**Murat Sertel • Arkadii Slinko**

# Ranking Committees, Income Streams or Multisets

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**Abstract:** Multisets are collections of objects which may include several copies of the same object. They may represent bundles of goods, committees formed of members of several political parties, or income streams. In this paper we investigate the ways in which a linear order on a finite set $A$ can be consistently extended to an order on the set of all multisets on $A$ of some given cardinality $k$ and when such an extension arises from a utility function on $A$. The condition of consistency that we introduce is a close relative of the de Finetti's condition that defines comparative probability orders. We prove that, when $A$ has three elements, any consistent linear order on multisets on $A$ of cardinality $k$ arises from a utility function and all such orders can be characterised by means of Farey fractions. This is not true when $A$ has cardinality four or greater. It is proved that, unlike linear orders that can be represented by a utility function, any non-representable order on the set of all multisets of cardinality $k$ cannot be extended to a consistent linear order on multisets of cardinality $K$ for sufficiently large $K$. We also discuss the concept of risk aversion arising in this context.

**Keywords:** committee, income stream, multiset, utility, representable linear order, risk aversion

**JEL Classification Numbers:** D71

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# 1 Introduction

One of the most fundamental questions in economics is when a preference ordering on bundles of goods has a representation induced by a numerical order-preserving function of utilities of those commodities. The set of all admissible commodity bundles can be naturally represented as a subset of the Cartesian product $X = \prod_{i=1}^{n} X_i$, where the sets $X_i$ vary, depending on the model. A very commonly used assumption is that all $X_i$ are isomorphic to the set of real numbers $\mathbb{R}$. This imposes a very rich structure on $X$, which in this case becomes the $n$-dimensional Euclidean space $\mathbb{R}^n$. This case was extensively studied by J. von Neumann and O. Morgenstern [35], J. Marshak [26], I.N. Herstein and J. Milnor [20] and others. Another particular case, when the $X_i$ are all finite and have no structure at all, was studied by P. Fishburn [12].

However, there are several economic situation which require an intermediate approach when $X_i$ are finite but have some structure on each of them. We will assume that all $X_i$ are isomorphic to the set $[k] = \{1, 2, \ldots, k\}$ with the partially defined operation of addition, i.e $i + j$ is the usual sum of integers, when $i + j \le k$, and not defined otherwise. Such model has a number of useful interpretations.

The first interpretation stipulates $n$ different types of goods, which are indivisible. Each $\mathbf{x} \in X$ is a commodity bundle $\mathbf{x} = (x_1, \ldots, x_n)$, where $i$ is the type of good and $x_i$ is its quantity — which, due to the indivisibility assumption, is a nonnegative integer. The total number of goods is $k = \sum_{i=1}^{n} x_i$, hence $x_i \in [k]$. The relation $\mathbf{x} \succeq \mathbf{y}$ means that a consumer thinks $\mathbf{x}$ is at least as good as $\mathbf{y}$. (See e.g. W.D. Katzner [21].)

The second interpretation is that of an income stream. Here $i$ is a moment in time and $x_i \in [k]$ is the dollar amount of income to be received at this moment. The total amount to be received is $k = \sum_{i=1}^{n} x_i$. The relation $\mathbf{x} \succeq \mathbf{y}$ means that a beneficiary thinks income stream $\mathbf{x}$ is at least as good as income stream $\mathbf{y}$. (See T.C. Koopmans [22].)

In the third, we consider the composition of a $k$-member parliament or a committee. There are $n$ political parties to which the members are affiliated. Here $i$ is the type of political party and $k_i$ is the number of members of this political party elected to the parliament. The total number of elected parliamentarians is $k = \sum_{i=1}^{n} k_i$. The relation $\mathbf{x} \succeq \mathbf{y}$ means that a voter thinks parliament $\mathbf{x}$ is at least as good as parliament $\mathbf{y}$. (See M.R. Sertel and E. Kalaycıoğlu [31].)

We will also assume that the following linear order on the set $[n]$

$$1 \succ 2 \succ \ldots \succ n, \tag{1}$$

reflects the following preferences of the agent. The consumer has preferences over the types of goods, beneficiary has preferences over the times of receiving money (a dollar today is better than a dollar tomorrow), and the voter has preferences over the existing political parties. We will assume that all these preferences are strict and we will also require that the relation $\mathbf{x} \succeq \mathbf{y}$ on $X$ is consistent with (1) in the way which will be explained later.

Hence this problem can be viewed as the problem of consistent extension of (1) to $X$. It is in this form that the question of the consistent extension of preferences on the set of political parties to the set of committees was proposed by Sertel in a series of lectures [29], as later also published in Sertel and Kalaycıoğlu [31]. In his lecture at the IHES, "Questions for Mathematicians in Economic Design", Jan. 19.1999, Sertel posed the question of how to extend a linear order on an alphabet to the free semigroup generated by that alphabet.

Mathematically speaking, in all these examples we are talking about multisets on the set $[n] = \{1, 2, \ldots, n\}$ (see, e.g., [32]). Unlike sets, multisets allow multiple entry of elements, so in each example the object under consideration — the bundle, the income stream or the committee — can be represented as a multiset

$$M = \{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\}, \tag{2}$$

where $i^{k_i}$ means that element $i$ enters the multiset $k_i$ times. The number $k_i$ is called the multiplicity of $i$ in $M$. The multiset $M$ can also be described as $([n], \mu)$ where $\mu \colon [n] \to \mathbb{N}$ is the multiplicity function given by $\mu(i) = k_i$ for all $i \in [n]$. The sum of multiplicities $k = k_1 + k_2 + \ldots + k_n$ is called the cardinality of $M$. In all three examples we deal with orderings of the set of all multisets of fixed cardinality $k$ on the set $[n]$, which will be denoted by $\mathcal{P}_k[n]$. We will also omit $[n]$ when this invites no confusion. The usual notions for sets can be carried over to multisets in a natural way. We say that a multiset $M = ([n], \mu)$ is a subset of a multiset $M' = ([n], \mu')$ and write $M \subseteq M'$ iff $\mu(i) \le \mu'(i)$ for all $i \in [n]$.

It has to be noted that orders on multisets and its subsets which extend a given linear order on the underlying set on which multisets are constructed have been instrumental in proofs of program termination, in equational reasoning algorithms based on term rewriting systems, in computer algebra, the theory of invariants, and the theory of partitions. We refer the reader to the two surveys by Martin [27] and Dershowitz [8] and to the book by Kreuzer and Robbiano [25] on these topics.

Following [2], any reflexive, complete and transitive relation will be called an *order*[1] and any antisymmetric order will be called a *linear order*. An order will typically be denoted as $\succeq$. In this case $\succ$ will be the strict preference relation of $\succeq$, i.e. $M \succ N$ will mean that $M \succeq N$ holds but $N \succeq M$ fails; and $M \sim N$ will be the indifference relation of $\succeq$, i.e. $M \sim N$ will mean that both $M \succeq N$ and $N \succeq M$ hold.

In each of the applications mentioned above, the orderings of multisets, which were useful, were those which were "consistent" with a given linear order (1) on the alphabet $[n]$. To define the concept of consistency suitable for our purposes, we need the following notation. Let $\succeq$ be an order on $\mathcal{P}_k[n]$, $k \ge 2$, and let $1 \le j < k$. Then for any $W \in \mathcal{P}_j[n]$ we may define an order $\succeq_W$ on $\mathcal{P}_{k-j}[n]$ by setting, for any $U, V \in \mathcal{P}_{k-j}[n]$,

$$U \succeq_W V \iff U \cup W \succeq V \cup W. \tag{3}$$

Now we can give the following recursive definition:

**Definition 1** (Consistency)**.** *The only* consistent *order on $\mathcal{P}_1[n]$ is induced by (1). Suppose that consistent orders on $\mathcal{P}_\ell[n]$ for $\ell < k$ are defined. An order $\succeq$ on $\mathcal{P}_k[n]$, $k \ge 2$, is said to be* consistent *iff for every $j = 1, 2, \ldots, k-1$, the orders $\succeq_W$ for all words $W$ of fixed cardinality $j$ coincide, and this common order is a consistent order of $\mathcal{P}_{k-j}[n]$.*

In other words, any consistent order $\succeq$ on $\mathcal{P}_k(A)$ stipulates, for $\ell = 1, 2, \ldots, k$, the existence of consistent orders $\succeq_\ell$ on $\mathcal{P}_\ell(A)$ with $\succeq_k = \succeq$, such that for any multiset $W$ of cardinality $j - i > 0$ and for any two multisets of $U, V \in \mathcal{P}_i(A)$

$$U \succeq_i V \iff U \cup W \succeq_j V \cup W, \tag{4}$$

provided $j \le k$. We might think of multisets geometrically, identifying the multiset (2) with the point $(k_1, k_2, \ldots, k_n)$ of the positive orthant of $\mathbb{R}^n$. Then the set $\mathcal{P}_i(A)$ will be represented as the layer of integer points of the positive orthant lying on the hyperplane $x_1 + \ldots + x_n = i$, the orders $\succeq_i$ will order these layers and the condition (4) will be responsible for the coordination of the layers. We will say that $\succeq_j$ extends $\succeq_i$ for $j > i$. In particular, $\succeq = \succeq_k$ extends a certain order on $\mathcal{P}_1(A)$. Since $\mathcal{P}_1(A)$ can be naturally identified with $A$ itself, we may think that $\succeq$ extends a certain order on $A$.

