

# On the classification of groups of prime-power order by coclass: The 3-groups of coclass 2

Bettina Eick, C.R. Leedham-Green, M.F. Newman and E.A. O'Brien

## Abstract

In this paper we take a significant step forward in the classification of 3-groups of coclass 2. Several new phenomena arise for the first time. Theoretical and computational tools have been developed to deal with them. We identify and are able to classify an important subset of the 3-groups of coclass 2. With this classification and further extensive computations, it is possible to predict the full classification. On the basis of the work here and earlier work on the  $p$ -groups of coclass 1, we formulate another general coclass conjecture. It implies that, given a prime  $p$  and a positive integer  $r$ , a finite computation suffices to determine the  $p$ -groups of coclass  $r$ .

## 1 Introduction

The *coclass* of a group of order  $p^n$  and nilpotency class  $c$  is defined as  $n - c$ . In 1980, Leedham-Green & Newman [16] made a series of conjectures about finite  $p$ -groups, using coclass as the primary invariant. A detailed account of the proofs of these conjectures, and the resultant program of study, can be found in [14].

The goal is to classify  $p$ -groups via coclass. We expect that it is possible to reduce the classification to a finite calculation, and that the  $p$ -groups of a given coclass can be partitioned into finitely many families, where the groups in a family share similar structure and can be described by a parametrised presentation. One approach to achieving this goal is to understand the structure of the *coclass graph*  $\mathcal{G}(p, r)$ . Its vertices are the  $p$ -groups of coclass  $r$ , one for each isomorphism type, and its edges are  $P \rightarrow Q$ , with  $Q$  isomorphic to the quotient  $P/L_c(P)$ , where  $L_c(P)$  is the last non-trivial term of the lower central series of  $P$ . If  $\mathcal{G}(p, r)$  can be constructed from a finite subgraph using a finite number of graph-theoretic operations, then this may assist in realising our goal.

---

We thank Heiko Dietrich for his careful reading, many comments, and for an illustration; and Marcus du Sautoy for useful discussions. Both Eick and O'Brien were partially supported by the Alexander von Humboldt Foundation. All authors were partially supported by the Marsden Fund of New Zealand *via* grant UOA1015.

Eick & Leedham-Green [11] proved that the graph for the 2-groups of a given coclass can be constructed from a finite subgraph by applying just one type of operation to the subgraph – and this operation has an analogue at the group-theoretic level. That the graph exhibits such a simple structure was first conjectured by Newman & O’Brien [18]. Their Conjecture  $P$  was proved by du Sautoy [7] and in a much sharper form in [11]. The results of [11] have already been applied to study the automorphism groups of 2-groups [9], and Schur multipliers of  $p$ -groups [10].

Blackburn’s classification [3] of the 3-groups of coclass 1 implies that these groups exhibit behaviour similar to that proved for 2-groups. But we know from other investigations that the results of [11] are not generally true. The 5-groups of coclass 1 have been investigated in [4, 5, 6, 15, 17]; while this work suggests that  $\mathcal{G}(5, 1)$  can be constructed from a finite subgraph, the above operation does not suffice.

The number of isomorphism classes of  $p$ -groups of coclass  $r$  of order  $p^n$ , for odd  $p$ , is bounded by a linear function of  $n$  precisely when  $(p, r)$  is one of  $(3, 1)$ ,  $(3, 2)$  or  $(5, 1)$ . We consider here the 3-groups of coclass 2. The study of  $\mathcal{G}(3, 2)$  goes back to the late 1970s (see [1, 2]) and early results played a role in the development of the original coclass conjectures. Our computations, reported in Section 8, show that the complete graph is very dense. We determine a significant subgraph: the *skeleton graph* (defined in Section 3.4). While this subgraph is comparatively sparse, it exhibits the broad structure of  $\mathcal{G}(3, 2)$ . Our computations suggest that the complete structure of  $\mathcal{G}(3, 2)$  can be determined from a finite subgraph.

The skeleton graph of  $\mathcal{G}(3, 2)$  exhibits some new features; we consider these in Section 7. Its determination required dealing with number-theoretic problems similar to those considered by Leedham-Green & McKay [15] in their investigation of skeleton graphs of  $\mathcal{G}(p, 1)$  for  $p \geq 5$ . That  $\mathcal{G}(3, 2)$  does not reveal all complexities that arise in classifying  $p$ -groups by coclass is demonstrated by Dietrich [6]. We conclude by stating a conjecture about graph-theoretic operations needed to describe  $\mathcal{G}(p, r)$  for arbitrary  $p$  and  $r$ .

## 2 Preliminaries

### 2.1 Coclass trees

By [14, Corollary 7.4.13], every infinite pro- $p$ -group  $G$  of coclass  $r$  is a  $p$ -adic pre-space group. Namely,  $G$  has a normal subgroup  $T$  which is a free, finitely generated module over the ring  $\mathbb{Z}_p$  of  $p$ -adic integers, and  $Q := G/T$  is a finite  $p$ -group that acts uniserially on  $T$ . While  $T$  is not unique, the rank  $d$  of  $T$  as  $\mathbb{Z}_p$ -module is an invariant of  $G$  called its *dimension*; it is a consequence of [14, Corollary 7.4.13 and Theorem 10.5.12] that  $d = (p - 1)p^s$  for some  $s \in \{0, \dots, r - 1\}$ .

The uniserial action implies that the subgroups defined by  $T_0 := T$  and  $T_{i+1} := [T_i, Q]$  form a chain  $T = T_0 > T_1 > \dots > T_i > T_{i+1} > \dots > \{0\}$  with  $[T_i : T_{i+1}] = p$  and  $T_{i+d} = pT_i$  for  $0 \leq i < \infty$ . We set  $T_\infty := \{0\}$ . This chain extends to a doubly

infinite series  $\cdots > T_{-2} > T_{-1} > T_0 > T_1 > T_2 > \cdots$  and again  $T_{i+1}$  has index  $p$  in  $T_i$  for all  $i$ .

If  $P$  and  $Q$  are groups in  $\mathcal{G}(p, r)$ , then  $P$  is a *descendant* of  $Q$  if there is a (possibly trivial) path in  $\mathcal{G}(p, r)$  from  $P$  to  $Q$ . The *descendant tree* of  $Q$  is the subtree of its descendants, and has root  $Q$ .

If  $G$  is an infinite pro- $p$ -group of coclass  $r$ , then  $G/L_{i+1}(G)$ , the quotient of  $G$  having class  $i$ , is a finite  $p$ -group of coclass at most  $r$  for all  $i > 0$ , and the coclass of  $G/L_{i+1}(G)$  is precisely  $r$  for all but finitely many values of  $i$ . Moreover, since there are only finitely many infinite pro- $p$ -groups of coclass  $r$  up to isomorphism [14, p. (viii)], for sufficiently large  $i$  the group  $G/L_{i+1}(G)$  is a quotient of only one infinite pro- $p$ -group of coclass  $r$ . Choose  $i$  minimal with respect to these properties. The *coclass tree*  $\mathcal{T}(G)$  is the descendant tree of  $G/L_{i+1}(G)$  in  $\mathcal{G}(p, r)$ .

There are only finitely many coclass trees in  $\mathcal{G}(p, r)$  and only finitely many groups in  $\mathcal{G}(p, r)$  are not contained in a coclass tree [13, Proposition 2.2]. Hence the study of the broad structure of  $\mathcal{G}(p, r)$  reduces to an investigation of its coclass trees.

## 2.2 Mainline and branches

Let  $G$  be an infinite pro- $p$ -group of coclass  $r$ , and let  $\mathcal{T}(G)$  be its coclass tree with root  $G/L_{i+1}(G)$ . The quotients  $G/L_{i+1}(G), G/L_{i+2}(G), \dots$  form a unique maximal infinite path, or *mainline*, in  $\mathcal{T}(G)$ .

For  $j \geq i$ , let  $\mathcal{B}_j$  denote the subtree of  $\mathcal{T}(G)$  consisting of  $G/L_{j+1}(G)$  and all of its descendants that are not descendants of  $G/L_{j+2}(G)$ . Thus  $\mathcal{B}_j$  is a finite subtree of  $\mathcal{T}(G)$ , and is its  *$j$ th branch*. Hence  $\mathcal{T}(G)$  consists of an infinite sequence of trees  $\mathcal{B}_i, \mathcal{B}_{i+1}, \dots$ , connected by the mainline. The subtree of all vertices in  $\mathcal{B}_j$  of distance at most  $k$  from  $G/L_{j+1}(G)$  is denoted by  $\mathcal{B}_{j,k}$ .

## 2.3 Periodicity

Eick & Leedham-Green [11, Theorem 29] prove the following.

**Theorem 2.1** *Let  $G$  be an infinite pro- $p$ -group of coclass  $r$  and dimension  $d$ . There exists an explicit function  $f$  such that, for every positive integer  $k$  and every  $j \geq f(k)$ , there is a graph isomorphism  $\pi_j : \mathcal{B}_{j,k} \rightarrow \mathcal{B}_{j+d,k}$ .*

We say that  $\mathcal{T}(G)$  has *period*  $d$  and *defect*  $f$ ; bounds for the latter appear in [11].

This theorem suggests that we arrange the infinitely many branches of a coclass tree  $\mathcal{T}(G)$  with root  $G/L_{i+1}(G)$  into  $d$  sequences  $(\mathcal{B}_{i+e}, \mathcal{B}_{i+e+d}, \mathcal{B}_{i+e+2d}, \dots)$  for  $0 \leq e < d$ .

The *depth* of a rooted tree is the length of a maximal path from a vertex to the root. A sequence of branches has *bounded depth* if the depths of its trees  $\mathcal{B}_{i+e+kd}$  are bounded by a constant. (If every sequence of branches has bounded depth, then  $\mathcal{T}(G)$  has bounded depth.) Theorem 2.1 implies that a sequence of branches of bounded depth is ultimately constant, and can therefore be constructed from a finite subsequence.

Every sequence of branches of a coclass tree in  $\mathcal{G}(2, r)$  or  $\mathcal{G}(3, 1)$  has bounded depth (see [14, Theorem 11.4.4]). In these cases Theorem 2.1 shows that  $\mathcal{B}_{j+d} \cong \mathcal{B}_j$  for large enough  $j$ . The proof in [11] of Theorem 2.1 is underpinned by an explicit group-theoretic construction. It defines families of  $p$ -groups of coclass  $r$  where the groups in a family share similar structure and are described by a parametrised presentation.

All coclass graphs other than  $\mathcal{G}(2, r)$  and  $\mathcal{G}(3, 1)$  contain coclass trees of unbounded depth (see [14, Theorem 11.4.4]) and so are not covered by Theorem 2.1. We show that both types of coclass trees occur in  $\mathcal{G}(3, 2)$ .

## 2.4 Notation

Much of our notation is standard. For consistency, if  $G$  is the split extension  $A \rtimes B$  or the non-split extension  $A \cdot B$ , then in both cases  $B$  is normal in  $G$ . We denote a term of the lower central series of  $G$  by  $L_i(G)$  for  $i > 0$ ; and a left-normed commutator  $[a, b, \dots, b]$  by  $[a, {}_i b]$ .

## 3 Skeletons

In this section we recall a construction by Leedham-Green & McKay [14, §8.4] that is central to the investigation of branches of unbounded depth. Throughout this section, let  $p$  be an odd prime.

Let  $G$  be an infinite pro- $p$ -group of coclass  $r$ . Recall that  $G$  is an extension of a  $d$ -dimensional  $\mathbb{Z}_p$ -module  $T$  by a finite  $p$ -group  $Q$  which acts uniserially on  $T$  with series  $T = T_0 > T_1 > \dots$ . The exterior square  $T \wedge T$  is a  $\mathbb{Z}_p Q$ -module under the diagonal action of  $Q$ . If  $i < j$  then we define  $T_i \wedge T_j = T_j \wedge T_i$  to be the  $\mathbb{Z}_p$ -submodule of  $T_i \wedge T_i$  spanned by  $\{s \wedge t \mid s \in T_i, t \in T_j\}$ .

### 3.1 Twisting homomorphisms

Let  $\gamma : T \wedge T \rightarrow T$  be a  $\mathbb{Z}_p Q$ -module homomorphism. Then  $\gamma(T_\ell \wedge T)$  is a  $Q$ -invariant subgroup of  $T$  for every  $\ell \geq 0$ . Let  $\gamma(T \wedge T) = T_j$  for  $j \geq 0$ , and let  $\gamma(T_j \wedge T) = T_k$ . If  $j \leq m \leq k$ , then  $\gamma$  induces a homomorphism  $\gamma_m : T/T_j \wedge T/T_j \rightarrow T_j/T_m$  defined by

$$\gamma_m(a + T_j \wedge b + T_j) = \gamma(a \wedge b) + T_m.$$

This induced homomorphism can be used to define a new group multiplication ‘ $\cdot$ ’ on  $T/T_m$  that turns the additive abelian group  $T/T_m$  into a multiplicative group of class at most 2. More precisely, for  $a, b \in T$  we define

$$(a + T_m) \cdot (b + T_m) = (a + b + T_m) + \frac{1}{2}\gamma_m(a + T_j \wedge b + T_j).$$

The resulting group  $T_{\gamma, m} := (T/T_m, \cdot)$  has order  $p^m$ . Commutators are evaluated easily in  $T_{\gamma, m}$  as

$$[a + T_m, b + T_m] = \gamma_m(a + T_j \wedge b + T_j).$$

If  $m = j$ , then  $\gamma_m$  is the trivial homomorphism, and  $T_{\gamma,m}$  is abelian. If  $j < m \leq k$  then  $T_{\gamma,m}$  has derived subgroup  $T_j/T_m$  and class precisely 2. Also  $T_{\gamma,n}$  is a quotient of  $T_{\gamma,m}$  if  $j \leq n \leq m$ .

**Lemma 3.1** *With the above notation, let  $\gamma(T \wedge T) = T_j$  and  $\gamma(T_j \wedge T) = T_k$ , and let  $d$  be the rank of  $T$ , as  $\mathbb{Z}_p Q$ -module.*

- (a) *If  $j$  is infinite, or equivalently  $\gamma = 0$ , then  $m$  is infinite and  $T_{\gamma,\infty} \cong (T, +)$ .*
- (b) *If  $j$  is finite, or equivalently  $\gamma \neq 0$ , then  $2j - d < k \leq 2j + d$ .*
- (c) *If  $j \leq m \leq k$ , then  $T_{p^i \gamma, m+2id}$  is defined for every  $i \geq 0$ .*

PROOF:

- (a) This follows directly from the definition.
- (b) Write  $j = id + e$  with  $0 \leq e < d$ . Then  $T_k = \gamma(T_j \wedge T) = \gamma(p^i T_e \wedge T) = p^i \gamma(T_e \wedge T)$ , and  $T_{j+d} = pT_j = \gamma(pT \wedge T) \leq \gamma(T_e \wedge T) \leq \gamma(T \wedge T) = T_j$ . Hence  $p^i T_{j+d} \leq T_k \leq p^i T_j$  or, equivalently,  $id + j + d \geq k \geq id + j$ . As  $id = j - e$ , this yields  $2j + (d - e) \geq k \geq 2j - e$ .
- (c) Note that  $p^i \gamma(T \wedge T) = p^i T_j = T_{j+id}$ , and

$$p^i \gamma(T_{j+id} \wedge T) = p^{2i} \gamma(T_j \wedge T) = p^{2i} T_k = T_{k+2id}.$$

Thus if  $j \leq m \leq k$ , then  $j + 2id \leq m + 2id \leq k + 2id$ , and the result follows. •

## 3.2 Skeleton groups

Assume that  $G$  splits over  $T$ . Let  $\gamma : T \wedge T \rightarrow T$  be a  $\mathbb{Z}_p Q$ -module homomorphism, where  $\gamma(T \wedge T) = T_j$  and  $\gamma(T_j \wedge T) = T_k$ , and  $j \leq m \leq k < \infty$ . Since the natural action of  $Q$  on  $T/T_m$  respects the new multiplication induced by  $\gamma$ , we can define a *skeleton group*  $G_{\gamma,m} := Q \rtimes T_{\gamma,m}$ . If  $j$  is sufficiently large, then  $G_{\gamma,m}$  is a group of depth  $m - j$  in the branch of  $\mathcal{T}(G)$  with root  $Q \rtimes T/T_j$ . Lemma 3.1(c) shows that the homomorphism  $p^i \gamma$  for  $i \geq 0$  defines a skeleton group  $G_{p^i \gamma, m+2id}$  of depth  $m - j + id$  in the branch with root  $Q \rtimes T/T_{j+id}$ . Thus the sequence of branches with roots  $Q \rtimes T/T_{j+id}$  for  $i = 0, 1, \dots$  has unbounded depth.

