Uncountably many groups with the same profinite completion

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We use the construction described in Section 13.4 of the book *Subgroup Growth.*

This starts with a rooted tree $\mathcal{T}$ in which each vertex of level $n \geq 1$ has valency $1 + l_n$ (and the root has valency l_0).

Given a sequence $(T_n)_{n\geq 0}$ with $T_n \leq \mathrm{Sym}(l_n)$ for each n we take

$$W = \varprojlim W_n$$

where $W_0 = T_0$ and $W_{n+1} = T_n \wr W_{n-1}$ for $n \geq 0$. We identify W as a subgroup of $\mathrm{Aut}(\mathcal{T})$ in the natural way.

On pages 262-263 of *Subgroup Growth* we define four elements $\xi, \eta,\ a$ and b of W, set $\Gamma = \langle \xi, \eta, a, b\rangle$, and prove that under certain conditions, Γ is both dense and satisfies the congruence subgroup property in W. Together, these imply that the natural homomorphism from $\widehat{\Gamma}$ to W is an isomorphism of profinite groups.

The conditions are as follows:

(i) T_n is a doubly transitive subgroup of $\mathrm{Sym}(l_n)$

(ii) there exist a two-generator perfect group $P = \langle x, y\rangle$ and for each n an epimorphism $\phi_n : P \to T_n$

(iii) the automorphisms $\xi, \eta,\ a$ and b are built in a particular way out of the

$$\alpha_n = x\phi_n,\ \beta_n = y\phi_n \in T_n \leq \mathrm{Sym}(l_n).$$

Now let's get down to specifics. Let $l_n = 5$ and $T_n = \mathrm{Alt}(5)$ for all n. Put

$$\alpha = (123),\ \beta = (12345).$$

Let $\lambda \in \{0, 1\}^{\mathbb{N}_0}$ and set

$$\alpha_n = \alpha, \beta_n = \beta \text{ if } \lambda_n = 0$$
$$\alpha_n = \beta, \beta_n = \alpha \text{ if } \lambda_n = 1.$$

Let $P = \mathrm{Alt}(5) \times \mathrm{Alt}(5)$, $x = (\alpha, \beta)$, $y = (\beta, \alpha) \in P$. It's an easy exercise to check that $P = \langle x, y \rangle$, and we define $\phi_n : P \to T_n$ by

$$x\phi_n = \alpha^{1-\lambda_n} \cdot \beta^{\lambda_n}$$
$$y\phi_n = \alpha^{\lambda_n} \cdot \beta^{1-\lambda_n}$$

(so ϕ_n is simply the projection of $P = \mathrm{Alt}(5) \times \mathrm{Alt}(5)$ onto either the first or the second direct factor).

Let $\Gamma(\lambda) = \langle \xi(\lambda), \eta(\lambda), a(\lambda), b(\lambda) \rangle$ denote the group Γ constructed as above using the sequence λ. There are $2^{\aleph_0}$ such sequences, so we have constructed $2^{\aleph_0}$ 4-generator subgroups of $\mathrm{Aut}(\mathcal{T})$ with profinite completion W. (These groups are of course residually finite since they $\mathrm{Aut}(\mathcal{T})$ is.)

Claim: *For each sequence λ, the set $S(\lambda) := \{\mu \mid \Gamma(\lambda) \cong \Gamma(\mu)\}$ is countable.*

The claim implies that the number of isomorphism classes among the groups $\Gamma(\lambda)$ is still $2^{\aleph_0}$.

Sketch proof of claim:

Suppose that $S(\lambda)$ is uncountable. For $\mu \in S(\lambda)$ let $\theta_\mu : \Gamma(\mu) \to \Gamma(\lambda)$ be an isomorphism. Then θ_μ extends to a continuous automorphism σ_μ of W (universal property of profinite completions).

Now the set

$$\{a(\mu)^\sigma \mid \mu \in S(\lambda),\ \sigma \in \mathrm{Aut}(W)\} \subseteq \Gamma(\lambda)$$

is countable because $\Gamma(\lambda)$ is a finitely generated group. Hence there exists $c \in \Gamma(\lambda)$ such that the set

$$X := \{\mu \in S(\lambda) \mid a(\mu)^{\sigma_m} = c\}$$

is uncountable (all we *need* is: of cardinality at least 2).

Let $\mu \neq \nu \in X$. Then

$$a(\mu)^{\sigma_\mu \sigma_\nu^{-1}} = a(\nu).$$

Now for some n we have $\mu_n \neq \nu_n$. Say $a(\mu)_n = \alpha$ and $a(\nu)_n = \beta$. The (continuous!) automorphism $\sigma_\mu \sigma_\nu^{-1}$ of W induces an automorphism τ on the quotient

$$W_n = \mathrm{Alt}(5)^{(5^n)} \rtimes W_{n-1}$$

(*exercise*!), sending the coset of $a(\mu)$ to that of $a(\nu)$:

$$(1, \ldots, 1, \alpha, 1, 1, 1, 1) \cdot u \stackrel{\tau}{\longmapsto} (1, \ldots, 1, \beta, 1, 1, 1, 1) \cdot v$$

in an obvious notation (here, u and v lie in the stabilizer of the point $5^n - 4$). Examining the structure of $\mathrm{Aut}(W_n)$ (this is not a hard exercise) we find that this forces

$$(1, \ldots, 1, \beta, 1, 1, 1, 1) = (*, \ldots, *, \alpha^z, *, \ldots, *)$$

for some automorphism z of Alt(5). This is impossible since α and β have coprime orders.

Thus $S(\lambda)$ must be countable.

So we have established

Theorem 1 *There are continuously many pairwise non-isomorphic* 4*-generator residually finite groups all having the iterated wreath product* W *as their profinite completion.*

Remarks

1. There is lots of flexibility in this construction - we could use any family of finite images of a fixed finitely generated perfect group, provided they each have a primitive faithful permutation representation; these are used for example in my paper 'The finite images of finitely generated groups', *Proc. London Math. Soc.* **82** (2001), 597–613.

2. Uncountably many 3-generator groups having the same profinite completion, a group of automorphisms of the binary rooted tree, were constructed by V. Nekrashevych in *arXiv*:**1303.5782v2** [math.GR] 26 Mar 2013.

3. Uncountably many 4-generator groups having the same profinite completion, the product of an infinite cyclic group and a family of finite alternating groups, were constructed by L. Pyber in 'Groups of intermediate subgroup growth and a problem of Grothendieck', *Duke Math. J.* **121** (2004), 169–188.