This condition of consistency is somewhat stronger than the multiset analogue of Bossert's condition of Responsiveness [5], but weaker than the multiset analogue of the Strong Extended Independence [2].

Orders induced by a utility function, where the word "utility" serves as a generic name for a number of related but distinct concepts, play an exceptional role in statistics [13, 28], the representational theory of measurement and decision making [11], as well as computer science [23] and related other areas. They will play an important role in this paper as well.

**Definition 2.** *We will say that an order $\succeq$ on $\mathcal{P}_k[n]$ is* (additively) representable *iff there exist nonnegative real numbers $u_1, \ldots, u_m$ such that, for all $M = ([n], \mu)$, $N = ([n], \nu)$ belonging to $\mathcal{P}$,*

$$M \succeq N \iff \sum_{i=1}^{n} \mu(i) u_i \ge \sum_{i=1}^{n} \nu(i) u_i. \tag{5}$$

We will also refer to the numbers $u_1, \ldots, u_n$ as the *utilities* of $1, 2, \ldots, n$. For any multiset $M = ([n], \mu)$, the number $u(M) = \sum_{i=1}^{n} \mu(i) u_i$ will be referrred to as the *total utility* of $M$. Obviously, any representable order on $\mathcal{P}_k[n]$ is consistent.

[1] equivalently, the expressions of "complete preorder" and "total preorder" are also used for the same purpose.

A similar concept is also used for defining orders on subsets of $A$ (see [24] and the survey [2]) and also on the Cartesian product $A_1 \times A_2 \times \ldots \times A_n$, where the utility of a tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is defined as $u(\mathbf{a}) = \sum_{i=1}^{n} u_i(a_i)$, with $u_i$ a utility function for $A_i$ (see Fishburn [10, 11, 12]).

The following class of linear orders on $\mathcal{P}_k[n]$ is also of interest.

**Definition 3.** *We will say that an order $\succeq$ on $\mathcal{P}_k[n]$ is* almost representable *iff there exist nonnegative real numbers $u_1, \ldots, u_m$ such that, for all $M = ([n], \mu)$, $N = ([n], \nu)$ belonging to $\mathcal{P}_k[n]$,*

$$M \succeq N \Longrightarrow u(M) \geq u(N). \tag{6}$$

If an order $\succeq$ is almost representable, then for some two multisets $M$ and $N$ it is possible to have $M \succ N$ while having $u(M) = u(N)$. Almost representable orders need not be representable (see Sect. 3) and the latter class is broader than the former.

In this paper we prove that, for all integers $k \geq 1$, all consistent orders on $\mathcal{P}_k[3]$ are representable. Moreover, all consistent orders on $\mathcal{P}_k[3]$ form a one-parametric family and with the parameter has characterising the degree of risk aversion in the case of a consumer or a voter and the degree of impatience in the case of a beneficiary. We classify consistent orders on $\mathcal{P}_k[3]$ in terms of Farey fractions and find the asymptotics for the number of consistent orders in $\mathcal{P}_k[3]$.

We show that there exist 12 consistent linear orders on $\mathcal{P}_2[4]$, and that two of them, namely $A_4$ and $E_4$ in Figure 1, are not representable. Moreover, none of these two linear orders can be extended to a consistent order on $\mathcal{P}_3[4]$. However $A_4$ and $E_4$ are almost representable. We also present an example of a consistent linear order on $\mathcal{P}_3[4]$ which is not even almost representable. As the full classification of consistent linear orders on $\mathcal{P}_3[4]$ shows there are 18 such orderings. There are 30 almost representable but not representable linear orders and 80 representable linear orders.

We prove that there exists a positive-integer-valued function $k \mapsto f(k) > k$ such that, for all $n \geq 4$, a linear order $\succeq$ on $\mathcal{P}_k[n]$ can be extended to a consistent linear order on $\mathcal{P}_{f(k)}[n]$ iff it is representable. This means that having a non-representable preference order on $\mathcal{P}_k[n]$ is an early sign of global inconsistency of the agent as she will not be able to rank consistently multisets of $\mathcal{P}_K[n]$ for some $K > k$.

Finally, for an arbitrary positive integer $n$, we give the lower bound for the number of representable linear orders on $\mathcal{P}_2[n]$ and the lower bound for the total number of consistent linear orders on $\mathcal{P}_2[n]$.

## 2 Orders on $\mathcal{P}_k[3]$

We will start with the following example.

**Example 1** (Sertel, [29, 31])**.** *Let us take $n = 3$ and $k = 2$. Then, in any consistent order $\succeq$ on $\mathcal{P}_2[3]$, the order $1 \succ 2 \succ 3$ determines all relations between the pairs (multisets from $\mathcal{P}_2[3]$) except the relation between $\{1, 3\}$ and $\{2, 2\}$. Thus, to construct a consistent order of $\mathcal{P}_2[3]$ we have one degree of freedom and hence we can have at most two different consistent linear orders on pairs. We will show that both opportunities materialise.*

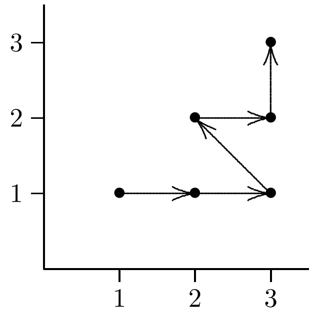
*Assuming $\{1, 3\} \succ \{2, 2\}$ we will obtain the "risk-loving" linear order,*

$$\{1, 1\} \succ \{1, 2\} \succ \{1, 3\} \succ \{2, 2\} \succ \{2, 3\} \succ \{3, 3\}, \tag{7}$$

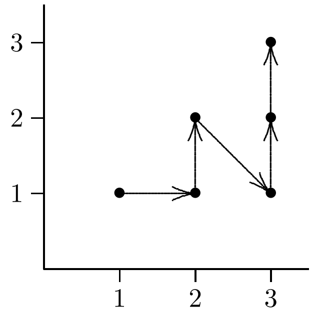
*and assuming $\{2, 2\} \succ \{1, 3\}$ we have the "risk-avoiding" one,*

$$\{1, 1\} \succ \{1, 2\} \succ \{2, 2\} \succ \{1, 3\} \succ \{2, 3\} \succ \{3, 3). \tag{8}$$

*They are graphically represented as follows:*



*Risk loving*



*Risk avoiding*

*Assuming* $\{1,3\} \sim \{2,2\}$ *we will obtain an order,*

$$\{1,1\} \succ \{1,2\} \succ \{1,3\} \sim \{2,2\} \succ \{2,3\} \succ \{3,3\}, \tag{9}$$

*which we call "risk-neutral." An agent with the first ranking is willing to forgo two instances of certain outcome* 2 *in favour of the possibility of getting the best choice* 1*, even together with the worst possible outcome* 3*. Clearly, this is not the case for the second ranking.*

*Consistency of these orders is easy to check. We observe that all three orders of* $\mathcal{P}_2[3]$*, the risk-loving, the risk-avoiding, and the risk-neutral, are representable with the utilities* $(u_1, u_2, u_3)$ *satisfying*

$$\frac{u_1+u_3}{2} > u_2, \quad \frac{u_1+u_3}{2} < u_2, \quad or \quad \frac{u_1+u_3}{2} = u_2,$$

*respectively. The utility function* $u\colon \{1,2,3\} \to \mathbb{R}$ *is concave up, concave down and linear respectively.*

*If our mutisets are interpreted as income streams, then an agent with the first ranking will prefer \$100 now and \$100 two month later to \$200 one month later. Clearly she puts a great emphasis on receiving something immediately, while an agent with the second ranking puts an emphasis on receiving the whole sum earlier. The first agent can be called impatient and the second can be called patient.*

*As in the case with "utility" the term "risk aversion" that we will use here is a generic name for a number of related but distinct concepts.*

In this section we prove a theorem that describes all consistent orders of $\mathcal{P}_k[3]$ for all $k$. First we prove that they are all representable and then we describe all representable orders in terms of the Farey fractions. We need to remind the reader of several definitions and concepts from Number Theory. The famous Farey sequence of fractions $\mathbf{F}_k$ is the increasing sequence of all fractions in lowest terms between 0 and 1 whose denominators do not exceed $k$. For example, the sequence $\mathbf{F}_6$ will be:

$$\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}.$$

Let $\phi\colon \mathbb{N} \to \mathbb{N}$ be the Euler totient function, for which $\phi(1) = 1$ and, for $h \geq 2$, $\phi(h)$ is the number of positive integers not exceeding $h$ and relatively prime to $h$. It is easy to see that passing from $\mathbf{F}_k$, to $\mathbf{F}_{k+1}$, there will be exactly $\phi(k+1)$ new Farey fractions added. Therefore there are exactly $\Phi(k)+1$ fractions in $\mathbf{F}_k$, where $\Phi(k) = \sum_{h=1}^{k} \phi(h)$. (The standard reference for Farey fractions is [19]. See also [9]).