Now assume that  $G$  is a non-split extension of  $T$  by  $Q$ . As described in [14, §10.4], there exists a unique minimal supergroup  $S$  of  $T$  such that  $G$  embeds in the infinite pro- $p$ -group  $H := Q \rtimes S$  of finite coclass. A finite upper bound to  $[H : G] = [S : T]$  is given in [14, Theorem 10.4.6]. Let  $\gamma : S \wedge S \rightarrow S$  be a  $\mathbb{Z}_p Q$ -module homomorphism where  $\gamma(S \wedge S) = S_j$  and  $\gamma(S_j \wedge S) = S_k$ , and  $j \leq m \leq k$ . Now  $H_{\gamma,m} = Q \rtimes S_{\gamma,m}$  is the skeleton group defined by  $\gamma$  and  $m$ . Assume that the largest mainline quotient of  $H_{\gamma,m}$  has class  $j$ , so  $H/L_{j+1}(H) \cong H_{\gamma,m}/L_{j+1}(H_{\gamma,m})$ . Assume also that  $j$  is large enough so that  $L_{j+1}(H) \leq G$ . Define  $G_{\gamma,m}$  as the full preimage in  $H_{\gamma,m}$  of  $G/L_{j+1}(H)$ . Then  $G_{\gamma,m}$  is the skeleton group for  $G$  defined by  $\gamma$  and  $m$ .

**Lemma 3.2** *Every constructible group in the sense of [14, Definition 8.4.9] is a skeleton group, and conversely.*

PROOF: This is straightforward if the infinite pro- $p$ -group  $G$  splits over  $T$ . If  $G$  is a non-split extension of  $T$  by  $Q$ , then a constructible group  $G_\alpha$  for  $G$  is defined as an extension determined by  $\alpha \in \text{Hom}_Q(T \wedge T, T)$  in [14, Definition 8.4.9]. This homomorphism extends to the minimal split supergroup  $S$  of  $T$  and defines a constructible group  $\overline{G}_\alpha$  for the pro- $p$ -group  $Q \times S$ . Since  $\overline{G}_\alpha$  is a skeleton group for  $Q \times S$  and contains  $G_\alpha$  as an appropriately embedded subgroup,  $G_\alpha$  is a skeleton group. •

### 3.3 The isomorphism problem for skeleton groups

Assume that  $T$  is a characteristic subgroup of  $G$ . (This assumption is always satisfied in our later applications.) Since  $T_i$  for  $i \geq 0$  is then characteristic in  $G$ , each  $\alpha \in \text{Aut}(G)$  induces an automorphism of  $T/T_i$ . Hence we can define an action of  $\text{Aut}(G)$  on the set of homomorphisms  $\gamma_m$  induced by surjections  $\gamma \in \text{Hom}_Q(T \wedge T, T_j)$ . Namely, for  $x, y \in T/T_j$  let

$$\alpha(\gamma_m)(x \wedge y) := \alpha(\gamma_m(\alpha^{-1}(x) \wedge \alpha^{-1}(y))).$$

**Lemma 3.3** *Let  $\gamma$  and  $\gamma'$  be two surjections in  $\text{Hom}_Q(T \wedge T, T_j)$ , and assume that there exists  $\alpha \in \text{Aut}(G)$  with  $\alpha(\gamma_m) = \gamma'_m$ . Then  $G_{\gamma, m} \cong G_{\gamma', m}$ .*

PROOF: First consider the case where  $G = Q \times T$ . Since  $T$  is characteristic in  $G$ , the automorphism  $\alpha$  induces automorphisms of  $Q$  and of  $T$ . The restriction of  $\alpha$  to  $T$  is a  $\mathbb{Z}_p$ -linear map. Hence for  $a, b \in T$ , if  $\cdot_\gamma$  and  $\cdot_{\gamma'}$  denote the twisted operations on  $T/T_m$  defined by  $\gamma$  and  $\gamma'$  respectively then

$$\begin{aligned} \alpha((a + T_m) \cdot_\gamma (b + T_m)) &= \alpha((a + b + T_m) + \frac{1}{2}\gamma_m(a + T_j \wedge b + T_j)) \\ &= \alpha(a + T_m) + \alpha(b + T_m) + \frac{1}{2}\alpha(\gamma_m(a + T_j \wedge b + T_j)) \\ &= \alpha(a + T_m) + \alpha(b + T_m) + \frac{1}{2}\gamma'_m(\alpha(a + T_j) \wedge \alpha(b + T_j)) \\ &= \alpha(a + T_m) \cdot_{\gamma'} \alpha(b + T_m). \end{aligned}$$

Thus  $\alpha$  induces an isomorphism  $T_{\gamma, m} \rightarrow T_{\gamma', m}$ . Since  $G = Q \times T$ , we deduce that  $G_{\gamma, m} = Q \times T_{\gamma, m}$  and the map  $G_{\gamma, m} \rightarrow G_{\gamma', m} : (g, x) \mapsto (\alpha(g), \alpha(x))$  is an isomorphism.

Now suppose that  $G$  is a non-split extension of  $T$  by  $Q$ , and let  $H = GS$  be a minimal split supergroup. An automorphism of  $G$  restricts to an automorphism of  $T$  and this, in turn, extends uniquely to  $S$ . Since  $H = GS$ , an automorphism of  $G$  extends to an automorphism of  $H$  which normalises  $G$ . The split case implies that  $H_{\gamma, m} \cong H_{\gamma', m}$  and thus  $G_{\gamma, m} \cong G_{\gamma', m}$ . •

The isomorphisms induced by this action of  $\text{Aut}(G)$  on skeleton groups are *orbit isomorphisms*. The determination of the orbit isomorphisms is an important step towards a solution of the isomorphism problem for the skeleton groups. Other isomorphisms can arise, as the study of 3-groups of coclass 2 shows. We call them *exceptional*. Their complete determination requires considerable care.

### 3.4 The skeleton graph

Let  $P$  be a skeleton group in  $\mathcal{T}(G)$  of class  $c$ , and let  $L_c(P)$  be the last non-trivial term of its lower central series. If  $P/L_c(P)$  is in  $\mathcal{T}(G)$ , then it is also a skeleton group. Thus the skeleton groups define a subgraph, the *skeleton graph*, of  $\mathcal{T}(G)$  which includes the mainline. The subtree of branch  $\mathcal{B}_j$  consisting of skeleton groups defines  $\mathcal{S}_j$ , the skeleton of  $\mathcal{B}_j$ .

The *twig* of  $P$  is the subtree of all descendants of  $P$  that are not descendants of any skeleton group that is a proper descendant of  $P$ . Thus  $\mathcal{T}(G)$  is partitioned into twigs, and the twigs are connected by the skeleton graph of  $\mathcal{T}(G)$ .

The following is a consequence of [14, Theorem 11.3.9] and Lemma 3.2.

**Theorem 3.4** *There is an absolute bound to the depth of the twigs in  $\mathcal{T}(G)$ .*

Hence the skeleton graph exhibits the broad structure of  $\mathcal{T}(G)$ , the twigs contain the fine detail. In particular, there are only finitely many isomorphism types of twigs. Conjecture W (Section 9) suggests that there are patterns in the isomorphism types of twigs which occur in a coclass tree.

## 4 The infinite pro-3-groups of coclass 2

We show that there are 16 infinite pro-3-groups of coclass 2 up to isomorphism, and identify the four coclass trees in  $\mathcal{G}(3, 2)$  that have unbounded depth.

**Theorem 4.1** *There are six isomorphism types of infinite pro-3-groups of coclass 2 with non-trivial centre. They have the following pro-3 presentations:*

$$\{a, t, z \mid a^3 = z^f, [t, t^a] = z^g, tt^a t^{a^2} = z^h, z^3 = [z, a] = [z, t] = 1\}$$

where  $(f, g, h)$  is one of  $(0, 0, 0)$ ,  $(0, 0, 1)$ ,  $(0, 1, 0)$ ,  $(0, 1, 2)$ ,  $(1, 0, 0)$ , or  $(1, 1, 2)$ .

PROOF: Every infinite pro-3-group of coclass 2 with non-trivial centre is a central extension of the cyclic group of order 3 by the (unique) infinite pro-3-group  $S$  of coclass 1 (see [14, §7.4]). This is reflected in the presentations, since  $\langle z \rangle$  is a central subgroup of order 3 with quotient  $S$ . The isomorphism types of infinite pro-3-groups of coclass 2 with non-trivial centre correspond one-to-one to the orbits of  $\text{Aut}(S) \times \text{Aut}(\mathbb{Z}/3\mathbb{Z})$  on  $H^2(S, \mathbb{Z}/3\mathbb{Z}) \cong (\mathbb{Z}/3\mathbb{Z})^3$ . •

Every infinite pro-3-group  $G$  of coclass 2 with trivial centre is a 3-adic space group of dimension  $d = 2 \cdot 3^s$  where  $s \in \{0, 1\}$ . As the unique 3-adic space group of dimension 2 has coclass 1, it follows that  $d = 6$ . Thus  $G$  is an extension of a free  $\mathbb{Z}_3$ -module of rank 6 by a finite 3-group acting faithfully and uniserially on the module.

We use number theory to describe the infinite pro-3-groups of coclass 2 in more detail. Let  $K$  be the ninth cyclotomic number field. Then  $K = \mathbb{Q}_3(\theta)$ , where  $\mathbb{Q}_3$  is the

field of 3-adic numbers, and  $\theta$  is a primitive ninth root of unity; so  $1 + \theta^3 + \theta^6 = 0$ . The ring of integers  $\mathcal{O}$  of  $K$  is a free  $\mathbb{Z}_3$ -module of rank 6 generated by  $1, \theta, \dots, \theta^5$ .

Let  $W$  be the group of  $\mathbb{Z}_3$ -linear maps of  $\mathcal{O}$  generated by the permutations  $a = (1, \theta, \dots, \theta^8)$  and  $y = (1, \theta^3, \theta^6)$ . Then  $W$  has order 81 and is isomorphic to the wreath product of two cyclic groups of order 3. Further,  $b = (\theta, \theta^4, \theta^7)(\theta^2, \theta^8, \theta^5) = a^5 y^2 a y$ , and so  $b \in W$ . The action of  $W$  on  $\mathcal{O}$  extends to a  $\mathbb{Q}_3$ -linear action on  $K$ . We write  $k^w$  for the image of  $k \in K$  under  $w \in W$ . Note that  $k^a = k\theta$ , and thus  $a$  acts as multiplication by  $\theta$ . Further,  $k^b = \sigma_4(k)$ , where, for  $i$  prime to 3, the map  $\sigma_i$  is the Galois automorphism of  $K$  defined by  $\theta \mapsto \theta^i$ .

The split extension  $W \rtimes (\mathcal{O}, +) = \{(w, o) \mid w \in W, o \in \mathcal{O}\}$  is a uniserial 3-adic space group of coclass 4 with translation subgroup  $\{(1, o) \mid o \in \mathcal{O}\} \cong (\mathcal{O}, +)$ . Let  $\mathfrak{p} = (\theta - 1)\mathcal{O}$  denote the unique maximal ideal in  $\mathcal{O}$ . The maximal  $W$ -invariant series in the translation subgroup  $(\mathcal{O}, +)$  is  $\mathcal{O} = \mathfrak{p}^0 > \mathfrak{p}^1 > \mathfrak{p}^2 > \dots$

It is sometimes useful to have a multiplicative version of  $(\mathcal{O}, +)$ . We denote this multiplicative group by  $T$  and write  $T_i$  for the subgroup corresponding to  $\mathfrak{p}^i$ . If  $t_0 \in T$  corresponds to  $1 \in \mathcal{O}$ , then  $t_i = [t_0, i a] \in T_i \setminus T_{i+1}$  corresponds to  $(\theta - 1)^i \in \mathfrak{p}^i \setminus \mathfrak{p}^{i+1}$ . In the multiplicative setting we write the split extension  $W \rtimes T$  as  $\{wt \mid w \in W, t \in T\}$ .

Every 3-adic space group of coclass 2 embeds as a subgroup of finite index in  $W \rtimes T$ . Define  $C := \langle a \rangle$  and  $D := \langle a, b \rangle$  as subgroups of  $W$ , and so of  $W \rtimes T$ .

**Theorem 4.2** *There are ten isomorphism types of infinite pro-3-groups of coclass 2 with trivial centre. They are:*

- (a)  $R := \langle a, t_0 \rangle$  so  $R = C \rtimes T$ .
- (b)  $G_i := \langle a, bt_{i-1} \rangle$  for  $i = 1, 2, 3$ , so  $G_i = D \cdot T_i$  and has index 3 in its minimal split supergroup  $H_{i-1} := \langle a, b, t_{i-1} \rangle$ .
- (c)  $\langle a, yt_{i-2} \rangle$  and  $\langle a, ya^{-1}t_{i-2} \rangle$  for  $i = 2, 3, 4$ . Each subgroup is isomorphic to  $W \cdot T_i$  and has index 9 in  $W \rtimes T_{i-2}$ .

That there are precisely ten isomorphism types follows from [8]; the algorithm given there can be used to obtain descriptions of the groups.

The following can be deduced from Corollary 11.4.2 and Theorem 11.4.4 of [14].

**Theorem 4.3** *The six groups of Theorem 4.1 and the six groups in part (c) of Theorem 4.2 have coclass trees whose sequences of branches all have bounded depth.*

Since the trees of bounded depth are covered by Theorem 2.1, we do not investigate them further. As we prove below, the remaining groups – those in parts (a) and (b) of Theorem 4.2 – have coclass trees with unbounded depth, and so we study them. In Sections 5 and 7 respectively, we study  $\mathcal{T}(R)$  and  $\mathcal{T}(G_i)$  for  $1 \leq i \leq 3$ ; in Section 6 we prepare for the latter by studying  $\mathcal{T}(H_i)$  for  $0 \leq i \leq 2$ . We first theoretically determine their skeleton graphs and then investigate their twigs.

## 5 The skeleton groups in $\mathcal{T}(R)$

The root of  $\mathcal{T}(R)$  has order  $3^9$ . Let  $\mathcal{B}_j$  denote the branch of  $\mathcal{T}(R)$  whose root has class  $j$  and order  $3^{j+2}$ , and let  $\mathcal{S}_j$  be the skeleton of  $\mathcal{B}_j$ .

In this section we determine  $\mathcal{S}_j$  for every  $j \geq 7$ . The general construction of skeleton groups is described in Section 3. In number-theoretic notation, it uses the surjections  $\gamma \in \text{Hom}_C(\mathcal{O} \wedge \mathcal{O}, \mathfrak{p}^j)$ , and their images  $\gamma_m$  in  $\text{Hom}_C(\mathcal{O}/\mathfrak{p}^j \wedge \mathcal{O}/\mathfrak{p}^j, \mathfrak{p}^j/\mathfrak{p}^m)$ , for  $j \leq m \leq k$ ; we determine the value of  $k$  in Lemma 5.3. The resulting skeleton groups  $R_{\gamma, m} := C \rtimes T_{\gamma, m}$  have order  $3^{m+2}$ , class  $m$ , and depth  $m - j$  in  $\mathcal{S}_j$ . We first describe  $\text{Hom}_C(\mathcal{O} \wedge \mathcal{O}, \mathcal{O})$ , then we determine all orbit isomorphisms, and finally show that there are no exceptional isomorphisms. We also give presentations for the skeleton groups and describe their automorphisms. To facilitate the determination of  $\mathcal{S}_j$ , we use a different representation of  $R$  for each  $j$ .

