of flippable pairs of $\preceq_{\mathbf{m}}$ is

$$g_n + h_n = \phi_{n+1} + \phi_n = \phi_{n+2}.$$

It remains to notice that $\preceq_{\mathbf{m}}$ is in $\mathcal{G}_{n+1}$. □

6 Simple games related to comparative probability orders

Let us consider a finite set X consisting of n elements (which are called *players*). For convenience, X can be taken to be the set $[n] = \{1, 2, \ldots, n\}$.

Definition 11 ([18, 16])**.** *A simple game is a pair $G = (X, W)$, where W is a subset of the power set 2^X satisfying the* monotonicity condition*: if $A \in W$ and $A \subset B \subseteq X$, then $B \in W$.*

Elements of the set W are called *winning coalitions*. We also define the complement $L = 2^X \setminus W$, and call the elements of this set *losing coalitions*. A winning coalition is said to be *minimal* if each of its proper subsets is losing. By the monotonicity condition, every simple game is fully determined by its set of minimal winning coalitions. Also for $A \subseteq X$, we will denote its complement $X \setminus A$ by A^c.

Definition 12. *A simple game is called* proper *if $A \in W$ implies that $A^c \in L$, and* strong *if $A \in L$ implies that $A^c \in W$. A simple game which is proper and strong is also called a* constant-sum game.

In a constant-sum game, there are exactly 2^{n-1} winning coalitions and exactly 2^{n-1} losing coalitions.

Definition 13. *A simple game G is called a* weighted majority game *if there exists a weight function $w\colon X \to \mathbb{R}^+$ (where $\mathbb{R}^+$ is the set of all non-negative reals) and a real number q, called the* quota, *such that $A \in W$ if and only if $\sum_{i\in A} w_i \geq q$.*

Associated with every simple game $G = (X, W)$ is a *desirability relation* $\preceq_G$ on X. This was defined by Lapidot and actively studied by Peleg (see [16]).

Definition 14. *Given a simple game G we say that a coalition $A \in 2^X$ is* less desirable *than a coalition $B \in 2^X$ if it has the property that whenever the coalition $A \cup C$ is winning for some coalition $C \in 2^X$ such that $C \cap (A \cup B) = \emptyset$, the coalition $B \cup C$ is winning as well. We denote this by $A \preceq_G B$, or by $A \preceq B$ when the game is clear from the context. Let us also write $A \sim_G B$ whenever $A \preceq_G B$ and $B \preceq_G A$.*

For an arbitrary simple game G, the relation $\preceq_G$ satisfies the following weak version of the de Finetti condition: for any subsets $A, B, C \in 2^X$ such that $C \cap (A \cup B) = \emptyset$,

$$A \preceq_G B \Longrightarrow A \cup C \preceq_G B \cup C. \qquad (18)$$

(Note that the arrow is only one-sided.) In other respects, this might not be a well-behaved relation. It might not be complete, and its strict companion $\prec_G$ could be cyclic (see [16]). For the class of games we will define, however, this relation is as nice as it can be. It is also quite natural in the light of (18).

Any (strict) comparative probability order $\leq$ on $X = [n]$ defines a constant-sum simple game $G(\leq)$. Indeed, all subsets of X are ordered according to $\leq$, say

$$\emptyset < A_1 < \ldots < A_{2^{n-1}-1} < A_{2^{n-1}} < \ldots < A_{2^n-1} < X.$$

Let us take $W = \{A_{2^{n-1}}, \ldots, X\}$, to obtain a constant-sum game $G(\leq)$. The pair $(A_{2^{n-1}-1}, A_{2^{n-1}})$ is the central pair of $\leq$, and as shown in [10], we have $A^c_{2^{n-1}-1} = A_{2^{n-1}}$. Also this pair is always flippable.

Proposition 8. *If $\leq$ is defined as above, then $\leq \subseteq \preceq_{G(\leq)}$. In particular, the desirability relation of such a game is complete, and the strict desirability relation is acyclic.*

Proof. Let A and B be two subsets of X, and suppose without of loss of generality that $A \leq B$. Now suppose also that $A \cup T \in W$ for some $T \cap (A \cup B) = \emptyset$. Then by de Finetti's axiom, $A \cup T \leq B \cup T$, which implies that $B \cup T \in W$ by definition of $G(\leq)$. Thus $A \preceq_{G(\leq)} B$. □

If $\leq$ is a representable comparative probability order, then $G(\leq)$ is a weighted majority game.

Peleg asked if any constant-sum simple game with complete desirability relation and acyclic strict desirability relation is a weighted majority game. This question was answered negatively in [17] (see also [16, Section 4.10]), but the cardinality of X in that counter-example is large (and not even specified). If our previous question is answered, it could provide us with a natural way of constructing such examples for smaller n. As we will see below, however, any non-representable comparative probability order that can be used for this purpose must have some very special properties in $\mathcal{P}_n$ relative to the flip relation. The following lemma explains why.

Lemma 5. *If the comparative probability order $\leq'$ is obtained from a comparative probability order $\leq$ by a flip over a flippable pair which is not central, then $G(\leq') = G(\leq)$.*

Proof. Suppose we flip over the flippable pair (A, B). Then for any $C \subset X$ such that $C \cap (A \cup B) = \emptyset$, the sets $A \cup C$ and $B \cup C$ are neighbours and cannot split the central pair. Hence in $G(\leq)$, either both $A \cup C$ and $B \cup C$ are winning, or both are losing, and thus $A \sim_{G(\leq)} B$. The same will happen in $G(\leq')$, and so $G(\leq') = G(\leq)$. □

Corollary 1. *Let $\leq$ be any comparative probability order in $\mathcal{P}_n$. If $\leq$ is connected to a representable comparative probability order by a sequence of flips, none of which changes the central pair of $\leq$, then $G(\leq)$ is a weighted majority game.*

Theorem 6. *If $\leq \in \mathcal{P}_5$ or $\leq \in \mathcal{P}_6$, then $G(\leq)$ is a weighted majority game.*

Proof. It is known (see [18]) that every constant sum game with five players is a weighted majority game. In the case of $\leq \in \mathcal{P}_5$, we can deduce this directly from our results. First, it can be seen from Table 1 that every comparative probability order $\leq$ in $\mathcal{P}_5$ has at least two representable neighbours. At least one of these must be flip-related to $\leq$ via a non-central pair, and hence the above lemma applies. This deals with the case $n = 5$. For $n = 6$, we have used MAGMA [1] to verify that for every $\leq$ in $\mathcal{P}_6$, the probability measure $\mathbf{p}$ of some representable order $\preceq \in \mathcal{L}_6$ gives a weight function w that makes $G(\leq)$ a weighted majority game. □

7 More Open Problems

Problem 3. *Is it true that $G(\leq)$ is always a weighted majority game?*

Problem 4. *Is Searles' conjecture true?*

Problem 5. *What is the minimum value of $|\mathrm{Irr}(C)|$ in $\mathcal{G}_n$?*

Problem 6. *Is $\mathcal{G}_n$ connected?*

It was checked in [13], and independently by us, that $\mathcal{G}_6$ is connected. As all representative orders form a connected subgraph in $\mathcal{G}_n$, it would be natural to try to prove that any order in $\mathcal{G}_n$ is connected to a representable order. This is not obvious. In $\mathcal{G}_6$, for example, there are vertices (orders) without representable neighbours. A stronger version of this problem which is required for extending Theorem 6 to all n is as follows.

Problem 7. *Is any non-representable order in $\mathcal{G}_n$ connected to a representable order by a sequence of non-central flips?*

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