For the proofs of Theorem 1 and Theorem 7 we need the following

**Lemma 1.** *Let* $\succeq$ *be a consistent order on* $\mathcal{P}_k[n]$ *and* $\succeq_i$, $i = 1, \ldots, k$, *be the corresponding orders on* $\mathcal{P}_i[n]$ *defined in Sect. 1.*

*(a) If* $U, V \in \mathcal{P}_\ell[n]$, $R, Q \in \mathcal{P}_h[n]$, *and* $U \succeq_\ell V$, $R \succeq_h Q$, $\ell + h \leq k$, *then* $U \cup R \succeq_{\ell+h} V \cup Q$.

*(b) If $U, V \in \mathcal{P}_\ell[n]$, and $U \succeq_\ell V$, $\ell h \le k$, for some integer $h$, then*

$$\underbrace{U \cup U \cup \ldots \cup U}_{h} \succeq_{\ell h} \underbrace{V \cup V \cup \ldots \cup V}_{h}.$$

*Proof.* (a) Due to consistency of $\succeq$, we get $U \cup R \succeq_{\ell+h} V \cup R \succeq_{\ell+h} V \cup Q$.
(b) follows from (a) by induction. $\square$

**Theorem 1.** *All consistent orders on $\mathcal{P}_k[3]$ are representable. Every such order can be represented by a vector of utilities $(u_1, u_2, u_3) = (1, \gamma, 0)$, where $\gamma \in (0, 1)$. Two vectors of utilities $(1, \gamma_1, 0)$ and $(1, \gamma_2, 0)$ determine the same order on $\mathcal{P}_k[3]$ iff $\gamma_1$ and $\gamma_2$ belong to the same interval $(f_i, f_{i+1})$ between two neighboring Farey fractions of $\mathbf{F}_k$. The vector of utilities $(1, \gamma, 0)$ determines a non-linear order iff $\gamma \in \mathbf{F}_k$.*

*Proof.* Here we will give only an outline of the proof and defer the rest of it to the Appendix. The proof is by induction on $k$. We assume that the statement is true for $\mathcal{P}_k[3]$. Firstly, we explore the ways in which a consistent order $\succeq$ on $\mathcal{P}_k[3]$ can be extended to an order on $\mathcal{P}_{k+1}[3]$. It appears that the only freedom we have is to choose the position of $\{2^{k+1}\}$ but sometimes we do not have freedom at all. We know that $\succeq$ corresponds to a certain vector of weights $(1, \gamma_1, 0)$ where $\gamma$ is a Farey fraction or belongs to the interval between two neighboring fractions $(f_i, f_{i+1})$. In the former case we have no freedom and the continuation is unique. It also appears that if none of the fractions

$$\frac{1}{k+1}, \; \frac{2}{k+1}, \; \ldots, \; \frac{k}{k+1}.$$

falls into $(f_i, f_{i+1})$, then continuation is unique again. Finally, if $\frac{j}{k+1}$ falls into $(f_i, f_{i+1})$ (and only one Farey fraction can fall), then there are three continuations: one corresponding the interval $\left(f_i, \frac{j}{k+1}\right)$, another to the interval $\left(\frac{j}{k+1}, f_{i+1}\right)$, and the third, when $\gamma = \frac{j}{k+1}$. The difference between the three will be in relative position of $\{2^{k+1}\}$ and $\{1^j, 3^{k+1-j}\}$. We will have

$$\{2^{k+1}\} \prec \{1^j, 3^{k+1-j}\}, \quad \{2^{k+1}\} \succ \{1^j, 3^{k+1-j}\}, \quad \{2^{k+1}\} \sim \{1^j, 3^{k+1-j}\},$$

respectively. $\square$

**Example 2.** *Suppose that we are trying to estimate the degree of impatience of an agent. Suppose that the triple $(a, b, c)$ denotes the income stream, when the agent receives \$a now, \$b one month later and \$c two months later. We can give her to compare the income stream $(0, 120, 0)$ with the income streams*

$$(30, 0, 90), \; (40, 0, 90), \; (60, 0, 60), \; (80, 0, 40), \; (90, 0, 30).$$

*Then, if the agent says that she is indifferent between $(0, 120, 0)$ and $(80, 0, 40)$, then her degree of impatience is exactly $2/3$. If she thinks that $(0, 120, 0)$ is better than $(60, 0, 60)$ but worse than $(80, 0, 40)$, then her degree of impatience is between $1/2$ and $2/3$ and in this case we might want to ask her to compare $(0, 120, 0)$ with $(72, 0, 48)$ to find out if it is between $1/2$ and $2/5$ or between $2/5$ and $2/3$ or equal to $2/5$ exactly.*

**Corollary 1.** *For any integer $k \ge 1$, there exist exactly $2\Phi(k) - 1$ different consistent orders of $\mathcal{P}_k[3]$ and $\Phi(k)$ of these are linear orders.*

Note that the proof of Theorem 1 is algorithmic and, given a consistent order on $\mathcal{P}_k[3]$, it should be easy to write a computer program to construct utilities for this order.

**Corollary 2.** *Asymptotically, there are*

$$N(k, 3) = \frac{6k^2}{\pi^2} + O(k \log k) \tag{10}$$

*different orders on $\mathcal{P}_k[3]$, and approximately half of them are linear.*[2]

*Proof.* A celebrated result by Franz Mertens (see, for example, [7], p. 59) establishes the asymptotics of the sum $\Phi(k)$, namely,

$$\Phi(k) = \frac{3k^2}{\pi^2} + O(k \log k). \tag{11}$$

From this and Theorem 1, our asymptotic formula immediately follows. $\square$

In Section 3 we will show that, for any $n \geq 4$, there exist linear orders on $\mathcal{P}_2[n]$ that fail to be representable. It is also worthwhile to note that one cannot replace in Theorem 1 multisets of cardinality $k$ with multisets of cardinality $\leq k$. Indeed, it can be easily checked that the linear order

$$\{1,1\} \succ \{1,2\} \succ \{2,2\} \succ \{1,3\} \succ \{1\} \succ \{2,3\} \succ \{3,3\} \succ \{2\} \succ \{3\} \succ \emptyset$$

is not representable.

# 3 Consistent Linear Orders on $\mathcal{P}_2[n]$

In this section we study consistent linear orders on $\mathcal{P}_2[n]$. These linear orders appear to be important in the recent study [6], where it was discovered that any linear order on subsets which satisfies Dominance and Conditional Independence can be obtained from a consistent order on multisets of cardinality 2 consisting for each subset of its best and its worst elements. As usual, we assume that (1) is satisfied. To describe consistent linear orders of multisets of cardinality two (pairs), even in the case of $\mathcal{P}_2[4]$, we need the concept of reducibility. In other words, we need to know which linear orders can be obtained by combining two or more simpler ones.

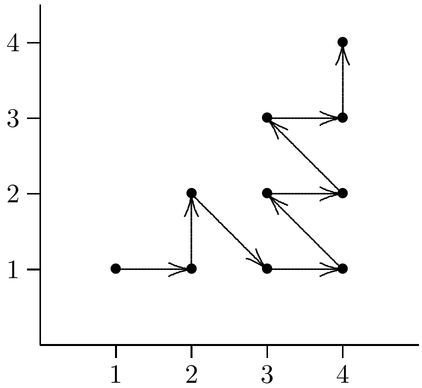
**Definition 4.** *Let $R_1$ and $R_2$ be two consistent linear orders on $\mathcal{P}_2[n_1]$ and $\mathcal{P}_2[n_2]$, respectively. We define a linear order $R = R_1 \times R_2 \in \mathcal{P}_2[n_1+n_2]$ as follows:*

1. *$\{i,j\}R\{p,q\} \Leftrightarrow \{i,j\}R_1\{p,q\}$ whenever $i,j,p,q \in [n_1]$;*
2. *$\{i,j\}R\{p,q\} \Leftrightarrow \{i-n_1, j-n_1\}R_2\{p-n_1, q-n_1\}$ whenever $i,j,p,q \in [n_1+n_2] \setminus [n_1]$;*
3. *$\{i,j\}R\{p,q\}$ is always true when $i,j \in [n_1]$ and either $p \notin [n_1]$ or $q \notin [n_1]$ (or both);*
4. *$\{i,j\}R\{p,q\}$ is always true when $p,q \in [n_1+n_2] \setminus [n_1]$ and either $i \in [n_1]$ or $j \in [n_1]$ (or both);*
5. *If $i,j \in [n_1]$ with $i \neq j$ and $p,q \in [n_1+n_2] \setminus [n_1]$, then $\{i,p\}R\{j,q\}$ if and only if $i < j$.*
6. *If $i \in [n_1]$ and $p,q \in [n_1+n_2] \setminus [n_1]$ with $p \neq q$, then $\{i,p\}R\{i,q\}$ if and only if $p < q$.*

*The linear order $R_1 \times R_2$ will be called the product of $R_1$ and $R_2$.*

**Example 3.** *Let us denote the only linear order on $\mathcal{P}_1[1]$ as $I$. Then the only consistent linear order of $\mathcal{P}_2[2]$ will be $I \times I$. The two consistent linear orders in Example 1 will be $I \times (I \times I)$ and $(I \times I) \times I$, respectively. The linear order $(I \times I) \times (I \times I)$ of $\mathcal{P}_2[4]$ will have the diagram shown below:*

[2] We write $g(n) = O(f(n))$ in case there is a positive constant $C$ such that $|g(n)| \leq Cf(n)$ for all sufficiently large values of $n$




*Example of a reducible order*

**Definition 5.** *We will call a linear order $R$ on $\mathcal{P}_2[n]$* irreducible *if it cannot be represented as $R = L \times M$ for linear orders $L$ and $M$ on $\mathcal{P}_2[n_1]$ and $\mathcal{P}_2[n_2]$, respectively, for any positive integers $n_1$ and $n_2$ with $n_1 + n_2 = n$. Otherwise, it will be called* reducible.