### 5.1 The homomorphism space

We now describe the space of homomorphisms which determine the skeleton groups in  $\mathcal{T}(R)$ . Recall that  $\sigma_i$  is the Galois automorphism of  $K$  defined by  $\theta \mapsto \theta^i$  where  $i$  is prime to 3.

**Lemma 5.1** *Let  $\vartheta : \mathcal{O} \wedge \mathcal{O} \rightarrow \mathcal{O} : x \wedge y \mapsto \sigma_2(x)\sigma_{-1}(y) - \sigma_{-1}(x)\sigma_2(y)$ . Then  $\vartheta \in \text{Hom}_C(\mathcal{O} \wedge \mathcal{O}, \mathcal{O})$ . If  $i$  and  $j$  are non-negative integers, then  $\vartheta$  maps  $\mathfrak{p}^i \wedge \mathfrak{p}^j$  onto  $\mathfrak{p}^{i+j+\varepsilon}$ , where  $\varepsilon = 3$  if  $i \equiv j \pmod{3}$ , and  $\varepsilon = 2$  otherwise.*

PROOF: Clearly  $\vartheta(\theta x \wedge \theta y) = \sigma_2(\theta x)\sigma_{-1}(\theta y) - \sigma_{-1}(\theta x)\sigma_2(\theta y) = \theta\vartheta(x \wedge y)$ . Hence  $\vartheta$  is compatible with the action of  $\theta$ , and thus  $\vartheta \in \text{Hom}_C(\mathcal{O} \wedge \mathcal{O}, \mathcal{O})$ . The image of  $\mathfrak{p}^i \wedge \mathfrak{p}^j$  under  $\vartheta$  is generated by  $\vartheta((\theta - 1)^i \theta^a \wedge (\theta - 1)^j \theta^b)$  for  $0 \leq a, b \leq 5$ , as  $\mathfrak{p}^i = (\theta - 1)^i \mathcal{O}$  and  $\mathcal{O}$  is generated by  $1, \theta, \dots, \theta^5$  as a  $\mathbb{Z}_3 C$ -module. Let  $k = i - j$  and  $l = a - b$ . Then

$$\begin{aligned} & \vartheta((\theta - 1)^i \theta^a \wedge (\theta - 1)^j \theta^b) \\ &= (\theta^2 - 1)^{j+k} \theta^{2(b+l)} (\theta^{-1} - 1)^j \theta^{-b} - (\theta^{-1} - 1)^{j+k} \theta^{-(b+l)} (\theta^2 - 1)^j \theta^{2b} \\ &= (\theta^2 - 1)^j (\theta^{-1} - 1)^j (\theta - 1)^k \theta^{b-l-k} [(1 + \theta)^k \theta^{3l+k} - (-1)^k] \\ &= (\theta - 1)^{i+j} c(i, j, a, b) [(1 + \theta)^k \theta^{3l+k} - (-1)^k], \end{aligned}$$

where  $c(i, j, a, b)$  is a unit. Consider the term  $(1 + \theta)^k \theta^{3l+k} - (-1)^k$ : if  $k \equiv 0 \pmod{3}$  it is in  $\mathfrak{p}^3$ ; if  $k \equiv 0 \pmod{3}$  and  $l \not\equiv 0 \pmod{3}$  it is in  $\mathfrak{p}^3 \setminus \mathfrak{p}^4$ ; if  $k \not\equiv 0 \pmod{3}$  it is in  $\mathfrak{p}^2 \setminus \mathfrak{p}^3$ . •

Let  $\mathcal{U}$  denote the unit group of the ring  $\mathcal{O}$ .

**Lemma 5.2** *Let  $i$  and  $j$  be non-negative integers.*

(a)  $\text{Hom}_C(\mathfrak{p}^i \wedge \mathfrak{p}^i, \mathfrak{p}^{i+j}) = \{c(\theta - 1)^{j-i-3}\vartheta \mid c \in \mathcal{O}\} = \mathfrak{p}^{j-i-3}\vartheta.$

(b)  $\text{Hom}_C(\mathfrak{p}^i \wedge \mathfrak{p}^i, \mathfrak{p}^{i+j}) \setminus \text{Hom}_C(\mathfrak{p}^i \wedge \mathfrak{p}^i, \mathfrak{p}^{i+j+1}) = (\theta - 1)^{j-i-3}\mathcal{U}\vartheta.$

PROOF: Lemma 5.1 shows that  $(\theta - 1)^{j-i-3}\vartheta$  is a surjective element of  $\text{Hom}_C(\mathfrak{p}^i \wedge \mathfrak{p}^i, \mathfrak{p}^{i+j})$ . By [14, Theorem 11.4.1], the set  $\{\vartheta\}$  is a  $K$ -basis of  $\text{Hom}_C(K \wedge K, K)$ . Hence  $\text{Hom}_C(\mathfrak{p}^i \wedge \mathfrak{p}^i, \mathfrak{p}^{i+j}) = \mathfrak{p}^{j-i-3}\vartheta$  for every  $j \geq 0$ . Also  $\mathfrak{p}^k \setminus \mathfrak{p}^{k+1} = (\theta - 1)^k \mathcal{U}$  for every  $k$ . •

## 5.2 A change of representation and notation

To describe  $\mathcal{S}_j$ , the skeleton of  $\mathcal{B}_j$ , we must determine the orbits of the action of  $\text{Aut}(R)$  on the set of homomorphisms induced by the surjections of Lemma 5.2 for  $i = 0$ . The term  $(\theta - 1)^{j-3}$  introduces technical complications to these computations. We avoid these by adjusting our notation.

Let  $R_h = C \rtimes T_h \cong C \rtimes \mathfrak{p}^h$  for  $h \geq 0$ . Then  $R_0 = R$  and  $R_h \cong R$  for  $h \geq 1$ . Using  $R_{j-3}$  instead of  $R_0$ , the skeleton groups in  $\mathcal{B}_j$  correspond to the  $\mathbb{Z}_3 C$ -module surjections

$$\gamma : \mathfrak{p}^{j-3} \wedge \mathfrak{p}^{j-3} \rightarrow \mathfrak{p}^{2j-3}.$$

Lemma 5.2 shows that these surjections can be written as  $c\vartheta$  for some unit  $c \in \mathcal{U}$ , avoiding  $(\theta - 1)^{j-3}$ .

If  $j \leq m \leq k$ , then the new surjection  $\gamma$  induces the homomorphism

$$\gamma_m : \mathfrak{p}^{j-3}/\mathfrak{p}^{2j-3} \wedge \mathfrak{p}^{j-3}/\mathfrak{p}^{2j-3} \rightarrow \mathfrak{p}^{2j-3}/\mathfrak{p}^{m+j-3}.$$

As with the previous notation,  $\gamma_m$  defines a multiplication ‘ $\cdot$ ’ on the set  $\mathfrak{p}^{j-3}/\mathfrak{p}^{m+j-3}$ . We denote the resulting group  $(\mathfrak{p}^{j-3}/\mathfrak{p}^{m+j-3}, \cdot)$  by  $T_{j-3,\gamma,m}$ . It has order  $p^m$  and derived subgroup  $\mathfrak{p}^{2j-3}/\mathfrak{p}^{m+j-3}$ . Now  $R_{j-3,\gamma,m} = C \rtimes T_{j-3,\gamma,m}$  is a skeleton group in  $\mathcal{S}_j$  of order  $3^{m+2}$ , class  $m$ , and depth  $m - j$ .

**Lemma 5.3** *For every  $j \geq 7$ , the skeleton  $\mathcal{S}_j$  has depth  $j - \chi_j$ , where  $\chi_j = 0$  if 3 divides  $j$ , and  $\chi_j = 1$  otherwise. In particular, every skeleton  $\mathcal{S}_j$  is non-trivial, and every sequence of branches in  $\mathcal{T}(R)$  has unbounded depth.*

PROOF: Lemma 5.1 implies that  $\vartheta$  maps  $\mathfrak{p}^{j-3} \wedge \mathfrak{p}^{j-3}$  onto  $\mathfrak{p}^{2j-3}$ , and  $\mathfrak{p}^{2j-3} \wedge \mathfrak{p}^{j-3}$  onto  $\mathfrak{p}^{3j-6+\varepsilon}$ , where  $\varepsilon = 3$  if 3 divides  $j$ , and  $\varepsilon = 2$  otherwise. Hence  $R_{j-3,\gamma,m}$  is defined for  $j \leq m \leq 2j - 3 + \varepsilon = 2j - \chi_j$ . •

Hence, for the remainder of Section 5, we assume that  $j \geq 7$  and  $j \leq m \leq 2j - \chi_j$ , where  $\chi_j$  is defined in the above lemma.

## 5.3 The automorphism group of $R_h$

We construct the automorphism group of  $R_h$  for  $h \geq 0$  from three subgroups.

*Galois automorphisms.* Observe that  $\sigma_2$  generates the Galois group  $\text{Gal}(K, \mathbb{Q}_3)$ , which is cyclic of order 6. Further,  $\sigma_2$  induces an automorphism of  $\mathcal{O}$ , and thus of  $\mathfrak{p}^h$ , that extends to an automorphism of  $R_h$ , also called  $\sigma_2$ , mapping  $a$  to  $a^2$ . Let

$$A_0 = \langle \sigma_2 \rangle \leq \text{Aut}(R_h).$$

*Unit automorphisms.* Multiplication by a unit  $u \in \mathcal{U}$  is an automorphism of the additive group  $\mathcal{O}$  that normalises  $\mathfrak{p}^h$ . Thus it extends to an automorphism  $\mu_u$  of  $R_h$  that fixes  $a$ . Let

$$A_1 = \langle \mu_u \mid u \in \mathcal{U} \rangle \leq \text{Aut}(R_h).$$

*Central automorphisms.* Viewing  $R_h$  as a subgroup of  $C \rtimes T_{h-1}$  allows an element of  $T_{h-1} \cong \mathfrak{p}^{h-1}$  to act as an automorphism by conjugation. This action of  $\phi \in \mathfrak{p}^{h-1}$  is denoted by  $\nu_\phi$ . Such automorphisms fix  $T_h$  and  $R_h/T_h$  pointwise. Let

$$A_2 = \langle \nu_\phi \mid \phi \in \mathfrak{p}^{h-1} \rangle \leq \text{Aut}(R_h).$$

**Lemma 5.4**  $\text{Aut}(R_h) = (A_0 \rtimes A_1) \rtimes A_2$ , and is isomorphic to  $(\text{Aut}(C) \rtimes \mathcal{U}) \rtimes T_{h-1}$ .

PROOF: Since  $\mathfrak{p}^h$  is a characteristic subgroup of  $R_h$ , it follows that  $\text{Aut}(R_h)$  maps into  $\text{Aut}(C)$ ; and this map is onto, since  $A_0$  maps onto  $\text{Aut}(C)$ . It remains to prove that  $A_1 \rtimes A_2$  is the kernel of this homomorphism. This kernel maps, by restriction, into the group of automorphisms of  $\mathfrak{p}^h$  as  $C$ -module; that is to say, as  $\mathcal{O}$ -module. But  $\mathfrak{p}^h$  is a free  $\mathcal{O}$ -module of rank 1, so this automorphism group is naturally isomorphic to the group of units  $\mathcal{U}$  of  $\mathcal{O}$ , and so the subgroup  $A_1$  shows that this restriction map is onto.

Finally, we need to verify that  $A_2$  is the kernel of this restriction map. This kernel, as it centralises both  $R_h/\mathfrak{p}^h$  and  $\mathfrak{p}^h$ , consists of the group of derivations of  $C$  into the  $C$ -module  $\mathfrak{p}^h$ . Now  $H^1(C, K) = 0$ , since  $C$  has order 3, and 3 is invertible in  $K$ . Thus, if  $R_h = C \rtimes \mathfrak{p}^h$  is embedded, in the natural way, into  $C \rtimes K$ , then every derivation of  $C$  into  $\mathfrak{p}^h$  becomes an inner derivation induced by an element of  $K$ . But the inner derivations induced by conjugation by elements of  $K$  that normalise  $R_h$  are those induced by conjugation by elements of  $\mathfrak{p}^{h-1}$ . The result follows.  $\bullet$

## 5.4 Some number theory

As the descriptions of  $\text{Aut}(R_{j-3})$  and  $\text{Hom}_C(\mathfrak{p}^{j-3} \wedge \mathfrak{p}^{j-3}, \mathfrak{p}^{2j-3})$  exhibit, number theory plays a role in the construction of skeleton groups. We now present some number-theoretic results that help to solve the isomorphism problem for skeleton groups in  $\mathcal{S}_j$ .

As  $\mathcal{O}$  is a local ring with unique maximal ideal  $\mathfrak{p}$ , its unit group  $\mathcal{U} = \mathcal{O} \setminus \mathfrak{p}$ . Recall the structure of  $\mathcal{U}$  from [12, Chapter 15].

**Lemma 5.5** For  $i > 0$ , let  $\mathcal{U}_i = 1 + \mathfrak{p}^i$  and  $\kappa_i = 1 + (\theta - 1)^i$ .

- (a)  $\mathcal{U} = \mathcal{U}_0 > \mathcal{U}_1 > \mathcal{U}_2 > \dots$  is a filtration of  $\mathcal{U}$  whose quotients are cyclic, and respectively generated by  $-1, \theta, \kappa_2, \theta^3$  and  $\kappa_i$  for  $i \geq 4$ .
- (b) The torsion subgroup of  $\mathcal{U}$  has order 18, and is generated by  $\theta$  and  $-1$ .
- (c) The exponential map defines an isomorphism  $\mathfrak{p}^4 \rightarrow \mathcal{U}_4$ .

Recall that a subgroup  $U$  of a group  $V$  covers a normal section  $A/B$  of  $V$  if  $A/B \leq UB/B$ .

**Lemma 5.6** *Let  $\rho : \mathcal{U} \rightarrow \mathcal{U} : u \mapsto \sigma_2(u)\sigma_{-1}(u)u^{-1}$ . Then  $\rho(\mathcal{U})$  is a subgroup of index  $3^4$  in  $\mathcal{U}$  that covers  $\mathcal{U}_i/\mathcal{U}_{i+1}$  if and only if  $i \notin \{1, 3, 5, 11\}$ .*

PROOF: First we consider  $\rho(\mathcal{U}_4)$ . Let  $\tau : \mathfrak{p}^4 \rightarrow \mathfrak{p}^4$  be defined by  $x \mapsto \sigma_2(x) + \sigma_{-1}(x) - x$ . The image  $\tau(\mathfrak{p}^4)$  maps under the exponential map onto  $\rho(\mathcal{U}_4)$ , so  $\mathfrak{p}^4/\tau(\mathfrak{p}^4) \cong \mathcal{U}_4/\rho(\mathcal{U}_4)$ . To determine the image of  $\tau$ , observe that  $\sigma_2$  is an automorphism of order 6. Thus it is diagonalisable with eigenvalues  $w^i$  for  $i = 0, \dots, 5$ , where  $w := -\theta^3$  is a primitive sixth root of unity. As  $\sigma_{-1} = \sigma_2^3$ , it follows that  $\tau$  is diagonalisable with eigenvalues  $\{w^i + w^{3i} - 1 \mid 0 \leq i \leq 5\}$ . Hence  $\det(\tau) = -9$  and the image of  $\tau$  has index  $3^2$  in  $\mathfrak{p}^4$ . Thus  $\rho(\mathcal{U}_4)$  has index  $3^2$  in  $\mathcal{U}_4$ .