**Theorem 2.** *If linear orders $L$ and $M$ on $\mathcal{P}_2[n_1]$ and $\mathcal{P}_2[n_2]$, respectively, are representable, then their product $L \times M$ is also a representable linear order.*

*Proof.* First we notice that, for all $k$ and $n$, a representable linear order of $\mathcal{P}_k[n]$ with the utilities $u_1, \ldots, u_n$ will stay unchanged if for some integers $a > 0$ and $b$, for all $i = 1, \ldots, n$, we undertake an affine transformation $u_i \mapsto u_i' = au_i + b$ of its utilities. Let $u_1, \ldots, u_{n_1}$ be a system of utilities for $L$ and $v_1, \ldots, v_{n_2}$ a system of utilities for $M$. Let $V$ be the sum of all utilities of the second system and $a, b$ be two sufficiently large integers such that the new system of utilities $u_1', \ldots, u_{n_1}'$, where $u_j' = au_j + b$, satisfies the following two conditions: $2u_{n_1}' > u_1' + V$ and $u_{k+1}' - u_k' > V$ for all $1 \le k < n_1 - 1$. Then the system of utilities $u_1', \ldots, u_{n_1}', v_1, \ldots, v_{n_2}$ will define $L \times M$. This proves the theorem. □

As we saw, all consistent linear orders of $\mathcal{P}_2[n]$ for $1 < n \le 3$ are reducible. In $\mathcal{P}_2[4]$ we will have seven irreducible ones. They are all described in the following theorem.

**Theorem 3.** *There are 12 distinct linear orders in $\mathcal{P}_2[4]$:*

1. *The five reducible consistent linear orders are*

$$
\begin{aligned}
R_{1,4} &= I \times (I \times (I \times I)), \\
R_{2,4} &= I \times ((I \times I) \times I), \\
R_{3,4} &= (I \times I) \times (I \times I), \\
R_{4,4} &= (I \times (I \times I)) \times I, \\
R_{5,4} &= ((I \times I) \times I) \times I,
\end{aligned}
$$

   *all of which are representable;*

2. *The seven irreducible consistent linear orders are $A_4$, $B_4$, $C_4$, $D_4$, $E_4$, $F_4$, $G_4$, given by their diagrams in Figure 1. Five of them apart from $A_4$ and $E_4$ are representable. $A_4$ and $E_4$ are not representable but they are almost representable.*

*Proof.* There are only five different arrangements of brackets that convert an associative word $x_1x_2x_3x_4$ of length four into a non-associative word. These non-associative words are $x_1(x_2(x_3x_4))$, $x_1((x_2x_3)x_4)$, $(x_1x_2)(x_3x_4)$, $(x_1(x_2x_3))x_4$, $((x_1x_2)x_3)x_4$. They are distinct elements of the free non-associative monoid (see, for example, [36], Chapter 1). Accordingly, we can construct five reducible consistent linear orders as listed in the theorem. They are all representable, due to Theorem 2. It is easy to check directly that they are distinct (but later we will prove a general statement in this respect). There are no other reducible consistent linear orders in $\mathcal{P}_2[4]$ since, if $R$ is reducible, then $R = P \times Q$, where $I$, $I \times I$, $I \times (I \times I)$, $(I \times I) \times I$ are the only possibilities for $P$ and $Q$.

The utilities for $B_4$, $C_4$, $D_4$, $F_4$, $G_4$ can be chosen according to Figure 2. Since any affine transformation $u_i' \mapsto au_i + b$ of the system of utilities does not change the order, when $a > 0$, we normalize these utilities so that $u_4 = 0$ and $u_1 = 1$. Then every point $(u_2, u_3)$ of the triangle shown on the figure represents a consistent order on $\mathcal{P}_2[4]$. The boundary between the orders for which $\{1,4\} \succ \{2,3\}$ and the orders for which $\{2,3\} \succ \{1,4\}$ will be the line $u_2 + u_3 = 1$. The boundary between the orders for which $\{2,2\} \succ \{1,3\}$ and the orders for which $\{1,3\} \succ \{2,2\}$ will be the line $2u_2 - u_3 = 1$, etc. After drawing all such lines we get ten regions which correspond to all ten representable linear orders on $\mathcal{P}_2[4]$.

The order $A_4$ cannot be representable for the following reasons: since we have

$$\{1,3\}A_4\{2,2\}, \quad \{2,3\}A_4\{1,4\}, \quad \{2,4\}A_4\{3,3\},$$

any system of utilities for $A_4$ would have

$$u_1 + u_3 > 2u_2, \quad u_2 + u_3 > u_1 + u_4, \quad u_2 + u_4 > 2u_3.$$

This system is inconsistent since adding the first and the third inequality gives us $u_2 + u_3 < u_1 + u_4$, i.e. just the opposite of the second inequality. Nevertheless the system of inequalities

$$u_1 + u_3 \geq 2u_2, \quad u_2 + u_3 \geq u_1 + u_4, \quad u_2 + u_4 \geq 2u_3.$$

is consistent with the only solution of it being the point $(\frac{2}{3}, \frac{1}{3})$. This means that $A_4$ is almost representable for the utilities $u_2 = \frac{2}{3}$ and $u_3 = \frac{1}{3}$.

Similar arguments apply also to $E_4$. Further, we are about to prove that $A_4$ and $E_4$ cannot be extended to a consistent order of $\mathcal{P}_3[4]$. $\square$

**Theorem 4.** *Let $\succeq$ be a linear order of $\mathcal{P}_2[n]$ and suppose that there exist indices $i, j, k, \ell$ satisfying at least one of the following two conditions:*

*(1) $\{i,i\} \succ \{j,k\}$, $\{j,\ell\} \succ \{i,k\}$ and $\{k,k\} \succ \{i,\ell\}$;*

*(2) $\{j,k\} \succ \{i,i\}$, $\{i,k\} \succ \{j,\ell\}$ and $\{i,\ell\} \succ \{k,k\}$;*

*Then this order cannot be extended to a consistent linear order on $\mathcal{P}_3[n]$.*

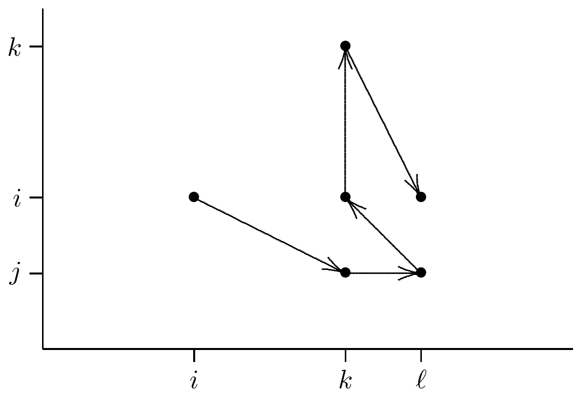



Illustration of the Condition (1)

*Proof.* We will show that the requirement of consistency for any extension of $\succ$ leads to intransitivity of the extension. Indeed, in the first case

$$\begin{array}{rcl} \{i^2\} \succ \{j,k\} & \Longrightarrow & \{i^2,\ell\} \succ \{j,k,\ell\} \\ \{j,\ell\} \succ \{i,k\} & \Longrightarrow & \{j,k,\ell\} \succ \{i,k^2\} \\ \{k^2\} \succ \{i,\ell\} & \Longrightarrow & \{i,k^2\} \succ \{i^2,\ell\}, \end{array}$$

so no consistent extension can be transitive. The second case is similar. $\square$

**Corollary 3.** *$A_4$ and $E_4$ cannot be extended to a consistent linear order on $\mathcal{P}_3[4]$.*

*Proof.* Indeed, in both cases we can spot two arrows going in one direction and an arrow between them going in the opposite direction. Thus, $E_4$ satisfies the first condition of Theorem 4 with $i = 2$, $j = 1$, $k = 3$, while $A_4$ satisfies the second condition for the same set of parameters. $\square$

We also need to remind the reader of the Catalan numbers (See, for example, [33] or [17], Ch. 20). We need them because, among other things, the $n$th Catalan number

$$c(n) = \frac{1}{n+1}\binom{2n}{n}$$

describes the number of ways in which brackets (parentheses) can be placed in an associative word $x_1x_2\dots x_n$ of length $n$ to determine the order in which the indeterminates must be multiplied. In other words $c(n)$ characterizes the number of different non-associative words that can be defined on the associative word $x_1x_2\dots x_n$. Let us denote the set of all such non-associative words by $\mathcal{W}$.