Next, we determine  $\rho(\mathcal{U})$  modulo  $\mathcal{U}_{12} = 1 + 9\mathcal{O}$ . A routine calculation shows:

$$\begin{aligned} \rho(-1) &= -1; \\ \rho(\theta) &= 1; \\ \rho(\kappa_2) &\equiv \kappa_2\theta^6\kappa_4\kappa_6\kappa_7\kappa_8\kappa_9 \pmod{\mathcal{U}_{12}}; \\ \rho(\theta^3) &= 1; \\ \rho(\kappa_4) &\equiv \kappa_4\kappa_5\kappa_6\kappa_{10}\kappa_{11} \pmod{\mathcal{U}_{12}}; \\ \rho(\kappa_5) &\equiv \kappa_6\kappa_7^2\kappa_{10}^2 \pmod{\mathcal{U}_{12}}; \\ \rho(\kappa_6) &\equiv \kappa_6\kappa_9^2 \pmod{\mathcal{U}_{12}}; \\ \rho(\kappa_7) &\equiv \kappa_8^2\kappa_9^2\kappa_{10}^2\kappa_{11} \pmod{\mathcal{U}_{12}}; \\ \rho(\kappa_8) &\equiv \kappa_8\kappa_9^2\kappa_{10}\kappa_{11} \pmod{\mathcal{U}_{12}}; \\ \rho(\kappa_9) &\equiv 1 \pmod{\mathcal{U}_{12}}; \\ \rho(\kappa_{10}) &\equiv \kappa_{10}\kappa_{11} \pmod{\mathcal{U}_{12}}; \\ \rho(\kappa_{11}) &\equiv 1 \pmod{\mathcal{U}_{12}}. \end{aligned}$$

Thus  $\rho(\mathcal{U}_4)$  covers neither  $\mathcal{U}_5/\mathcal{U}_6$  nor  $\mathcal{U}_{11}/\mathcal{U}_{12}$ . Since  $\rho(\mathcal{U}_4)$  has index  $3^2$  in  $\mathcal{U}_4$ , it contains  $\mathcal{U}_{12}$ . The result follows. •

Hence  $\mathcal{U}/\rho(\mathcal{U})$  has order 81 and is generated by the cosets with representatives  $\theta, \kappa_5, \kappa_{11}$ . A routine calculation shows that  $\kappa_5^3 \equiv \kappa_{11}^2 \pmod{\mathcal{U}_{12}}$ , so  $\theta$  and  $\kappa_5$  suffice. Defining  $V := (\mathbb{Z}/9\mathbb{Z})^2$ , we obtain an isomorphism of abelian groups  $\varphi : \mathcal{U}/\rho(\mathcal{U}) \rightarrow V$  defined by

$$\theta^{u_1}\kappa_5^{u_2}\rho(\mathcal{U}) \mapsto (u_1, u_2).$$

**Lemma 5.7** *The Galois automorphism  $\sigma_2$  acts on  $\mathcal{U}/\rho(\mathcal{U})$  as*

$$V \rightarrow V : (u_1, u_2) \mapsto (2u_1, 2u_2).$$

PROOF: By definition,  $\sigma_2(\theta) = \theta^2$ . A routine calculation shows that

$$\sigma_2(\kappa_5) \equiv \kappa_5^2\kappa_6^2\kappa_7^2\kappa_8\kappa_9\kappa_{11} \pmod{\mathcal{U}_{12}}.$$

This yields  $\sigma_2(\kappa_5) \equiv \kappa_5^2 \pmod{\rho(\mathcal{U})}$ . •

## 5.5 A solution of the isomorphism problem

We show that orbit isomorphisms solve the isomorphism problem completely for the skeleton groups in  $\mathcal{S}_j$ .

**Lemma 5.8** *Let  $\gamma$  and  $\gamma'$  be two surjections in  $\text{Hom}_C(\mathfrak{p}^{j-3} \wedge \mathfrak{p}^{j-3}, \mathfrak{p}^{2j-3})$ . Then  $R_{j-3, \gamma, m}$  and  $R_{j-3, \gamma', m}$  are isomorphic if and only if there exists  $\alpha \in \text{Aut}(R_{j-3})$  with  $\alpha(\gamma_m) = \gamma'_m$ .*

PROOF: By Lemma 3.3, we only have to show that, if  $R_{j-3, \gamma, m}$  and  $R_{j-3, \gamma', m}$  are isomorphic, then there exists an automorphism  $\alpha$  of  $R_{j-3}$  with  $\alpha(\gamma_m) = \gamma'_m$ . But  $T_{j-3}/T_{m+j-3}$  is characteristic in  $R_{j-3, \gamma, m}$  and  $R_{j-3, \gamma', m}$ . Thus, if  $R_{j-3, \gamma, m}$  and  $R_{j-3, \gamma', m}$  are isomorphic, then an isomorphism between them induces automorphisms  $\alpha_1$  and  $\alpha_2$  of  $C = \langle a \rangle$  and  $T_{j-3}/T_{2j-3}$  respectively. These automorphisms form a compatible pair; namely,  $a^{-\alpha_1} t^{\alpha_2} a^{\alpha_1} = (a^{-1} t a)^{\alpha_2}$  for all  $t$  in  $T_{j-3}/T_{2j-3}$ . Since  $T_{j-3}$  is a free  $\mathcal{O}$ -module,  $\alpha_2$  lifts to an automorphism  $\alpha_3$  of  $R_{j-3}$  such that  $\alpha_1$  and  $\alpha_3$  form a compatible pair, and hence define an automorphism  $\alpha$  of  $R_{j-3}$ . Clearly  $\alpha$  satisfies  $\alpha(\gamma_m) = \gamma'_m$ .  $\bullet$

We must investigate the action of  $\text{Aut}(R_{j-3})$  on the homomorphisms induced by the surjections to solve the isomorphism problem for skeleton groups. Observe that  $A_2 \leq \text{Aut}(R_{j-3})$  acts trivially on them. Hence it remains to determine the  $(A_0 \times A_1)$ -orbits on the image of  $\text{Hom}_C(\mathfrak{p}^{j-3} \wedge \mathfrak{p}^{j-3}, \mathfrak{p}^{2j-3})$  in  $\text{Hom}_C(\mathfrak{p}^{j-3}/\mathfrak{p}^{2j-3} \wedge \mathfrak{p}^{j-3}/\mathfrak{p}^{2j-3}, \mathfrak{p}^{2j-3}/\mathfrak{p}^{m+j-3})$ . Throughout let  $n = m - j$ , so  $R_{j-3, \gamma, m}$  is a group of depth  $n$  in  $\mathcal{S}_j$ .

**Lemma 5.9** *Let  $c, c' \in \mathcal{U}$ . The surjections  $c\vartheta$  and  $c'\vartheta$  induce the same element of  $\text{Hom}_C(\mathfrak{p}^{j-3}/\mathfrak{p}^{2j-3} \wedge \mathfrak{p}^{j-3}/\mathfrak{p}^{2j-3}, \mathfrak{p}^{2j-3}/\mathfrak{p}^{2j-3+n})$  if and only if  $c \equiv c' \pmod{\mathcal{U}_n}$ .*

PROOF: Let  $c \in \mathcal{U}_n$ , so  $c = 1 + e$  for  $e \in \mathfrak{p}^n$ . If  $x, y \in \mathfrak{p}^{j-3}$ , then  $c\vartheta(x \wedge y) = \vartheta(x \wedge y) + e\vartheta(x \wedge y)$  and  $e\vartheta(x \wedge y) \in \mathfrak{p}^{2j-3+n}$ . The converse is similar.  $\bullet$

By Lemma 5.9, the desired orbits correspond to the  $(A_0 \times A_1)$ -orbits on

$$\Omega_n = \mathcal{U}/\mathcal{U}_n.$$

We first consider the action of  $A_1$ . Using the definition of  $\rho$  from Lemma 5.6, for  $\mu_u \in A_1$  and  $c \in \mathcal{U}$

$$\begin{aligned} (\mu_u(c\vartheta))(x \wedge y) &= (c\vartheta(xu^{-1} \wedge yu^{-1}))u \\ &= c(\sigma_2(xu^{-1})\sigma_{-1}(yu^{-1}) - \sigma_2(yu^{-1})\sigma_{-1}(xu^{-1}))u \\ &= c(\sigma_2(x)\sigma_2(u^{-1})\sigma_{-1}(y)\sigma_{-1}(u^{-1}) - \sigma_2(y)\sigma_2(u^{-1})\sigma_{-1}(x)\sigma_{-1}(u^{-1}))u \\ &= c\sigma_2(u^{-1})\sigma_{-1}(u^{-1})u(\sigma_2(x)\sigma_{-1}(y) - \sigma_2(y)\sigma_{-1}(x)) \\ &= c\rho(u^{-1})\vartheta(x \wedge y). \end{aligned}$$

Thus  $A_1$  acts on  $\Omega_n$  via multiplication by  $\rho(\mathcal{U})$ . The orbits of this action correspond to the cosets

$$\Delta_n := \mathcal{U}/\rho(\mathcal{U})\mathcal{U}_n.$$

Lemma 5.6 shows that  $\Delta_n$  has at most  $3^4$  elements for every  $n$ . As  $A_1$  is normal in  $A_0 \times A_1$ , the orbits under the action of  $A_1$  are blocks for the orbits of  $A_0 \times A_1$ . It remains to determine the orbits of  $A_1$  on the elements of  $\Delta_n$ . For  $c \in \mathcal{U}$

$$\begin{aligned} (\sigma_2(c\vartheta))(x \wedge y) &= \sigma_2(c(\vartheta(\sigma_2^{-1}(x) \wedge \sigma_2^{-1}(y)))) \\ &= \sigma_2(c\sigma_2^{-1}(\vartheta(x \wedge y))) \\ &= \sigma_2(c)\vartheta(x \wedge y). \end{aligned}$$

Lemma 5.7 shows that  $\sigma_2$  acts as multiplication by the diagonal matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

on  $V \cong \mathcal{U}/\rho(\mathcal{U})$ . This allows us to read off the orbits of  $A_0$  on  $\Delta_n$ .

**Theorem 5.10** *The skeleton  $\mathcal{S}_j$  is isomorphic to the first  $j - \chi_j$  levels in Figure 1, where  $j \geq 7$  and  $\chi_j = 0$  if 3 divides  $j$  and  $\chi_j = 1$  otherwise.*

PROOF: The root of this tree corresponds to a mainline group, and the nodes at depth  $n$  correspond to groups defined by  $\gamma = c\vartheta$  for  $c \in \mathcal{U}/\mathcal{U}_n$ , or rather to orbits of such parameters by Lemmas 5.8 and 5.9. Thus the vertex of depth 1 corresponds to  $c = \pm 1 \pmod{\mathcal{U}_1}$ . These two values of  $c$  lie in the same orbit, as they are in the same coset modulo  $\rho(\mathcal{U})$ . The two vertices of depth 2 arise from the parameters  $c \in \{1, \theta, \theta^2\} \pmod{\mathcal{U}_2}$ . These last two are in the same orbit under  $\sigma_2$ . The left vertex of depth 2 corresponds to  $c = \theta$ , and the right vertex to  $c = 1$ . The three vertices of depth 4 arise from  $c = \theta^k \pmod{\mathcal{U}_4}$  with  $k$  determined modulo 9. The leftmost node we take to be defined by  $\theta$ , and may equally be defined by  $\theta^k$  for any  $k$  prime to 1 modulo 3. The central node we take to be defined by  $c = \theta^3$ , the alternative  $c = \theta^6$  being in the same orbit as this value by the action of  $\sigma_2$ . The rightmost node is defined by  $c = 1$ . The eight nodes of depth 6, from left to right, we take to be defined by  $c = \theta\kappa_5, \theta, \theta\kappa_5^2; \theta^3\kappa_5, \theta^3, \theta^3\kappa_5^2; \kappa_5, 1 \pmod{\mathcal{U}_6}$ . The groups of depth 12 are defined by  $c = \theta\kappa_5^4, \theta\kappa_5, \theta\kappa_5^7; \theta\kappa_5^3, \theta, \theta\kappa_5^6; \theta\kappa_5^2, \theta\kappa_5^5, \theta\kappa_5^8; \theta^3\kappa_5, \theta^3\kappa_5^3, \theta^3, \theta^3\kappa_5^6; \theta^3\kappa_5^2; \kappa_5; \kappa_5^3, 1 \pmod{\mathcal{U}_{12}}$ . •

Hence the 17 isomorphism types of groups of depth at least 12 in  $\mathcal{S}_j$  are obtained by using the homomorphism  $\gamma = \theta^{u_1}\kappa_5^{u_2}\vartheta$  with the values of  $u_1$  and  $u_2$  listed in Table 1.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$u_1$	1	1	1	1	1	1	1	1	1	3	3	3	3	3	0	0	0
$u_2$	4	1	7	3	0	6	2	5	8	1	3	0	6	2	1	3	0

Table 1: Representative units for the groups in  $\mathcal{S}_j$  of depth at least 12

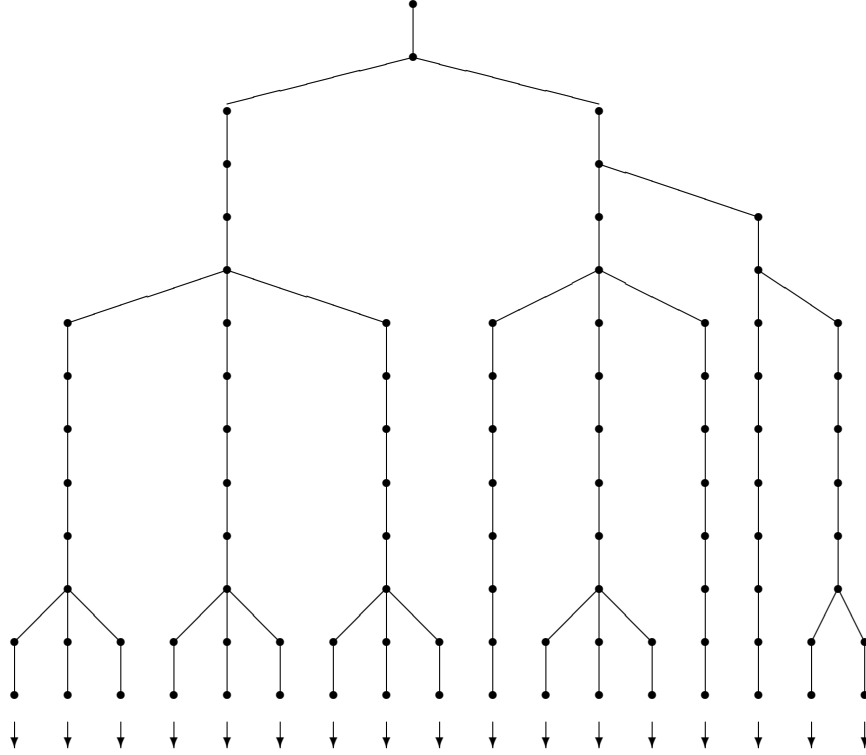


Figure 1: The skeleton  $\mathcal{S}_j$  of  $\mathcal{B}_j$

## 5.6 Presentations for the skeleton groups in $\mathcal{B}_j$

**Lemma 5.11** *Let  $\gamma = \theta^{u_1} \kappa_5^{u_2} \vartheta$  and let  $7 \leq j \leq m \leq 2j - \chi_j$ . Let*

$$f(x) = (-1)^{j-3} x^{2j-5} (x+1)^{(u_1+2+8j)} (x+2)^{j-3} (x^2+3x+3)(x^5+1)^{u_2} \in \mathbb{Z}[x],$$

*and let  $a_i$  denote the coefficient of  $x^i$  in  $f(x)$  for  $2j-5 \leq i \leq m+j-4$ . Let  $\alpha$  and  $\tau$  be two abstract group generators and let  $\tau_i = [\tau, \alpha^i]$  for  $i \geq 0$ . Let*

$$r = [\tau_1, \tau_0] \left( \prod_{i=2j-5}^{m+j-4} \tau_{i-j+3}^{a_i} \right)^{-1}.$$

*Then  $R_{j-3, \gamma, m}$  has a presentation*

$$\{\alpha, \tau \mid \alpha^9 = (\tau \alpha^3)^3 = [\tau, \alpha^3, \tau] = [\tau, \tau^{\alpha^4}] [\tau^\alpha, \tau^{\alpha^3}] = [\tau, \tau^{\alpha^5}] [\tau^{\alpha^2}, \tau^{\alpha^3}] = [\tau, \tau_m \alpha] = r = 1\}.$$

**PROOF:** Let  $F$  denote the free group on  $\{\alpha, \tau\}$ , and  $T$  the normal closure of  $\{\tau\}$  in  $F$ . Let  $G$  denote the group defined by the presentation, and let  $H = G/L_{j-2}(G)$ . We use the same notation for elements and subgroups of  $F$ , and the images of these elements and subgroups in  $G$  and  $H$ ; the context will resolve ambiguities.

We first check that the relations are satisfied in  $R_{j-3,\gamma,m}$ , when  $\alpha$  and  $\tau$  stand for  $a$  and  $(\theta - 1)^{j-3}$  respectively.

Clearly  $a^9 = 1$ , and so the relation  $(\tau\alpha^3)^3 = 1$  reduces to  $s \cdot s^{\theta^3} \cdot s^{\theta^6} = 0$ , where  $s = (\theta - 1)^{j-3}$ , and the twisted operation defined by  $\gamma$  is denoted by  $\cdot$ . But  $\vartheta(u \wedge u\theta^3) = 0$  for every  $u \in \mathfrak{p}^{j-3}$ , so  $s \cdot s^{\theta^3} \cdot s^{\theta^6} = s + s^{\theta^3} + s^{\theta^6} = 0$ , and the relation  $[\tau, \alpha^3, \tau] = 1$  follows from the same identity. Similarly  $[\tau, \tau^{\alpha^4}][\tau^\alpha, \tau^{\alpha^3}] = [\tau, \tau^{\alpha^5}][\tau^{\alpha^2}, \tau^{\alpha^3}] = 1$  follows from the identity  $\vartheta(u \wedge \theta^4 u) + \vartheta(\theta u \wedge \theta^3 u) = \vartheta(u \wedge \theta^5 u) + \vartheta(\theta^2 u \wedge \theta^3 u) = 0$ . Finally, to check the relation  $r = 1$ , we need to calculate  $\gamma((\theta - 1)^{j-4} \wedge (\theta - 1)^{j-3})$

$$\begin{aligned}
&= \theta^{u_1}(1 + (\theta - 1)^5)^{u_2} \vartheta((\theta - 1)^{j-4} \wedge (\theta - 1)^{j-3}) \\
&= \theta^{u_1}(1 + (\theta - 1)^5)^{u_2} ((\theta^2 - 1)^{j-4} (\theta^{-1} - 1)^{j-3} - (\theta^{-1} - 1)^{j-4} (\theta^2 - 1)^{j-3}) \\
&= \theta^{u_1}(1 + (\theta - 1)^5)^{u_2} ((\theta - 1)^{j-4} (\theta + 1)^{j-4} (\theta - 1)^{j-3} (-1)^{j-3} \theta^{-(j-3)} \\
&\quad - (\theta - 1)^{j-4} \theta^{-(j-4)} (-1)^{j-4} (\theta - 1)^{j-3} (\theta + 1)^{j+3}) \\
&= \theta^{u_1}(1 + (\theta - 1)^5)^{u_2} (\theta - 1)^{2j-7} (\theta + 1)^{j-3} (-1)^{j-3} \theta^{-(j-3)} (\theta + 1 + \theta^{-1}) \\
&= \theta^{u_1-j+2} (1 + (\theta - 1)^5)^{u_2} (\theta - 1)^{2j-7} (\theta + 1)^{j-3} (-1)^{j-3} (\theta^2 + \theta + 1).
\end{aligned}$$

Now substituting  $x$  for  $\theta - 1$  gives rise to the polynomial  $f(x)$ , as required. Note that the two leading coefficients  $a_{2j-5}$  and  $a_{2j-4}$  of  $f(x)$  are both multiples of 3, and  $a_{2j-3}$  is not a multiple of 3, reflecting the fact that  $\gamma$  maps  $\mathfrak{p}^{j-3} \wedge \mathfrak{p}^{j-3}$  onto  $\mathfrak{p}^{2j-3}$ .

We now consider  $H$ . Define  $\tau^{(k)} = \tau^{\alpha^k}$  for  $k \geq 0$ . Observe that  $T/T'$  is generated by  $\{\tau^{(k)} : 0 \leq k \leq 5\}$ , since the relations  $\alpha^9 = (\tau\alpha^3)^3 = 1$  imply that  $T/T'$  is a homomorphic image of the additive group of integers in the ninth cyclotomic number field. The next step is to prove that  $T$  is abelian. The relation  $r = 1$  reduces, in  $H$ , to the relation  $[\tau_1, \tau_0] = 1$ , or, equivalently, to  $[\tau^{(1)}, \tau] = 1$ . Let  $I$  be the set of pairs  $\{(k, \ell) : 0 \leq k < \ell \leq 5\}$  such that  $[\tau^{(\ell)}, \tau^{(k)}] = 1$ . So  $(k, \ell) \in I$  if  $\ell = k + 1$ ; in particular,  $(2, 3) \in I$ ; so the relation  $[\tau, \tau^{(5)}][\tau^{(2)}, \tau^{(3)}] = 1$  implies that  $(5, 0) \in I$ . So  $\tau^{(6)}$  and  $\tau^{(0)}$  commute with  $\tau^{(5)}$ . But the relation  $(\alpha^3 \tau)^3 = 1$  implies that  $\tau \tau^{(3)} \tau^{(6)} = 1$ , so  $(5, 3) \in I$ , and  $(k, \ell) \in I$  if  $\ell = k + 2$ . The relation  $[\tau, \alpha^3, \tau] = 1$  implies that  $(k, \ell) \in I$  if  $\ell = k + 3$ . Since  $(k, \ell) \in I$  if  $\ell = k + 1$ , or if  $\ell = k + 2$ , the relations  $[\tau, \tau^{(4)}][\tau^{(1)}, \tau^{(3)}] = [\tau, \tau^{(5)}][\tau^{(2)}, \tau^{(3)}] = 1$  imply that  $(k, \ell) \in I$  if  $\ell = k + 4$  or  $\ell = k + 5$ . Thus  $T$  is abelian, as required.

It follows that  $H$  is a mainline quotient of  $R$ , and that  $G$  lies in the branch with root  $H$ . To verify that  $G$  is isomorphic to  $R_{j-3,\gamma,n}$  it suffices to check that  $r$  corresponds to the image of  $\gamma(\theta - 1 \wedge 1)$ , a routine calculation. •

## 5.7 The automorphism groups of the skeleton groups in $\mathcal{B}_j$

The classification, up to isomorphism, of the skeleton groups in  $\mathcal{B}_j$  translates to the calculation of certain orbits. The stabilisers of the orbit representatives determine the automorphism groups of the skeleton groups.

$n$	$\delta(n)$	$\zeta(n)$
1	0	(6)
2, 3	1	(3, 6)
4, 5	2	(1, 3, 6)
6, ..., 11	3	(1, 1, 1, 3, 3, 3, 3, 6)
12, ..., $j-1$	4	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3, 1, 1, 3, 6)

Table 2: Definitions of functions  $\delta$  and  $\zeta$

**Theorem 5.12** *Let  $7 \leq j \leq m \leq 2j-3$ , and let  $\gamma : \mathfrak{p}^{j-3} \wedge \mathfrak{p}^{j-3} \rightarrow \mathfrak{p}^{2j-3}$  be a surjection. Suppose that  $R_{j-3, \gamma, m}$  is isomorphic to the skeleton group of depth  $n := m - j$  in  $\mathcal{B}_j$ , determined by the column labelled by  $k$  in Table 1.*

- (a)  $\text{Aut}(R_{j-3, \gamma, m})$  has order  $3^{m+j+\delta(n)}\zeta(n, k)$ , where  $\delta$  and  $\zeta$  are defined in Table 2, and  $\zeta(n, k)$  is the  $k$ th entry in  $\zeta(n)$ .
- (b)  $\text{Aut}(R_{j-3, \gamma, m})$  is isomorphic to a subgroup of  $(C_6 \times \mathcal{U}/\mathcal{U}_m) \times T_{j-4, \gamma', m}$ , where  $C_6$  is the cyclic group of order 6 and  $\gamma'$  is the map of  $\mathfrak{p}^{j-4} \wedge \mathfrak{p}^{j-4}$  onto  $\mathfrak{p}^{2j-5}$  induced by  $\gamma$ .

PROOF: Since  $T_{j-3, \gamma, m}$  is characteristic in  $R_{j-3, \gamma, m}$ , there are homomorphisms  $\mu$  and  $\nu$  where

$$\begin{aligned} \text{Aut}(R_{j-3, \gamma, m}) &\xrightarrow{\mu} \text{Aut}(C) \times \text{Aut}(T_{j-3, \gamma, m}) : \xi \mapsto (\xi|_{R_{j-3, \gamma, m}/T_{j-3, \gamma, m}}, \xi|_{T_{j-3, \gamma, m}}), \\ \text{Aut}(R_{j-3, \gamma, m}) &\xrightarrow{\nu} \text{Aut}(C) : \xi \mapsto \xi|_{R_{j-3, \gamma, m}/T_{j-3, \gamma, m}}. \end{aligned}$$

Let  $K_0$  denote the kernel of  $\mu$ , and  $K_2$  the kernel of  $\nu$ . Recall that  $C = \langle a \rangle$ . Observe that  $C_{K_2}(a)$ , the centraliser of  $a$  in  $K_2$ , is a complement to  $K_0$  in  $K_2$ , and it acts faithfully on  $T_{j-3, \gamma, m}$ . Thus  $C_{K_2}(a)$  can be identified with the stabiliser in  $\mathcal{U}/\mathcal{U}_m$  of  $\gamma_m$  under the action defined by Lemma 5.6. Let  $U$  be the subgroup of  $C_{K_2}(a)$  corresponding to  $\mathcal{U}_n/\mathcal{U}_m$ . Define  $K_1 := U \times K_0$  to obtain a chain of normal subgroups

$$K_0 \leq K_1 \leq K_2 \leq \text{Aut}(R_{j-3, \gamma, m}).$$

Using a construction dual to the orbit computations for the isomorphism problem, we deduce that  $K_0$  has order  $3^m$ ; the index of  $K_0$  in  $K_1$  equals the order of  $U \cong \mathcal{U}_n/\mathcal{U}_m$ , and thus, by Lemma 5.5, is  $3^{m-n} = 3^j$ ; the index of  $K_1$  in  $K_2$  is  $3^{\delta(n)}$ ; the index of  $K_2$  in  $\text{Aut}(R_{j-3, \gamma, m})$  is the order  $\zeta(n, k)$  of a subgroup of  $\text{Aut}(C) \cong C_6$ , and is 1, 3 or 6. •

## 6 The skeleton groups in $\mathcal{T}(H_i)$

For  $i \geq 0$  let  $H_i = D \times T_i$ , where  $D = \langle a, b \rangle$  as defined in Section 4. The isomorphism type of  $H_i$  depends only on the value of  $i$  modulo 3. We thus consider  $i \in \{0, 1, 2\}$  and

investigate the skeleton groups in  $\mathcal{T}(H_i)$  as an intermediate step towards understanding those in  $\mathcal{T}(G_{i+1})$ .

In number-theoretic notation, the skeleton groups in  $\mathcal{T}(H_i)$  are determined by surjections  $\gamma \in \text{Hom}_D(\mathfrak{p}^i \wedge \mathfrak{p}^i, \mathfrak{p}^{i+j})$  and their images  $\gamma_m \in \text{Hom}_D(\mathfrak{p}^i / \mathfrak{p}^{i+j} \wedge \mathfrak{p}^i / \mathfrak{p}^{i+j}, \mathfrak{p}^{i+j} / \mathfrak{p}^{i+m})$  for  $j \leq m \leq k$  where  $k$  is the maximum class of a skeleton group in  $\mathcal{B}_j(H_i)$ , the branch with root of class  $j$  in  $\mathcal{T}(H_i)$ ; we determine the value of  $k$  in Lemma 6.2. The resulting skeleton groups  $H_{i,\gamma,m} := D \rtimes T_{i,\gamma,m}$  have order  $3^{m+3}$ , coclass 3 and class  $m$ . If  $\mathcal{S}_{i,j}$  is the skeleton of  $\mathcal{B}_j(H_i)$ , then  $H_{i,\gamma,m}$  is a group of depth  $n := m - j$  in  $\mathcal{S}_{i,j}$ .

We use the following notation. Let  $\omega := \theta^3$  be a primitive cube root of unity. Now  $K_3 = \mathbb{Q}_3(\omega)$  is the third cyclotomic number field with ring of integers  $\mathcal{O}_{(3)}$ . Let  $\mathfrak{p}_{(3)} := (\omega - 1)\mathcal{O}_{(3)}$  be the unique maximal ideal of  $\mathcal{O}_{(3)}$ , and let  $\mathcal{U}_{(3)}$  be its group of units. The Galois group of  $K_3$  over  $\mathbb{Q}_3$  is cyclic of order 2, and is generated by  $\sigma_2 : \omega \mapsto \omega^2$ .

## 6.1 The homomorphism space

We now determine the space of homomorphisms describing the skeleton groups in  $\mathcal{T}(H_i)$ . Recall the definition of  $\vartheta$  from Lemma 5.1. It follows from Section 5.5 that  $\vartheta$  is compatible with the action of the Galois automorphism  $\sigma_4 = \sigma_2^2$ , and thus  $\vartheta \in \text{Hom}_D(\mathcal{O} \wedge \mathcal{O}, \mathcal{O})$ .

**Lemma 6.1** *If  $i$  and  $j$  are non-negative integers, then  $\text{Hom}_D(\mathfrak{p}^i \wedge \mathfrak{p}^i, \mathfrak{p}^{i+j}) = \mathfrak{p}_{(3)}^{\lceil (j-i-3)/3 \rceil} \vartheta$ . Thus if  $j \not\equiv i \pmod{3}$  there are no surjections in  $\text{Hom}_D(\mathfrak{p}^i \wedge \mathfrak{p}^i, \mathfrak{p}^{i+j})$ ; otherwise*

$$\text{Hom}_D(\mathfrak{p}^i \wedge \mathfrak{p}^i, \mathfrak{p}^{i+j}) \setminus \text{Hom}_D(\mathfrak{p}^i \wedge \mathfrak{p}^i, \mathfrak{p}^{i+j+1}) = (\omega - 1)^{(j-i-3)/3} \mathcal{U}_{(3)} \vartheta.$$

PROOF: Observe that  $D$  acts as  $\langle C, \sigma_4 \rangle$  on  $\mathcal{O}$ ; thus  $\text{Hom}_D(\mathfrak{p}^i \wedge \mathfrak{p}^i, \mathfrak{p}^{i+j})$  corresponds to the space of fixed points of  $\sigma_4$  in  $\text{Hom}_C(\mathfrak{p}^i \wedge \mathfrak{p}^i, \mathfrak{p}^{i+j})$ . Lemma 5.2 shows that  $\text{Hom}_C(\mathfrak{p}^i \wedge \mathfrak{p}^i, \mathfrak{p}^{i+j}) = \mathfrak{p}^{j-i-3} \vartheta$ . From Section 5.5 we deduce that  $\sigma_4(c\vartheta) = (\sigma_4(c))\vartheta$  for every  $c \in K$ . The fixed points of  $\sigma_4$  in  $K$  are the elements of  $K_3$ . Finally,  $\mathfrak{p}^{j-i-3} \cap K_3 = \mathfrak{p}_{(3)}^{\lceil (j-i-3)/3 \rceil}$ . The lemma follows.  $\bullet$

Lemma 6.1 implies the following.