Let $w = (x_1x_2\dots x_n)_q$ be a non-associative word belonging to $\mathcal{W}$ with the arrangement of brackets $q$. Then we can construct a linear order on $\mathcal{P}_2[n]$

$$w(I) = (I \times I \times \dots \times I)_q$$

where the operation is the product of linear orders given in Definition 4.

**Lemma 2.** *Let $w_1 = (x_1x_2\dots x_n)_{q_1}$ and $w_2 = (x_1x_2\dots x_n)_{q_2}$ be two non-associative words belonging to $\mathcal{W}$ with the arrangements of brackets $q_1$ and $q_2$, respectively. Let $w_1(I)$ and $w_2(I)$ be the corresponding linear orders. Then $w_1(I) = w_2(I)$ if and only if $q_1 = q_2$.*

*Proof.* We will prove this statement by induction. As we saw in Example 1 for $n = 3$ we have only two different arrangements of brackets and they correspond to different linear orders. This gives us a basis for our induction.

As is known (see, for example, [36]) any non-associative word has a unique representation as a product of two non-associative words of smaller length. Suppose now that

$$\begin{aligned} w_1 &= (x_1\dots x_k\,)_{r_1}(x_{k+1}\dots x_n)_{r_2}, \\ w_2 &= (x_1\dots x_m)_{s_1}(x_{m+1}\dots x_n)_{s_2}, \end{aligned}$$

where $r_1, r_2, s_1, s_2$ are certain arrangements of brackets. We denote $\succ_1 = w_1(I)$ and $\succ_2 = w_2(I)$. If $k = m$, then either $r_1 \neq s_1$ or $r_2 \neq s_2$ and we may apply the induction hypothesis. Suppose now that $m > k$ (we can do this without loss of generality). Then $x_{k+1}$ can be found in the first bracket of $w_2$. This will lead to $\{1, n\} \succ_1 \{k+1, k+1\}$ but $\{k+1, k+1\} \succ_2 \{1, n\}$. This shows that $\succ_1$ and $\succ_2$ are different. The theorem is proved. $\square$

**Theorem 5.** *There exist at least $c(n)$ representable reducible linear orders on $\mathcal{P}_2[n]$. In total, there are at least $2^{2n-5}$ consistent linear orders on $\mathcal{P}_2[n]$.*

*Proof.* By Lemma 2 we can produce as many representable reducible linear orders on $\mathcal{P}_2[n]$ as claimed, using the trivial order $I$ and the product operation defined above. Indeed, we have one such linear order for any non-associative arrangement of brackets on a word $x_1x_2\dots x_n$ of length $n$.

Let us see now what we can do if we drop reducibility and do not assume our linear orders to be representable but only consistent. Let $i \in [n]$. By a "diagonal" let us mean any set of pairs satisfying one of the two following properties:

1. $\{i, i\}, \{i-1, i+1\}, \dots, \{1, 2i-1\}$, in case $2i - 1 \le n$,

2. $\{i, i\}, \{i-1, i+1\}, \dots, \{2i-n, n\}$, in case $2i - 1 > n$.

For each diagonal, we independently choose a direction of arrows and follow it through the whole diagonal. For example, we may choose

$$\{i,i\} \succ \{i-1,i+1\} \succ \ldots \succ \{1,2i-1\}$$

or

$$\{i,i\} \prec \{i-1,i+1\} \prec \ldots \prec \{1,2i-1\}.$$

By choice of directions on all diagonals, a linear order is defined uniquely, and clearly, this will be a consistent order. Since we have $2n-5$ such diagonals and their directions are chosen independently, we can construct at least $2^{2n-5}$ consistent linear orders. The theorem is proved. □

To compare the two bounds we note that asymptotically the $n$th Catalan number is

$$c(n) \sim \frac{1}{\sqrt{\pi}} \frac{2^{2n}}{n^{3/2}} = \frac{1}{\sqrt{\pi}} 2^{2n-\frac{3}{2}\log_2 n}.$$

This can be found in [34]. As we see the second bound is only slightly better than the first.

# 4 Consistent Linear Orders on $\mathcal{P}_3[4]$

The phenomenon that we encounter here is the existence of a consistent linear order which is not even almost representable. We start with such an example.

**Example 4.** *Let us consider the following consistent linear order $R$ on $\mathcal{P}_3[4]$:*

$\{1^3\} \succ \{1^2,2\} \succ \{1^2,3\} \succ \{1^2,4\} \succ \{1,2^2\} \succ \{1,2,3\} \succ \{1,2,4\} \succ \{1,3^2\} \succ \{1,3,4\} \succ \{2^3\} \succ \{2^2,3\} \succ \{1,4^2\} \succ \{2^2,4\} \succ \{2,3^2\} \succ \{2,3,4\} \succ \{2,4^2\} \succ \{3^3\} \succ \{3^2,4\} \succ \{3,4^2\} \succ \{4^3\}$

*which extends the lexicographic order on $\mathcal{P}_2[4]$ (or $R_{1,4}$ in our notation on Figure 2) and fails to be almost representable. Indeed we have:*

$$\{2^2,3\} \succ \{1,4^2\}, \tag{12}$$
$$\{2,4^2\} \succ \{3^3\}, \tag{13}$$
$$\{1,3,4\} \succ \{2^3\}. \tag{14}$$

*If this ranking were almost representable then the respective system of inequalities*

$$\begin{array}{rcl} 2u_2+u_3 & \geq & 1 \\ u_2 & \geq & 3u_3 \\ 1+u_3 & \geq & 3u_2 \end{array}$$

*would be consistent, but it is not. These inequalities imply $0 \geq u_3$, which contradicts our assumption that $u_3 > u_4 = 0$.*

To demystify this example, we note that if in $R$ we change the signs in (12,13,14) to the opposite, effectively swapping three pairs of neighboring multisets, we will obtain a consistent linear order. Moreover, it is representable: one can take, for example, $u_2 = \frac{337}{840}$, and $u_3 = \frac{131}{840}$. Since the pairs of multisets do not have common elements, the order after the swap will be consistent if and only if the original one was consistent. A full classification of consistent linear orders on $\mathcal{P}_3[4]$ is contained in the following theorem.

**Theorem 6.** *There are 128 consistent linear orders on $\mathcal{P}_3[4]$. Among them 80 are representable, 30 are almost representable but not representable. The remaining 18 are not almost representable.*

*Proof.* The classification was obtained first by hand and then checked with the help of the computer algebra system Magma [3]. □

## 5 Nonextendability of Linear Orders That Are Not Representable

**Theorem 7.** *For any two positive integers $n \geq 4$ and $k$, there exists an integer $f(n,k) > k$ such that a linear order $\succeq$ on $\mathcal{P}_k[n]$ can be extended to a consistent linear order on $\mathcal{P}_{f(n,k)}[n]$ if and only if it is representable.*

*Proof.* It is clear that any representable order $\succeq$ on $\mathcal{P}_k[n]$ can be extended to a representable order of $\mathcal{P}_m[n]$ for all $m \geq k$. It is enough to take the order with the same utilities.

Suppose now that $\succeq$ is not representable. Under this assumption, we will prove that $\succeq$ cannot be extended to a consistent order of $\mathcal{P}_{f(n,k)}[n]$ for $f(n,k) = k^{n+1}n^{n/2+1}$. This number is huge but sufficient for our purposes. We do not attempt to find a smaller one.

Let $N = \binom{k+n-1}{k}$, which is the total number of multisets in $\mathcal{P}_k[n]$. Let us enumerate all multisets $Q_i$, $(i = 1, \ldots, N)$ of $\mathcal{P}_k[n]$ so that

$$Q_1 \succ Q_2 \succ \ldots \succ Q_N. \tag{15}$$

Let $\mu_i$ be the multiplicity function of $Q_i$. To each multiset $Q_i$ we assign a linear form $f_i \colon \mathbb{R}^n \to \mathbb{R}$ such that for any $\mathbf{x} \in \mathbb{R}^n$

$$f_i(\mathbf{x}) = \sum_{j=1}^{n} \mu_i(j) x_j \qquad (i = 1, \ldots, N). \tag{16}$$

Then the following system of $N-1$ linear inequalities

$$\begin{aligned} f_1(\mathbf{x}) - f_2(\mathbf{x}) &> 0 \\ f_2(\mathbf{x}) - f_3(\mathbf{x}) &> 0 \\ &\ldots \\ f_{N-1}(\mathbf{x}) - f_N(\mathbf{x}) &> 0 \end{aligned}$$

cannot be consistent (otherwise we would be able to find utilities for $\succeq$). The $i$th inequality $f_i(\mathbf{x}) - f_{i+1}(\mathbf{x}) > 0$ defines a half-space $H_i$ in $\mathbb{R}^n$ determined by the corresponding hyperplane $f_i(\mathbf{x}) - f_{i+1}(\mathbf{x}) = 0$. For each $i \in \{1, \ldots, N-1\}$ we define the vector

$$\mathbf{v}_i = (\mu_i(1) - \mu_{i+1}(1), \ldots, \mu_i(n) - \mu_{i+1}(n))^t.$$

This is an inner normal vector of $H_i$, i.e. $\mathbf{x} \in H_i$ iff $(\mathbf{v}_i, \mathbf{x}) > 0$. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{N-1}$ have coefficients ranging from $k$ to $-k$, and the sum of all coefficients is zero for each of them. A standard linear-algebraic argument tells us that inconsistency of the system above is equivalent to the existence of a nontrivial linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{N-1}$ with nonnegative coefficients summing up to zero (see, for example, Theorem 2.9 of [16], page 48).