**Lemma 6.2** *The skeletons  $\mathcal{S}_{i,j}$  with  $j \not\equiv i \pmod{3}$  have depth 0. The skeletons  $\mathcal{S}_{i,j}$  with  $j \equiv i \pmod{3}$  have depth  $j - \chi_j$ , where  $\chi_j = 0$  if 3 divides  $j$  and  $\chi_j = 1$  otherwise. In particular, two of the six sequences of branches in  $\mathcal{T}(H_i)$  have unbounded depth.*

It remains to investigate  $\mathcal{S}_{i,j}$  for  $j \equiv i \pmod{3}$  in more detail. Since  $H_{j-3} \cong H_i$ , we adjust our notation as in Section 5.2, and use  $H_{j-3}$  instead of  $H_i$ . Thus the homomorphisms used to construct skeleton groups now have the form  $\gamma = c\vartheta$  for some unit  $c \in \mathcal{U}_{(3)}$ .

## 6.2 The automorphism group of $H_i$

**Theorem 6.3**  $\text{Aut}(H_i)$  is isomorphic to  $(\text{Aut}(D) \cdot \mathcal{U}_{(3)}) \rtimes T_{i-1}$ .

PROOF: Since  $T_i$  is the Fitting subgroup of  $H_i$ , it is characteristic in  $H_i$ ; so  $\text{Aut}(H_i)$  maps into  $\text{Aut}(D)$ , a group of order 54. To show that this map is onto, consider first the group  $W$  defined in Section 4; it is isomorphic to  $C_3 \wr C_3$ , and acts on  $\mathfrak{p}^i$  in such a way that  $W \rtimes \mathfrak{p}^i$  contains  $H_i = D \rtimes \mathfrak{p}^i$  as a normal subgroup of index 3. Thus  $W$  may be regarded as a subgroup of  $\text{Aut}(H_i)$  that maps onto a subgroup of order 27 of  $\text{Aut}(D)$ . Now  $\sigma_2$  acts naturally on  $\mathfrak{p}^i$ , and this action extends to an automorphism of  $H_i$  of even order that maps  $a \in D$  to  $a^2$  and centralises  $b$ . Thus  $\text{Aut}(H_i)$  maps onto  $\text{Aut}(D)$ . If  $\sigma_2$  and  $W$  are regarded as lying in  $\text{Aut}(H_i)$ , they generate a group of order 162 that maps onto  $\text{Aut}(D)$ .

The kernel of the homomorphism of  $\text{Aut}(H_i)$  onto  $\text{Aut}(D)$  acts as multiplication by elements of  $\mathcal{U}_{(3)}$  on  $T_i$ . Thus  $\text{Aut}(H_i)$  acts on  $T_i$  as an extension of  $\mathcal{U}_{(3)}$  by  $\text{Aut}(D)$ .

The kernel of the action of  $\text{Aut}(H_i)$  on  $T_i$ , and on  $H_i/T_i$ , is isomorphic to the additive group of derivations of  $D$  into  $T_i$ . Since 3 is invertible in the field  $K$ , we deduce that  $H^1(D, K) = 0$ . Thus the desired derivations can be realised as conjugation by elements of  $\mathfrak{p}^{i-1}$ . •

## 6.3 Some number theory

Recall the structure of  $\mathcal{U}_{(3)}$  from [12, Chapter 15].

**Lemma 6.4** For  $k > 0$ , let  $\mathcal{U}_{(3,k)} = 1 + \mathfrak{p}_{(3)}^k$  and  $\lambda_k = 1 + (\omega - 1)^k$ .

- (a)  $\mathcal{U}_{(3)} > \mathcal{U}_{(3,1)} > \mathcal{U}_{(3,2)} > \dots$  is a filtration of  $\mathcal{U}_{(3)}$  whose quotients are cyclic, and respectively generated by  $-1, \omega$  and  $\lambda_k$  for  $k \geq 2$ .
- (b) The torsion subgroup of  $\mathcal{U}_{(3)}$  has order 6, and is generated by  $\omega$  and  $-1$ .
- (c) The exponential map defines an isomorphism  $\mathfrak{p}_{(3)}^2 \rightarrow \mathcal{U}_{(3,2)}$ .

**Lemma 6.5**  $\rho(\mathcal{U}_{(3)})$  is a subgroup of index  $3^2$  in  $\mathcal{U}_{(3)}$  that covers  $\mathcal{U}_{(3,k)}/\mathcal{U}_{(3,k+1)}$  if and only if  $k \notin \{1, 3\}$ .

PROOF: First we consider  $\rho(\mathcal{U}_{(3,4)})$ . Let  $\tau : \mathfrak{p}_{(3)}^4 \rightarrow \mathfrak{p}_{(3)}^4$  be defined by  $x \mapsto \sigma_2(x) + \sigma_{-1}(x) - x$ . Then  $\tau$  is an endomorphism of  $\mathfrak{p}_{(3)}^4$  which has determinant  $-3$ . Thus  $\rho(\mathcal{U}_{(3,4)})$  has index 3 in  $\mathcal{U}_{(3,4)}$ . (Note that  $\mathcal{O}$  is now a module over  $\mathbb{Z}_3(\omega)$  whereas  $\mathcal{O}$  in Lemma 5.6 is a module over  $\mathbb{Z}_3$ .)

Next, we determine  $\rho(\mathcal{U}_{(3)})$  modulo  $\mathcal{U}_{(3,6)} = 1 + 27\mathcal{O}_{(3)}$ . A routine calculation shows:

$$\begin{aligned} \rho(-1) &= -1; \\ \rho(\omega) &= 1; \end{aligned}$$

$$\begin{aligned}
\rho(\lambda_2) &\equiv \lambda_2 \lambda_3^2 \lambda_5 \pmod{\mathcal{U}_{(3,6)}}; \\
\rho(\lambda_3) &\equiv \lambda_5 \pmod{\mathcal{U}_{(3,6)}}; \\
\rho(\lambda_4) &\equiv \lambda_4 \lambda_5 \pmod{\mathcal{U}_{(3,6)}}; \\
\rho(\lambda_5) &\equiv 1 \pmod{\mathcal{U}_{(3,6)}}.
\end{aligned}$$

Thus  $\rho(\mathcal{U}_{(3,4)})$  does not cover  $\mathcal{U}_{(3,5)}/\mathcal{U}_{(3,6)}$ . Since  $\rho(\mathcal{U}_{(3,4)})$  has index 3 in  $\mathcal{U}_{(3,4)}$ , it contains  $\mathcal{U}_{(3,6)}$ . The result follows.  $\bullet$

Hence  $\mathcal{U}_{(3)}/\rho(\mathcal{U}_{(3)})$  has order 9 and is generated by the cosets with representatives  $\omega$  and  $\lambda_3$ . Defining  $V := (\mathbb{Z}/3\mathbb{Z})^2$ , we obtain an isomorphism of abelian groups  $\varphi : \mathcal{U}_{(3)}/\rho(\mathcal{U}_{(3)}) \rightarrow V$  defined by

$$\omega^{u_1} \lambda_3^{u_2} \rho(\mathcal{U}_{(3)}) \mapsto (u_1, u_2).$$

### Lemma 6.6

(a) *The Galois automorphism  $\sigma_2$  acts on  $\mathcal{U}_{(3)}/\rho(\mathcal{U}_{(3)})$  as*

$$V \rightarrow V : (u_1, u_2) \mapsto (2u_1, 2u_2).$$

(b) *Multiplication by  $\omega$  translates to the action on  $\mathcal{U}_{(3)}/\rho(\mathcal{U}_{(3)})$  via*

$$V \rightarrow V : (u_1, u_2) \mapsto (u_1 + 1, u_2).$$

PROOF: Clearly  $\sigma_2(\omega) = \omega^2$  and  $\sigma_2(\lambda_3) = 1 - (\omega - 1)^3 \equiv \lambda_3^2 \pmod{\rho(\mathcal{U}_{(3)})}$ . Thus  $\sigma_2$  acts as multiplication by  $2I_2$  on  $V$ . That (b) holds is obvious.  $\bullet$

## 6.4 A solution of the isomorphism problem

We show that orbit isomorphisms solve the isomorphism problem completely for the skeleton groups in  $\mathcal{S}_{i,j}$ .

**Lemma 6.7** *Let  $j \geq 7$ , and let  $\xi$  be a  $\mathbb{Z}_3 D$ -automorphism of  $\mathfrak{p}^i/\mathfrak{p}^{i+j}$ . There is a  $\mathbb{Z}_3 D$ -automorphism  $\zeta$  of  $\mathfrak{p}^i$  such that, if  $\hat{\zeta}$  is the automorphism of  $\mathfrak{p}^i/\mathfrak{p}^{i+j}$  induced by  $\zeta$ , then  $\xi - \hat{\zeta}$  maps  $\mathfrak{p}^i/\mathfrak{p}^{i+j}$  into  $\mathfrak{p}^{i+j-4}/\mathfrak{p}^{i+j}$ .*

PROOF: Since  $\mathfrak{p}^i$  is a projective  $\mathcal{O}$ -module,  $\xi$  can be lifted to an  $\mathcal{O}$ -module automorphism  $\kappa$  of  $\mathfrak{p}^i$ . Since  $\xi$  is a  $D$ -module automorphism,  $\sigma_4(\kappa) - \kappa$  maps  $\mathfrak{p}^i$  into  $\mathfrak{p}^{i+j}$ . Now  $\kappa$  is multiplication by some  $\eta \in \mathcal{O}$ , and  $\eta = \eta_0 + \eta_1\theta + \eta_2\theta^2$  where  $\eta_e \in \mathbb{Q}_3(\omega)$  for  $0 \leq e \leq 2$ . Then  $\sigma_4(\eta) - \eta = \eta_1(\theta^4 - \theta) + \eta_2(\theta^8 - \theta^2) = (\omega - 1)(\eta_1\theta + \eta_3\theta^2)$ , where  $\eta_3 = \eta_2(\omega + 1) \in \mathbb{Q}_3(\omega)$ . Since  $\eta_1$  and  $\eta_3$  are elements of  $\mathbb{Q}_3(\omega)$  and  $\sigma_4(\eta) - \eta \in \mathfrak{p}^j$ , it follows that  $\eta_1$  and  $\eta_3$  are elements of  $\mathfrak{p}^{j-4}$ , and so  $\eta_2 \in \mathfrak{p}^{j-4}$ . Taking  $\zeta$  to be multiplication by  $\eta_0$ , the result follows.  $\bullet$

**Theorem 6.8** *Let  $7 \leq j \leq m \leq 2j - \chi_j$ . Let  $\gamma$  and  $\gamma'$  be two surjections in  $\text{Hom}_D(\mathfrak{p}^{j-3} \wedge \mathfrak{p}^{j-3}, \mathfrak{p}^{2j-3})$ . Then  $H_{j-3,\gamma,m}$  and  $H_{j-3,\gamma',m}$  are isomorphic if and only if there exists  $\alpha \in \text{Aut}(H_{j-3})$  with  $\alpha(\gamma_m) = \gamma'_m$ .*

PROOF: If  $\gamma$  and  $\gamma'$  are in the same  $\text{Aut}(H_{j-3})$ -orbit, then  $H_{j-3,\gamma,m}$  and  $H_{j-3,\gamma',m}$  are isomorphic by Lemma 3.3. Conversely, let  $\alpha : H_{j-3,\gamma,m} \rightarrow H_{j-3,\gamma',m}$  be an isomorphism. Theorem 6.3 shows that  $\text{Aut}(H_{j-3})$  maps onto  $\text{Aut}(D)$ , so we may assume that  $\alpha$  acts as the identity on  $D$ . Then  $\alpha$  restricts to an isomorphism from  $T_{j-3,\gamma,m}$  to  $T_{j-3,\gamma',m}$ , which is also an automorphism of the  $D$ -module  $T_{j-3}/T_{j-3+m}$ . Hence this automorphism commutes with the action of  $\langle a \rangle$ , and, since  $T_{j-3}$  is projective as  $\mathcal{O}$ -module, lifts to an automorphism  $\alpha'$  of  $T_{j-3}$  which also commutes with the action of  $a$ . Now  $\alpha'$  is multiplication by a unit  $u \in \mathcal{U}$ . Lemma 6.7 shows that  $u = u_1 u_2$ , where  $u_1 \in \mathcal{U}_{(3)}$  and  $u_2 \in \mathcal{U}_{m-4}$ . Multiplication by  $u_1$  induces an automorphism of  $H_{j-3,\gamma,m}$  taking  $\gamma_m$  to  $\rho(u_1)\gamma_m$ . It remains to show that  $\rho(u_1)\gamma_m = \gamma'_m$ . But  $\rho(u)\gamma_m = \gamma'_m$ , so it suffices to prove that  $\rho(u_2)$  centralises  $\gamma_m$ . But  $1 - \rho(u_2) \in \mathfrak{p}^{m-4}$ , and  $\mathfrak{p}^{m-4}\gamma_m = 0$  since  $\gamma_m \in \text{Hom}_D(\mathfrak{p}^{j-3}/\mathfrak{p}^{2j-3} \wedge \mathfrak{p}^{j-3}/\mathfrak{p}^{2j-3}, \mathfrak{p}^{2j-3}/\mathfrak{p}^{j-3+m})$ , and  $2j - 3 + m - 4 \geq j - 3 + m$ . •

We now solve the isomorphism problem for the skeleton groups by describing the orbit isomorphisms. The description of  $\text{Aut}(H_{j-3})$  in Theorem 6.3 shows that  $T_{j-4}$  acts trivially on the homomorphisms induced by the surjections. Further,  $D$  acts as the inner automorphisms of  $D$ , and so also acts trivially. Hence  $\text{Aut}(H_{j-3})$  acts as  $\text{Out}(D) \cdot \mathcal{U}_{(3)}$ .

Throughout, we assume that  $i \equiv j \pmod{3}$  and  $n = m - j$ , so that  $H_{j-3,\gamma,m}$  is a group of depth  $n$  in  $\mathcal{S}_{i,j}$ . The following is proved in a manner similar to Lemma 5.9.

**Lemma 6.9** *Let  $c, c' \in \mathcal{U}_{(3)}$ . The surjections  $c\vartheta$  and  $c'\vartheta$  induce the same element of  $\text{Hom}_D(\mathfrak{p}^{j-3}/\mathfrak{p}^{2j-3} \wedge \mathfrak{p}^{j-3}/\mathfrak{p}^{2j-3}, \mathfrak{p}^{2j-3}/\mathfrak{p}^{2j-3+n})$  if and only if  $c \equiv c' \pmod{\mathcal{U}_{(3,[n/3])}}$ .*

We must determine the orbits of  $\text{Out}(D) \cdot \mathcal{U}_{(3)}$  on

$$\Omega_n := \mathcal{U}_{(3)}/\mathcal{U}_{(3,[n/3])}.$$

As in Section 5.5, the normal subgroup  $\mathcal{U}_{(3)}$  acts on  $\Omega_n$  as multiplication by  $\rho(\mathcal{U}_{(3)})$ . The orbits of this action correspond to the cosets

$$\Delta_n := \mathcal{U}_{(3)}/\rho(\mathcal{U}_{(3)})\mathcal{U}_{(3,[n/3])}.$$

Lemma 6.5 shows that  $\Delta_n$  has at most  $3^2$  elements. As  $\mathcal{U}_{(3)}$  is normal in  $\text{Out}(D) \cdot \mathcal{U}_{(3)}$ , its orbits are blocks for the desired orbits. It remains to determine the action of  $\text{Out}(D)$  on  $\Delta_n$ . Recall that  $\mathcal{U}_{(3)}/\rho(\mathcal{U}_{(3)})$  is isomorphic to  $V = (\mathbb{Z}/3\mathbb{Z})^2$  via  $\varphi$ .

Recall that  $D$  is a normal subgroup of index 3 in the group  $W$  of  $\mathbb{Z}_p$ -linear maps of  $\mathcal{O}$  defined in Section 4, and hence is normalised by the element  $\beta$  of  $W$  that permutes the powers of  $\theta$  by the permutation  $(\theta^2, \theta^5, \theta^8)$ . Observe that  $\text{Out}(D)$  has order 6, and is generated by the images of  $\beta$  and  $\sigma_2$ .