Since the sum of all coefficients of every $\mathbf{v}_i$ is zero, they all lie in a subspace of lower dimension than $n$. Another elementary linear-algebraic argument shows that there exist $m \leq n-1$ vectors $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_m}$ among $\mathbf{v}_1, \ldots, \mathbf{v}_{N-1}$, which are linearly dependent with positive coefficients and such that no proper subset of $\{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_m}\}$ is linearly dependent (see, for example, Lemma 5.1 in [11]). This means, in particular, that the linear dependency between $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_m}$ is unique up to a scalar multiple, that is, if $a_1\mathbf{v}_{i_1} + \ldots + a_m\mathbf{v}_{i_m} = 0$ and $b_1\mathbf{v}_{i_1} + \ldots + b_m\mathbf{v}_{i_m} = 0$ are two nontrivial linear combinations that vanish, then there exists a scalar $c \neq 0$ such that $a_i = cb_i$ for all $i = 1, 2, \ldots, m$. In particular, any linear combination $b_1\mathbf{v}_{i_1} + \ldots + b_m\mathbf{v}_{i_m} = 0$ has all its coefficients of the same sign, either all positive or all negative.

Without loss of generality we may assume that $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_m}$ are the first $m$ vectors of the system $\{\mathbf{v}_1, \ldots, \mathbf{v}_{N-1}\}$. Let us consider the matrix $V = (\mathbf{v}_1 \mid \ldots \mid \mathbf{v}_m)$, whose columns are the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Suppose, without loss of generality, that the first $m-1$ rows of $V$ are linearly independent. Let $A$ be a square $m \times m$ matrix whose $m$ rows are the upper $m$ rows of $V$. Let $A = (\mathbf{w}_1 \mid \ldots \mid \mathbf{w}_m)$, where $\mathbf{w}_1, \ldots, \mathbf{w}_m$ are the columns of $A$. It is clear that $b_1\mathbf{v}_1 + \cdots + b_m\mathbf{v}_m = 0$ if and only if $b_1\mathbf{w}_1 + \cdots + b_m\mathbf{w}_m = 0$. Since $\det A = 0$, we get $A_{11}\mathbf{w}_1 + A_{12}\mathbf{w}_2 + \cdots + A_{1m}\mathbf{w}_m = 0$, where $A_{ij}$ is the $(i,j)$-cofactor of matrix $A$. Since the entries of every cofactor are integers between $-k$ and $k$, the maximal value of such a determinant can be no greater than $k^m m^{m/2} \le k^n n^{n/2}$. (This immediately follows from an important theorem of Hadamard [18] which states that if $A$ is any real $n \times n$ matrix with $-1 \le a_{ij} \le 1$, then $|\det A| \le n^{n/2}$.) As was mentioned above, this implies $A_{11}\mathbf{v}_1 + A_{12}\mathbf{v}_2 + \cdots + A_{1m}\mathbf{v}_m = 0$, and all coefficients can be made positive due to the comment made earlier. So we assume that we have a linear combination

$$a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m = 0 \tag{17}$$

with nonnegative integer coefficients $a_i$ such that $0 \le a_i \le g(n,k)$, where $g(n,k) = k^n n^{n/2}$. Returning to our original notation will mean that

$$a_1\mathbf{v}_{i_1} + \ldots + a_m\mathbf{v}_{i_m} = 0. \tag{18}$$

Now we recollect that each $\mathbf{v}_k$ is the inner normal of the half-space $H_k$ defined by the inequality

$$\sum_{j=1}^{n} \mu_{i_k}(j)x_j - \sum_{j=1}^{n} \mu_{i_k+1}(j)x_j > 0, \tag{19}$$

where $\mu_{i_k}$ and $\mu_{i_k+1}$ are the multiplicity functions of $Q_{i_k}$ and $Q_{i_k+1}$, respectively. We denote $M_k = Q_{i_k}$ and $N_k = Q_{i_k+1}$ and note that $M_k \succ N_k$. Let us consider now the two multisets:

$$M = \bigcup_{i=1}^{m} \underbrace{M_i \cup \ldots \cup M_i}_{a_i}, \qquad N = \bigcup_{i=1}^{m} \underbrace{N_i \cup \ldots \cup N_i}_{a_i} \tag{20}$$

where $a_i$, $i = 1, \ldots, m$, are the coefficients of (18). The common cardinality $r$ of $M$ and $N$ is no greater than $(\sum_{i=1}^{m} a_i)k \le kng(n,k) = f(n,k)$. Hence both $M$ and $N$ are from $\mathcal{P}_r[n]$. If we assume that $\succeq$ can be extended to a consistent order on $\mathcal{P}_{f(n,k)}[n]$, then it can be also extended to a consistent order on $\mathcal{P}_r[n]$ and Lemma 1 will imply that $M \succ N$. On the other hand, (18) implies that $M = N$ as these two sets consist of the same elements taken with the same multiplicities. This contradiction proves the theorem. □

# 6 Further Research

It would be interesting to find analogues of the max-min and min-max rankings that were introduced in [4] and axiomatically characterised in [1] and find their axiomatic characterisations.

In relation to income streams it will be interesting to find necessary and sufficient conditions for representability of a ranking on $\mathcal{P}_k[n]$ by a system of weights $w_i = \alpha^{i-1}$, $i = 1, 2, \ldots, n$, where $\alpha$ is a discount rate. A further step might involve rankings of investment projects which will be a continuation of the paper [15], where conditions were given under which one investment project dominates a second investment project irrespective of the discout rate.

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# 7 Appendix: Proof of Theorem 1

*Proof.* When $k = 2$, we have $\phi(2) = 1$, $\Phi(2) = 2$, and

$$\mathbf{F}_2 = \{f_0 = 0, f_1 = 1/2, f_2 = 1\}.$$

The risk-loving linear order occurs precisely when $\gamma < 1/2$ or $\gamma \in (f_0, f_1)$, while the risk-avoiding linear order occurs precisely when $\gamma \in (f_1, f_2)$. We will have also one non-antisymmetric order when $\gamma = f_1$. Therefore we get three different orders, two of which are linear. This gives a basis for our induction.

Let us prove by induction that, for all $k$, there will be as many consistent orders on $\mathcal{P}_k[3]$ as the number of Farey fractions in $\mathbf{F}_k$, different from 0 and 1, plus the number of intervals into which the fractions of $\mathbf{F}_k$ split $[0, 1]$, and that these orders are all representable with each of them arising when $\gamma \in (f_i, f_{i+1})$ for some two neighboring Farey fractions $f_i$ and $f_{i+1}$ (in which case we obtain a linear order) or when $\gamma = f_i \notin \{0, 1\}$ (in which case the order is not antisymmetric).

Assuming that this is true for $\mathcal{P}_k[3]$, let us consider $\mathcal{P}_{k+1}[3]$. By the induction hypothesis, all consistent orders of $\mathcal{P}_k[3]$ are representable and therefore each has at least one extension to an order of $\mathcal{P}_{k+1}[3]$, in particular the one with the same utilities. Let us explore where we can obtain more than one extension of a consistent order $\succeq$ on $\mathcal{P}_k[3]$ to a representable order on $\mathcal{P}_{k+1}[3]$. If $\gamma = f_i$, for some $i$, the extension is, of course, unique. Therefore we may assume that $\succeq$ is determined by the utilities such that $\gamma \in (f_i, f_{i+1})$.

If any two multisets of $\mathcal{P}_{k+1}[3]$ contain an element in common, then their order in any consistent extension is already determined by $\succeq$. The only thing which is not determined by $\succeq$ and the consistency of the extension is the position of the multiset $T = \{2^{k+1}\}$ in the sequence of the following $k$ multisets:

$$\{3^{k+1}\} \prec \{1, 3^k\} \prec \ldots \prec \{1^k, 3\} \prec \{1^{k+1}\}. \tag{21}$$

These multisets will be arranged in the order (21) for every consistent order $\succeq$ of $\mathcal{P}_{k+1}[3]$. Let us denote $S_i = \{1^i, 3^{k+1-i}\}$. Two different values of $\gamma$ from the same interval $(f_i, f_{i+1})$ might position $T$ relative to the sequence (21) differently. This happens if and only if one of the fractions

$$\frac{1}{k+1},\ \frac{2}{k+1},\ \ldots,\ \frac{k}{k+1}, \tag{22}$$

falls into the interval $(f_i, f_{i+1})$. Indeed, if no such fraction falls into $(f_i, f_{i+1})$, then for some $j \in \{1, \ldots, k-1\}$ we have

$$\frac{j}{k+1} \le f_i < f_{i+1} \le \frac{j+1}{k+1}.$$

This means that for every $\gamma \in (f_i, f_{i+1})$ we actually have

$$\frac{j}{k+1} < \gamma < \frac{j+1}{k+1}$$

or $u_\gamma(S_j) < u_\gamma(T) < u_\gamma(S_{j+1})$, where $u_\gamma(X)$ denotes the total utility of $X$ calculated for the vector of utilities $(1, \gamma, 0)$. This means that every representable extension of $\succeq$ ranks $T$ between $S_j$ and $S_{j+1}$. Hence, there is a unique extension of $\succeq$ to a representable linear order on $\mathcal{P}_{k+1}[3]$.