**Lemma 6.10**  $\beta$  acts on  $c\vartheta$  via multiplication by  $\omega$  for  $c \in \mathcal{U}_{(3)}$ .

PROOF: Let  $X, Y$  and  $Z$  be the  $\mathbb{Z}_3(\omega)$ -submodules of  $\mathcal{O}$  generated by  $\{1, \theta^3\}$ ,  $\{\theta, \theta^4\}$ , and  $\{\theta^2, \theta^5\}$  respectively. Now  $X = \mathcal{O}_{(3)}$  and  $\mathcal{O} = X \oplus Y \oplus Z$ . Also  $\beta$  centralises  $X$  and  $Y$ , and acts as multiplication by  $\omega$  on  $Z$ .

Both  $\sigma_2$  and  $\sigma_{-1}$  normalise  $X$  and interchange  $Y$  with  $Z$ . Thus

$$\begin{aligned} \sigma_2(X)\sigma_{-1}(Y) &\subseteq XZ \subseteq Z & \text{and} & \quad \sigma_2(Y)\sigma_{-1}(X) \subseteq ZX \subseteq Z; \\ \sigma_2(X)\sigma_{-1}(Z) &\subseteq XY \subseteq Y & \text{and} & \quad \sigma_2(Z)\sigma_{-1}(X) \subseteq YX \subseteq Y; \\ \sigma_2(Y)\sigma_{-1}(Z) &\subseteq ZY \subseteq X & \text{and} & \quad \sigma_2(Z)\sigma_{-1}(Y) \subseteq YZ \subseteq X. \end{aligned}$$

Observe that  $\mathcal{O} \wedge \mathcal{O} = (X \wedge Y) \oplus (X \wedge Z) \oplus (Y \wedge Z)$ . Let  $x, y$  and  $z$  be elements of  $X, Y$  and  $Z$  respectively. We can determine the action of  $\beta$  on  $\vartheta$  as follows:

$$\begin{aligned} \beta(\vartheta)(x \wedge y) &= \beta[\sigma_2(\beta^{-1}(x))\sigma_{-1}(\beta^{-1}(y)) - \sigma_2(\beta^{-1}(y))\sigma_{-1}(\beta^{-1}(x))] \\ &= \beta[\sigma_2(x)\sigma_{-1}(y) - \sigma_2(y)\sigma_{-1}(x)] \\ &= \omega\vartheta(x \wedge y) \\ \beta(\vartheta)(x \wedge z) &= \beta[\sigma_2(\beta^{-1}(x))\sigma_{-1}(\beta^{-1}(z)) - \sigma_2(\beta^{-1}(z))\sigma_{-1}(\beta^{-1}(x))] \\ &= \beta[\sigma_2(x)\sigma_{-1}(\omega^{-1}z) - \sigma_2(\omega^{-1}z)\sigma_{-1}(x)] \\ &= \omega\beta[\sigma_2(x)\sigma_{-1}(z) - \sigma_2(z)\sigma_{-1}(x)] \\ &= \omega\vartheta(x \wedge z) \\ \beta(\vartheta)(y \wedge z) &= \beta[\sigma_2(\beta^{-1}(y))\sigma_{-1}(\beta^{-1}(z)) - \sigma_2(\beta^{-1}(z))\sigma_{-1}(\beta^{-1}(y))] \\ &= \beta[\sigma_2(y)\sigma_{-1}(\omega^{-1}z) - \sigma_2(\omega^{-1}z)\sigma_{-1}(y)] \\ &= \omega\beta[\sigma_2(y)\sigma_{-1}(z) - \sigma_2(z)\sigma_{-1}(y)] \\ &= \omega\vartheta(x \wedge z). \end{aligned}$$

Thus  $\beta$  acts as multiplication by  $\omega$  on  $\vartheta$  and so on  $c\vartheta$  for  $c \in \mathcal{U}_{(3)}$ . •

Hence the action of  $\text{Out}(D) = \langle \sigma_2, \beta \rangle$  on the space of homomorphisms translates to the action of  $\sigma_2$  and multiplication by  $\omega$  on  $\Delta_n$ . We summarise the action determined in Lemma 6.6.

**Corollary 6.11**  $\text{Out}(D) = \langle \sigma_2, \beta \rangle$  acts on  $V$  as the affine matrix group  $M = \langle a, b \rangle$  with

$$a = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Namely, the image of  $(u_1, u_2) \in V$  under  $m \in M$  is determined by the vector-matrix multiplication  $(u_1, u_2, 1)m$ .

This allows us to read off the orbits of  $\text{Out}(D)$  on  $V$  and thus on  $\Delta_n$ . The orbits on  $V$  are  $\{(0, 0), (1, 0), (2, 0)\}$  and  $\{(0, 1), (1, 1), (2, 1), (0, 2), (1, 2), (2, 2)\}$ . These amalgamate to one orbit in  $V/V_1$  for  $V_1 = \langle (0, 1) \rangle$ . We now summarise the classification of skeleton groups in  $\mathcal{T}(H_i)$ .



other than the relation  $r = 1$ , all hold, as in the proof of Lemma 5.11, and the relations involving  $\beta$  clearly hold. We now verify the relation  $r = 1$ .

If  $w = 0$ , then  $\gamma = \vartheta$ . Since  $t = (\theta^3 - 1)^x \theta^{3x}$ , we can evaluate  $[t, t^a]$  by computing

$$\vartheta((\theta^3 - 1)^x \wedge (\theta^3 - 1)^x \theta) = (\theta^6 - 1)^{2x} (\theta^{-1} - \theta^2) = (\theta^3 - 1)^{2x} \theta^{3x} (\theta^{-1} - \theta^2)$$

using the identity  $(\theta^3 + 1)^2 = (\theta^6)^2 = \theta^3$ . Thus  $[[t, a], t] t^{(a^3-1)^x (a^{-1}-a^2)} = 1$ .

If  $w = 1$ , then  $\gamma = c\vartheta$  for  $c = 1 + 3(\theta^3 - 1)$ . Writing  $w_x$  for  $t^{(a^3-1)^x (a^{-1}-a^2)}$ , we deduce, as in the first case, that  $[[t, a], t] w_x^{1+3(a^3-1)} = 1$ .

It now follows, exactly as in the proof of Lemma 5.11, that the group generated by  $\{\alpha, \tau\}$ , subject to the above relations that do not involve  $\beta$ , is isomorphic to  $R_{j-3, \gamma, m}$ .

It is easy to see that the relations not involving  $\tau$  give a presentation for  $D$ . It remains to prove that the given presentation defines the action of  $\beta$  on the normal closure of  $\langle \tau \rangle$  in  $\langle \alpha, \tau \rangle$ . But the presentation implies that  $(\tau^{\alpha^i})^\beta = \tau^{\alpha^{4i}}$ , and the proof is complete.  $\bullet$

Lemma 6.13 yields a presentation for the skeleton group  $H_{0, \gamma, m}$ . When  $i = 1$  or  $2$ , we do not give presentations for the corresponding skeleton groups, but instead exhibit them as subgroups of low index in  $H_{0, \gamma, m}$ .

**Remark 6.14** *Let  $H_{0, \gamma, m} = \langle a, b, t \rangle$  be a group of depth  $n := m - j$  in  $\mathcal{B}_j(H_0)$ .*

- (a) *The subgroup  $\langle a, b, [t, a] \rangle$  of index 3 in  $H_{0, \gamma, m}$  is the skeleton group for  $H_1$  defined by  $\gamma$  and  $m$ . It has depth  $n - 2$  in the skeleton with root of class  $j + 1$ .*
- (b) *The subgroup  $\langle a, b, [t, a, a] \rangle$  of index 9 in  $H_{0, \gamma, m}$  is the skeleton group for  $H_2$  defined by  $\gamma$  and  $m$ . It has depth  $n - 4$  in the skeleton with root of class  $j + 2$ .*

## 7 The skeleton groups in $\mathcal{T}(G_{i+1})$

The aim of this section is to determine the skeleton groups in  $\mathcal{T}(G_{i+1})$  with  $i \in \{0, 1, 2\}$ . The roots of  $\mathcal{T}(G_{i+1})$  have orders  $3^9$ ,  $3^8$  and  $3^7$  respectively.

Our main result is the following.

**Theorem 7.1** *Let  $i \in \{0, 1, 2\}$  and  $h \geq 9$ . Let  $\mathcal{S}_{i+1, h}$  denote the skeleton of the branch of  $G_{i+1}$  with root of order  $3^{h+2}$  and class  $h$ . Then  $\mathcal{S}_{i+1, h}$  is non-trivial if and only if  $h \equiv 0 \pmod{3}$ .*

*Let  $\mathcal{F}_{l, n}$  and  $\mathcal{F}_{r, n}$  denote the rooted subtrees of depth  $n$  of the left and right graphs in Figure 3, respectively.*

- (a)  $\mathcal{S}_{1, h} \cong \mathcal{F}_{r, h}$  if  $h \equiv 0 \pmod{6}$  and  $\mathcal{S}_{1, h} \cong \mathcal{F}_{l, h}$  if  $h \equiv 3 \pmod{6}$ .
- (b)  $\mathcal{S}_{2, h} \cong \mathcal{F}_{r, h-5}$  if  $h \equiv 0 \pmod{6}$  and  $\mathcal{S}_{2, h} \cong \mathcal{F}_{l, h-5}$  if  $h \equiv 3 \pmod{6}$ .
- (c)  $\mathcal{S}_{3, h} \cong \mathcal{F}_{l, h-3}$  if  $h \equiv 0 \pmod{6}$  and  $\mathcal{S}_{3, h} \cong \mathcal{F}_{r, h-3}$  if  $h \equiv 3 \pmod{6}$ .

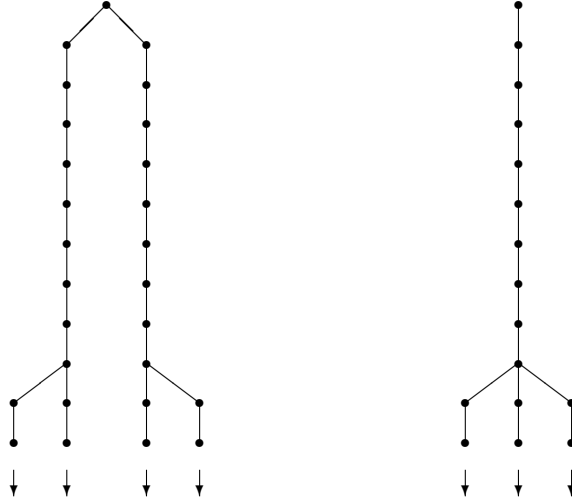


Figure 3: Two skeletons of branches in  $\mathcal{T}(G_{i+1})$

The proof of this theorem differs significantly from the other cases. We first describe  $\text{Aut}(G_{i+1})$ . We describe the skeleton groups in  $\mathcal{T}(G_{i+1})$  in Section 7.2. Determining their isomorphism types is challenging. We classify the skeleton groups up to orbit isomorphism in Section 7.3. For the first time, there are exceptional isomorphisms among skeleton groups. We determine these in Section 7.4. Finally, in Section 7.5, we identify isomorphism types of skeleton groups in  $\mathcal{T}(G_{i+1})$  with subgroups of representative skeleton groups in  $\mathcal{T}(H_i)$ .

## 7.1 The automorphism group of $G_{i+1}$

Recall, from Section 4, that  $D = \langle a, b \rangle$  and  $t_{i+1} = [t_i, a]$  for  $i \in \{0, 1, 2\}$ . The infinite pro-3-group  $G_{i+1} = \langle a, bt_i \rangle = D \cdot T_{i+1}$  embeds as a subgroup of index 3 in its minimal split supergroup  $H_i = \langle a, b, t_i \rangle = D \rtimes T_i$ .

**Theorem 7.2**  *$\text{Aut}(G_{i+1})$  is the normaliser of  $G_{i+1}$  in  $\text{Aut}(H_i)$ . It has index 6 in  $\text{Aut}(H_i)$ , and is isomorphic to  $(\text{Aut}(D) \cdot \mathcal{U}_{(3,1)}) \rtimes T_i$ .*

PROOF: Recall, from Theorem 6.3, that  $\text{Aut}(H_i) \cong (\text{Aut}(D) \cdot \mathcal{U}_{(3)}) \rtimes T_{i-1}$ . Every  $D$ -automorphism of  $T_{i+1}$  extends uniquely to a  $D$ -automorphism of  $T_i$ , so  $\text{Aut}(G_{i+1})$  is the normaliser of  $G_{i+1}$  in  $\text{Aut}(H_i)$ . Nine of the 13 maximal subgroups of  $H_i$  have coclass 2. Of these, three are isomorphic to  $R$ . The remaining six, all isomorphic to  $G_{i+1}$ , are  $\langle at_i^\epsilon, bt_i^\delta \rangle$  for  $\epsilon \in \{0, 1, 2\}$  and  $\delta = \pm 1$ . These are conjugate under the action of  $\text{Aut}(H_i)$ : the three values of  $\epsilon$  are permuted transitively by conjugating by  $t_{i-1}$ , and the two values of  $\delta$  are exchanged by multiplication by  $-1 \in \mathcal{U}_{(3)} \setminus \mathcal{U}_{(3,1)}$ . The result follows. •

Note that  $\text{Aut}(H_i)$  contains an element  $\sigma_2$  corresponding to the action of  $\sigma_2$  on  $\mathcal{O}$ , and an element  $\mu$  corresponding to multiplication by  $-1$  on  $\mathcal{O}$ . Theorem 7.2 shows that  $\mu \notin \text{Aut}(G_{i+1})$ . If  $i+1$  is odd, then  $\sigma_2 \in \text{Aut}(G_{i+1})$ ; otherwise  $\mu\sigma_2 \in \text{Aut}(G_{i+1})$ .

## 7.2 Descriptions of the skeleton groups in $\mathcal{T}(G_{i+1})$

Let  $j \geq 7$ , and let  $\gamma \in \text{Hom}_D(\mathfrak{p}^i \wedge \mathfrak{p}^i, \mathfrak{p}^{i+j})$  be surjective, so  $j \equiv i \pmod{3}$ . Then  $\gamma$  defines a skeleton group  $H_{i,\gamma,m} := D \rtimes T_{i,\gamma,m}$  for each  $m$  with  $j \leq m \leq 2j - \chi_j$  as shown in Lemma 6.2. The skeleton group  $H_{i,\gamma,m}$  has order  $3^{m+3}$ , class  $m$  and it is contained in the  $j$ th branch of  $\mathcal{T}(H_i)$ .

As before, we identify  $t_i \in T$  with its corresponding element in  $T_{i,\gamma,m}$  and thus obtain that  $H_{i,\gamma,m} = \langle a, b, t_i \rangle$ . With this notation,  $G_{i+1,\gamma,m} = \langle a, bt_i \rangle$  is the skeleton group for  $G_{i+1}$  defined by  $\gamma$  and  $m$ . It has order  $3^{m+2}$  and class  $m$ . However, it is non-trivial to read off the branch of  $\mathcal{T}(G_{i+1})$  containing  $G_{i+1,\gamma,m}$ . The following lemma provides bounds for the branch.

**Lemma 7.3** *Let  $j \geq 7$ , let  $\gamma \in \text{Hom}_D(\mathfrak{p}^i \wedge \mathfrak{p}^i, \mathfrak{p}^{i+j})$  be a surjection and let  $j \leq m \leq 2j - \chi_j$ . If  $G_{i+1,\gamma,m}$  is contained in the skeleton  $\mathcal{S}_{i+1,h}$ , then  $j \leq h \leq j+2$ .*

PROOF: We must determine the maximal mainline quotient of  $G_{i+1,\gamma,m}$ . By construction, the class  $j$  quotient is on the mainline. Since  $\gamma(\mathfrak{p}^{i+1} \wedge \mathfrak{p}^{i+1}) = \mathfrak{p}^{i+j+2}$ , the class  $j+3$  quotient is not. Thus  $j \leq h \leq j+2$ .  $\bullet$

It can happen that  $H_{i,\gamma,m}$  is isomorphic to  $H_{i,\gamma',m}$ , yet  $G_{i+1,\gamma,m}$  is not isomorphic to  $G_{i+1,\gamma',m}$ . Thus, to construct the skeleton groups in  $\mathcal{T}(G_{i+1})$ , we have to consider *all* homomorphisms  $\gamma$ , and cannot restrict to isomorphism type representatives for the skeleton groups in  $\mathcal{T}(H_i)$ .