On the other hand, when one of the fractions (22), say $f = \frac{j}{k+1}$, falls into $(f_i, f_{i+1})$, there can be three different extensions of $\succeq$ to a representable order on $\mathcal{P}_{k+1}[3]$. One will occur when $u\gamma \in (f_i, f)$, another when $\gamma = f$, and the last one when $\gamma \in (f, f_{i+1})$. They will position $T$ between $S_{j-1}$ and $S_j$, force a tie $T \sim S_j$, or position $T$ between $S_j$ and $S_{j+1}$, respectively. There can be no more than one new Farey fraction in $(f_i, f_{i+1})$ since it is known that, for any two neighboring Farey fractions $\frac{a}{b}$ and $\frac{c}{d}$, we have $|ad - bc| = 1$ (see [19], Theorem 28), which is not true for $\frac{a}{b} = \frac{j}{k+1}$ and $\frac{c}{d} = \frac{j+1}{k+1}$ so they have to be separated by $f_i$ or $f_{i+1}$. This shows that we will have as many more consistent representable orders of $\mathcal{P}_{k+1}[3]$ in comparison to $\mathcal{P}_k[3]$ as twice the number of Farey fractions from $\mathbf{F}_{k+1} \setminus \mathbf{F}_k$, namely $2\phi(k+1)$ new orders. Half of these additional orders are linear. Thus, $2\Phi(k) - 1$ consistent orders on $\mathcal{P}_k[3]$ generate $2\Phi(k+1) - 1$ consistent representable orders on $\mathcal{P}_{k+1}[3]$.

It remains to prove that every consistent order on $\mathcal{P}_{k+1}[3]$ appears as an extension of one of the consistent representable orders on $\mathcal{P}_k[3]$ in the way which has been just described. This will prove that all orders on $\mathcal{P}_{k+1}[3]$ are representable.

Let us assume now that $\succeq$ is a consistent order of $\mathcal{P}_{k+1}[3]$. Then it induces a consistent order $\succeq_i$ on $\mathcal{P}_i[3]$ for each $i \in \{1, 2, \ldots, k\}$. By the induction hypothesis, all orders $\succeq_i$ are representable and, with no loss of generality, we may assume that they are determined by the same set of utilities, i.e. have the same $\gamma$. We know that $\gamma$ either belongs to the interval $(f_i, f_{i+1})$ for some two neighboring terms $f_i$ and $f_{i+1}$ of $\mathbf{F}_k$ or coincides with $f_i$ for some $i$. As Case 1 let us assume that $\succeq$ positions the multiset $T = \{2^{k+1}\}$ against the elements of the sequence (21) as follows: $S_j \prec T \prec S_{j+1}$.

**Case 1a**. $\gamma \in (f_i, f_{i+1})$. We know that $S_j \prec T \prec S_{j+1}$ for all representable orders with $\gamma \in \left(\frac{j}{k+1}, \frac{j+1}{k+1}\right)$. Hence we need to prove that

$$(f_i, f_{i+1}) \cap \left(\frac{j}{k+1}, \frac{j+1}{k+1}\right) \neq \emptyset, \tag{23}$$

for then $\gamma$ can be adjusted, if needed, to obtain a representable order which coincides with $\succeq$. Suppose to the contrary that (23) does not hold and that the intersection there is empty. Without loss of generality, we assume that $f_{i+1} \le \frac{j}{k+1}$. (The case of $f_i \ge \frac{j+1}{k+1}$ can be handled similarly.) Then $\gamma < f_{i+1} \le \frac{j}{k+1}$, and $\gamma < \frac{j}{k+1}$.

Suppose, first, that the fraction $\frac{j}{k+1}$ is not in its lowest terms, i.e. $1 < d = \gcd(j, k+1)$. Let $\ell = \frac{j}{d}$ and $h = \frac{k+1}{d}$. Then $\gamma < \frac{\ell}{h} = \frac{j}{k+1}$ and hence

$$\{2^h\} \prec_h \{1^\ell, 3^{h-\ell}\}.$$

Since $hd = k+1$ and $\ell d = j$, this immediately implies, due to consistency of $\succeq$ and Lemma 1(b), that

$$T = \{2^{k+1}\} = \{2^{hd}\} \prec \{1^{\ell d}, 3^{(h-\ell)d}\} = \{1^j, 3^{k+1-j}\} = S_j,$$

which is a contradiction.

So assume that the fraction $\frac{j}{k+1}$ is in its lowest terms, i.e. $1 = \gcd(j, k+1)$. Then we consider the two neighboring Farey fractions $\frac{s}{t}$ and $\frac{\ell}{h}$ of $\frac{j}{k+1}$ in $\mathbf{F}_{k+1}$ such that

$$\frac{s}{t} < \frac{j}{k+1} < \frac{\ell}{h}. \tag{24}$$

Since the fraction $\frac{j}{k+1}$ is in its lowest terms, $f_{i+1} \neq \frac{j}{k+1}$.

As we assumed the contrary to (23) we have $\gamma < f_{i+1} \leq \frac{s}{t} < \frac{\ell}{h}$. Then one of the main theorems about Farey fractions states that in this case

$$s + \ell = j \quad \text{and} \quad t + h = k + 1 \tag{25}$$

(see [19], Theorem 29 or [7] Theorem 9). In particular, $t < k + 1$ and $h < k + 1$. Hence it follows that

$$\begin{array}{rcl} \{2^h\} & \prec_h & \{1^\ell, 3^{h-\ell}\}, \\ \{2^t\} & \prec_t & \{1^s, 3^{t-s}\}. \end{array}$$

By Lemma 1(a) and and (25) we get

$$T = \{2^{k+1}\} = \{2^h\} \cup \{2^t\} \prec \{1^\ell, 3^{h-\ell}\} \cup \{1^s, 3^{t-s}\} = S_j,$$

which is a contradiction. So (23) in this subcase holds.

**Case 1b**. $\gamma = f_i$ for some $i$. We need to show that in this case

$$f_i \in \left( \frac{j}{k+1}, \frac{j+1}{k+1} \right). \tag{26}$$

The argument is very similar to the previous subcase. We assume the contrary to (26) and with no loss of generality we may assume that $f_i \leq \frac{j}{k+1}$. Assuming that $\frac{j}{k+1}$ is not in its lowest terms we get a contradiction, proving that $T \prec S_j$ as in the Case 1a. Assuming that $\frac{j}{k+1}$ is in its lowest terms, we get $f_i \neq \frac{j}{k+1}$ and hence $f_i < \frac{j}{k+1}$. We again take the two neighboring Farey fractions $\frac{s}{t}$ and $\frac{\ell}{h}$ of $\frac{j}{k+1}$ in $\mathbf{F}_{k+1}$ satisfying (24). As $f_i \leq \frac{s}{t} < \frac{\ell}{h}$, we also get (16), (17) and (18) from which we deduce $T \prec S_j$ and again obtain a contradiction with $S_j \prec T$. This proves (26) in this subcase as well.

As Case 2, we take $T \sim S_j$ for some $j$. We again have to consider two subcases. We leave these to the reader.

So much proves that $\succeq$ is representable. This completes the proof of the induction hypothesis. $\square$

**Irreducible orders of $\mathcal{P}_2[4]$**

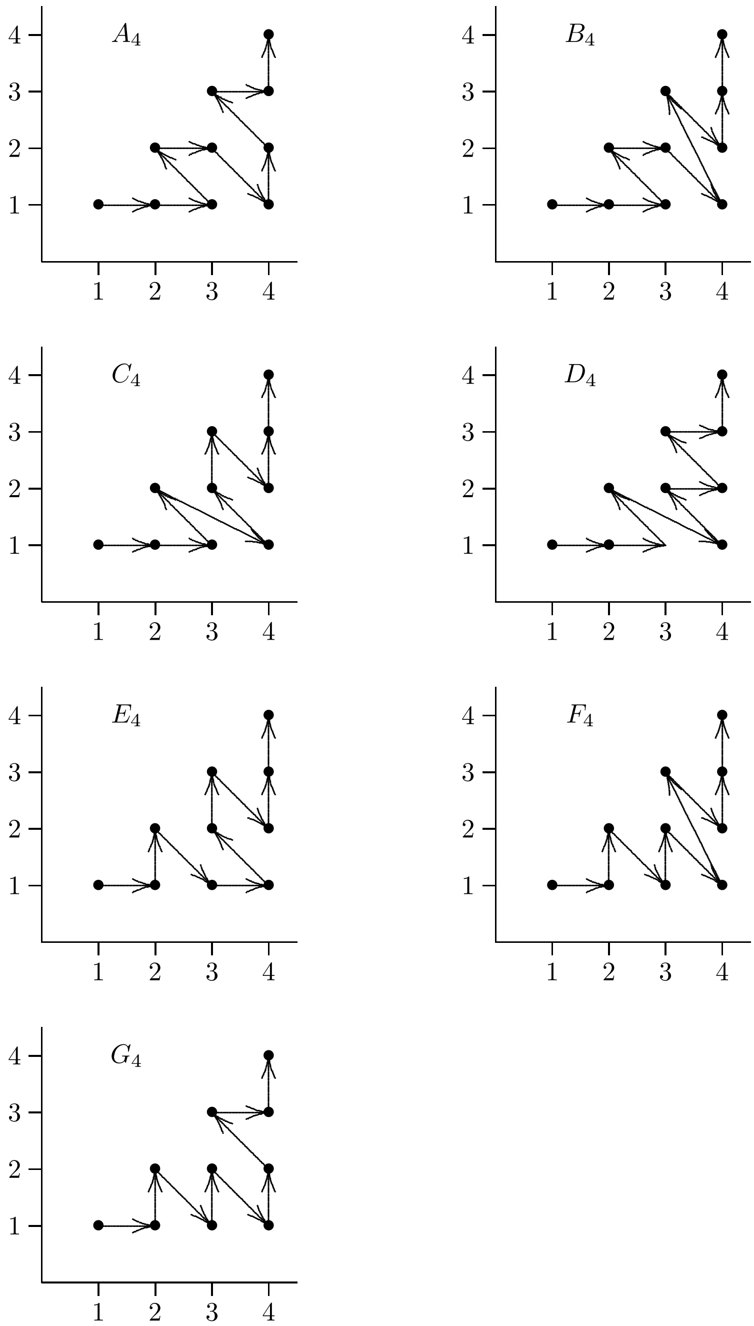


**Figure 1**

**Classification of representable linear orders on $\mathcal{P}_2[4]$**
**according to their values of $u_2$ and $u_3$**
( $u_1 = 1$ and $u_4 = 0$)

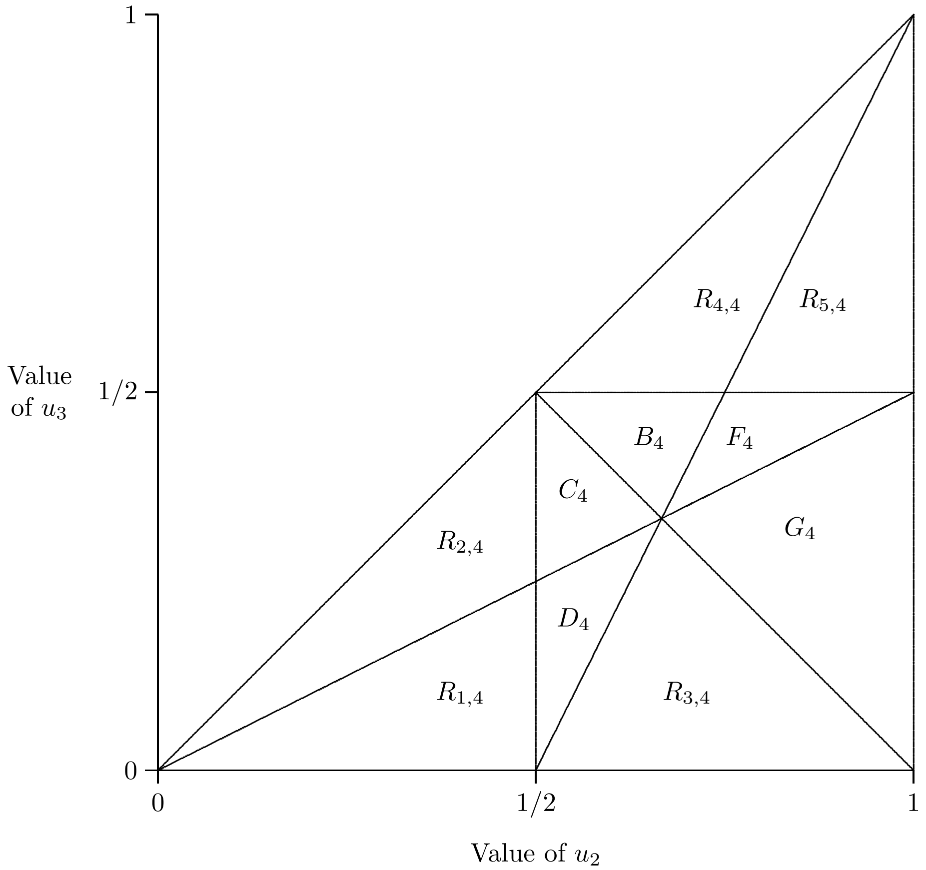


**Figure 2**

Table 2. Types of consistent rankings $\mathcal{P}_3[4]$

| Order on pairs | Representable extensions | Almost representable extensions | Not almost representable extensions | Total |
|---|---|---|---|---|
| $R_{1,4}$ | 10 | 3 | 1 | 14 |
| $R_{2,4}$ | 12 | 3 | 1 | 17 |
| $R_{3,4}$ | 10 | 7 | 7 | 24 |
| $R_{4,4}$ | 12 | 3 | 1 | 17 |
| $R_{5,4}$ | 10 | 3 | 1 | 14 |
| $B_4$ | 5 | 1 | 0 | 7 |
| $C_4$ | 5 | 1 | 0 | 7 |
| $D_4$ | 3 | 1 | 0 | 4 |
| $F_4$ | 3 | 1 | 0 | 4 |
| $G_4$ | 10 | 7 | 7 | 24 |
| Total | 80 | 30 | 18 | 128 |

**Classification of representable rankings from $\mathcal{P}_3[4]$ according to their values of $w_2$ and $w_3$.**

($u_1 = 1$ and $u_4 = 0$)

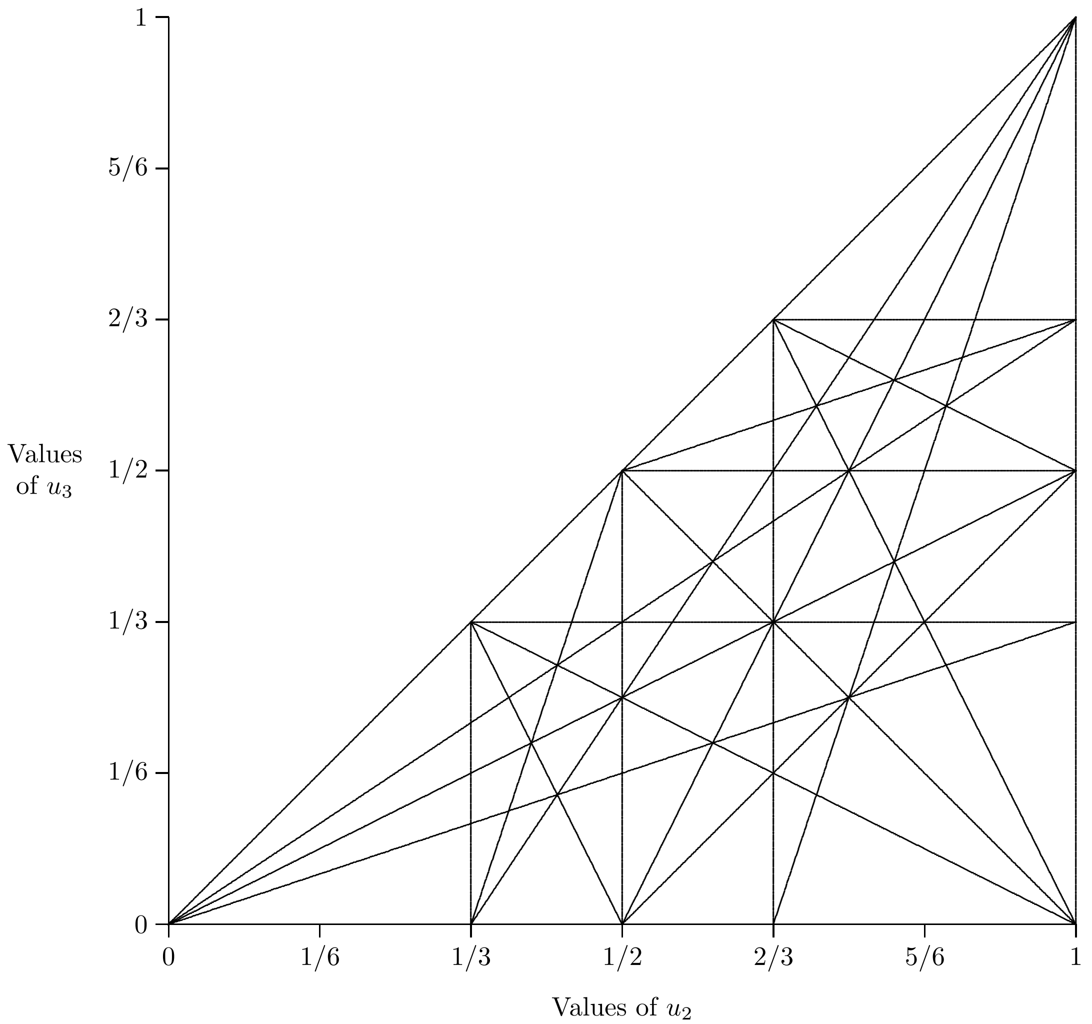


**Figure 3.**