To solve the isomorphism problem for the skeleton groups in  $\mathcal{T}(G_{i+1})$ , we proceed in two steps. First, we consider surjections  $\gamma \in \text{Hom}_D(\mathfrak{p}^i \wedge \mathfrak{p}^i, \mathfrak{p}^{i+j})$  and reduce the homomorphisms induced by them under the action of  $\text{Aut}(G_{i+1})$ . This yields representatives for skeleton groups in  $\mathcal{T}(G_{i+1})$  up to orbit isomorphism. We then determine exceptional isomorphisms among these.

## 7.3 Orbit isomorphisms

The orbit isomorphisms for  $G_{i+1}$  are determined by the orbits of  $\text{Aut}(G_{i+1})$  on the homomorphisms induced by surjections in  $\text{Hom}_D(\mathfrak{p}^i \wedge \mathfrak{p}^i, \mathfrak{p}^{i+j})$ . Lemma 6.1 shows that such surjections exist if and only if  $i \equiv j \pmod{3}$ . Again, we consider the case  $i = j - 3$  to simplify the number-theoretic translation.

Let  $n = m - j$ . Observe that  $\text{Aut}(G_{j-2})$  acts as  $\text{Out}(D) \cdot \mathcal{U}_{(3,1)}$  on

$$\Omega_n = \mathcal{U}_{(3)}/\mathcal{U}_{(3, \lceil n/3 \rceil)}.$$

As in Section 6.4, the normal subgroup  $\mathcal{U}_{(3,1)}$  acts on  $\Omega_n$  as multiplication by  $\rho(\mathcal{U}_{(3,1)})$ . The orbits under this action can be determined readily, as in Lemma 6.5. We summarise the result.

**Lemma 7.4**  $\rho(\mathcal{U}_{(3,1)})$  is a subgroup of index  $2 \cdot 3^2$  in  $\mathcal{U}_{(3)}$  that covers  $\mathcal{U}_{(3,k)}/\mathcal{U}_{(3,k+1)}$  if and only if  $k \notin \{0, 1, 3\}$ .

It remains to determine the orbits of  $\text{Out}(D)$  on

$$\Delta_n := \mathcal{U}_{(3)}/\rho(\mathcal{U}_{(3,1)})\mathcal{U}_{(3, \lceil n/3 \rceil)}.$$

Defining  $V := \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^2$ , we obtain an isomorphism of abelian groups  $\varphi : \mathcal{U}_{(3)}/\rho(\mathcal{U}_{(3,1)}) \rightarrow V$  defined by

$$(-1)^{u_0} \omega^{u_1} \lambda_3^{u_2} \rho(\mathcal{U}_{(3,1)}) \mapsto (u_0, u_1, u_2).$$

If  $V_1 = \langle (0, 1, 0), (0, 0, 1) \rangle$  and  $V_2 = \langle (0, 0, 1) \rangle$ , then

$$\Delta_n \cong \begin{cases} V/V_1 & \text{for } 1 \leq n \leq 3 \\ V/V_2 & \text{for } 3 < n \leq 9 \\ V & \text{for } 9 < n. \end{cases}$$

**Theorem 7.5** Let  $j \geq 7$ .

- (a) If  $j$  is odd, then the action of  $\text{Out}(D)$  on  $V$  yields four orbits, with representatives  $(0, 0, 0), (0, 0, 1), (1, 0, 0), (1, 0, 1)$ . These orbits amalgamate to two orbits in  $V/V_1$  and in  $V/V_2$ . Thus the orbits of  $\text{Out}(D)$  on  $\Delta_n$  for  $1 \leq n \leq \ell$  define a graph isomorphic to the first  $\ell$  levels of the left graph in Figure 3.
- (b) If  $j$  is even, then the action of  $\text{Out}(D)$  on  $V$  yields three orbits, with representatives  $(0, 0, 0), (0, 0, 1), (0, 0, 2)$ . These orbits amalgamate to one orbit in  $V/V_1$  and in  $V/V_2$ . Thus the orbits of  $\text{Out}(D)$  on  $\Delta_n$  for  $1 \leq n \leq \ell$  define a graph isomorphic to the first  $\ell$  levels of the right graph in Figure 3.

**PROOF:** Recall  $\text{Out}(D) = \langle \sigma_2, \beta \rangle$ . Lemma 6.10 shows that  $\beta$  acts via multiplication by  $\omega$  on the space of surjections. This translates to the action  $V \rightarrow V : (u_0, u_1, u_2) \mapsto (u_0, u_1 + 1, u_2)$ .

If  $j$  is odd, then  $\text{Aut}(G_{j-2})$  contains  $\sigma_2$ . This acts via  $\sigma_2(-1) = -1$ ,  $\sigma_2(\omega) = \omega^2$  and  $\sigma_2(\lambda_3) \equiv \lambda_3^2 \pmod{\rho(\mathcal{U}_{(3,1)})}$ , which translates to  $V \rightarrow V : (u_0, u_1, u_2) \mapsto (u_0, 2u_1, 2u_2)$ . Together with the action of  $\beta$  this produces four orbits on  $V$ :

$$\begin{aligned} & \{(0, x, 0) \mid x \in \{0, 1, 2\}\}, & \{(1, x, 0) \mid x \in \{0, 1, 2\}\}, \\ & \{(0, x, y) \mid x \in \{0, 1, 2\}, y \in \{1, 2\}\}, & \{(1, x, y) \mid x \in \{0, 1, 2\}, y \in \{1, 2\}\}. \end{aligned}$$

These amalgamate to two orbits in  $V/V_1$  and in  $V/V_2$  and so yield the result in (a).

If  $j$  is even, then  $\text{Aut}(G_{j-2})$  contains  $\mu\sigma_2$ , where  $\mu$  acts as multiplication by  $-1$ . This translates to the action  $V \rightarrow V : (u_0, u_1, u_2) \mapsto (u_0 + 1, 2u_1, 2u_2)$ . Together with the action of  $\beta$  this produces three orbits on  $V$ :  $\{(u, x, 0) \mid u \in \{0, 1\}, x \in \{0, 1, 2\}\}$  and  $\{(0, x, 1), (1, x, 2) \mid x \in \{0, 1, 2\}\}$  and  $\{(0, x, 2), (1, x, 1) \mid x \in \{0, 1, 2\}\}$ . These amalgamate to one orbit in  $V/V_1$  and in  $V/V_2$  and so yield the result in (b).  $\bullet$

Theorem 7.5 yields a complete list of skeleton groups  $G_{j-2,\gamma,m}$  up to orbit isomorphisms, where  $\gamma = c\vartheta$  and  $c$  is determined by an orbit representative listed there. We define a graph  $\mathcal{S}_{i+1,j}^*$  on these orbit isomorphism types of groups, with edges  $G_{j-2,c\vartheta,m} \rightarrow G_{j-2,c'\vartheta,m-1}$  if  $G_{j-2,c\vartheta,m-1}$  is orbit isomorphic to  $G_{j-2,c'\vartheta,m-1}$ . Then  $\mathcal{S}_{i+1,j}^*$  contains the mainline group of class  $j$ . Theorem 7.5 asserts that  $\mathcal{S}_{i+1,j}^*$  is non-trivial if and only if  $i \equiv j \pmod{3}$ , and in this case has the shape displayed in Figure 3.

## 7.4 Exceptional isomorphisms

In this section we determine the exceptional isomorphisms among skeleton groups for  $G_{i+1}$ . We show that there are no exceptional isomorphisms for  $G_1$ . However such isomorphisms occur, for the first time, for each of  $G_2$  and  $G_3$ .

As a first step, we observe that the determination of all exceptional isomorphisms can be reduced to a finite calculation.

**Theorem 7.6** *Let  $0 \leq i \leq 2$  and let  $\mathcal{S}_{i+1,h,k}$  be the tree consisting of all groups of depth at most  $k$  in  $\mathcal{S}_{i+1,h}$ .*

- (a) *For  $h > 61$ ,  $\mathcal{S}_{i+1,h,10}$  is isomorphic to  $\mathcal{S}_{i+1,h-6,10}$ .*
- (b) *For  $h \geq 10$ , if  $\mathcal{S}_{i+1,h,10}$  is isomorphic to  $\mathcal{F}_{\star,10}$  where  $\star$  is  $l$  or  $r$ , then  $\mathcal{S}_{i+1,h}$  is isomorphic to  $\mathcal{F}_{\star,h}$ .*

PROOF: (a) The construction in [11] of the isomorphism  $\pi_h$  of Theorem 2.1 ensures that it maps a skeleton group  $G_{i+1,\gamma,m}$  in  $\mathcal{B}_{h,10}(G_{i+1})$  to the skeleton group  $G_{i+1,3\gamma,m}$  in  $\mathcal{B}_{h+6,10}(G_{i+1})$  for all  $h \geq f(10)$ , where  $f$  is the defect function of Theorem 2.1. This induces a graph isomorphism  $\mathcal{S}_{i+1,h,10} \cong \mathcal{S}_{i+1,h+6,10}$  for all  $h \geq f(10)$ . Hence the skeleton of  $\mathcal{B}_{h,10}(G_{i+1})$  for all  $h$  can be obtained from those for  $h \leq f(10) + 6$  using  $\pi_h$ .

It remains to determine an explicit bound for  $f(10)$ . An application of [11, Theorem 29] yields 686 as a bound for  $f(10)$ . This can be improved significantly for skeleton groups as follows. By Lemma 7.3, we have that  $j \leq h \leq j + 2$ . Thus a group of depth at most 10 in a skeleton satisfies  $m \leq h + 10 \leq (j + 2) + 10$ .

We now employ the notation of [11]. Let  $P$  be the class 6 quotient of the infinite pro-3-group  $G_{i+1}$ . Recall that  $L_7(G_{i+1,\gamma,m})$  corresponds as a set to  $T_{i+6}/T_{i+m}$ . As  $\gamma$  maps  $T_{i+6} \wedge T_{i+6}$  into  $T_{i+j+12}$  and  $m \leq j + 12$ , it follows that  $L_7(G_{i+1,\gamma,m})$  is isomorphic as  $P$ -module to  $L_7(G_{i+1})/L_{m+1}(G_{i+1})$ . Hence  $G_{i+1,\gamma,m}$  can be considered as an extension of  $L_7(G_{i+1})/L_{m+1}(G_{i+1})$  by  $P$ . This implies that we can choose  $l = 7$ .

If  $K$  is the kernel of the action of  $P$  on  $L_7(G_{i+1})$ , then  $K$  is elementary abelian of order  $3^5$ . Thus  $3^a = \exp(K/K') = 3$  and  $3^b = 3$ , the exponent of the Schur multiplier of  $K$ . So  $a = b = 1$ . Since  $P$  has order  $3^e = 3^8$ , we deduce that  $n = \max\{2(a + b + 1)d, ed\} = \max\{36, 48\} = 48$ . An upper bound for  $f(10)$  is now  $l + n = 55$ .

(b) Consider two groups  $G_{i+1,\gamma,m}$  and  $G_{i+1,\gamma',m}$  of depth at least 10 in  $\mathcal{S}_{i+1,h}$ . If they are isomorphic, then their quotients at depth precisely 10 are isomorphic. The hypothesis

and Theorem 7.5 imply that at depth 10 isomorphism coincides with orbit isomorphism. Hence the two quotients at depth 10 are orbit isomorphic. Since there is no further branching below depth 10 in the graph  $\mathcal{S}_{i+1,j}^*$ , it follows that  $G_{i+1,\gamma,m}$  and  $G_{i+1,\gamma',m}$  are orbit isomorphic. Hence, orbit isomorphism and isomorphism coincide at all depths at least 10, and so  $\mathcal{S}_{i+1,h}$  can be read off.  $\bullet$

We use Lemma 7.3 and Theorem 7.5 to construct a finite list of groups of the form  $G_{i+1,\gamma,m}$  that contains a complete set of skeleton groups for  $\mathcal{S}_{i+1,h,10}$  and then reduce this list up to isomorphism using the algorithm of [20]. Theorem 7.6(a) implies that it suffices to perform the finite calculation for (b) for all values of  $h \leq 61$ .

We determined  $\mathcal{S}_{i+1,h,10}$  by computer for  $h \leq 61$ . If  $3|h$ , then this is the tree suggested by Theorem 7.1 up to depth 10; otherwise it is an isolated vertex, the mainline group. This computation is sufficient to prove Theorem 7.1.

In the remainder of this section we provide an alternative theoretical description of the exceptional isomorphisms. Our proof provides insight into the exceptional isomorphisms. It proceeds in two steps. First, given a skeleton group  $G_{i+1,\gamma,m}$ , we determine the skeleton  $\mathcal{S}_{i+1,h}$  containing it; so we determine the precise class  $h$  of the maximal mainline quotient of  $G_{i+1,\gamma,m}$ . Secondly, we determine the precise shape of  $\mathcal{S}_{i+1,h,10}$  and apply Theorem 7.6(b) to deduce the shape of  $\mathcal{S}_{i+1,h}$ . Before we embark on this, we present some preliminary results.

#### 7.4.1 Some preliminary results

Our overall approach towards the investigation of the exceptional isomorphisms is to consider each skeleton group  $G_{i+1,\gamma,m} = \langle a, bt_i \rangle$  as a cyclic extension of  $K_{i+1,\gamma,m} := \langle a, t_{i+1} \rangle \leq G_{i+1,\gamma,m}$  by a cyclic group of order 3.

**Lemma 7.7** *Aut( $G_{i+1}$ ) acts transitively on the subset  $\{\langle a(bt_i)^k, t_{i+1} \rangle \mid 0 \leq k \leq 2\}$  of the set of maximal subgroups of  $G_{i+1}$ , and each element of this subset is isomorphic to  $R_{i+1} = C \times T_{i+1}$ .*

PROOF: There is an automorphism of  $D$  that maps  $a$  to  $ab$  and centralises  $b$ . As in the proof of Theorem 6.3, this automorphism lifts to an element of  $W$  that may be regarded as an automorphism  $\delta_1$  of  $H_i$ . So  $\delta_1$  maps  $a$  to  $ab$ , centralises  $b$ , and centralises  $t_i$  modulo  $T_{i+1}$ . The same theorem shows that  $\text{Aut}(H_i)$  contains an automorphism  $\delta_2$ , corresponding to conjugation by  $t_{i-1}^2$ , that maps  $a$  to  $at_i^{-2}$ , centralises  $b$  modulo  $T_{i+2}$ , and centralises  $t_i$ . Then  $\delta := \delta_1\delta_2$  is an automorphism of  $H_i$  that sends  $\langle a, t_{i+1} \rangle$  to  $\langle abt_i, t_{i+1} \rangle$ , and normalises  $G_{i+1}$ . Hence  $\delta \in \text{Aut}(G_{i+1})$ . Since  $\delta^2$  maps  $\langle a, t_{i+1} \rangle$  to  $\langle a(bt_i)^2, t_{i+1} \rangle$ , the result follows.  $\bullet$

**Remark 7.8** *The subgroup  $K_{i+1,\gamma,m}$  is a skeleton group for  $R_{i+1}$ . Lemma 5.1 implies that  $\vartheta(\mathfrak{p}^i \wedge \mathfrak{p}^i) = \mathfrak{p}^{2i+3}$ ; so, if  $\gamma : \mathfrak{p}^i \wedge \mathfrak{p}^i \mapsto \mathfrak{p}^{i+j}$  is surjective, then  $\gamma(\mathfrak{p}^{i+1} \wedge \mathfrak{p}^{i+1}) = \mathfrak{p}^{i+j+2}$ . Hence  $K_{i+1,\gamma,j+n}$  is isomorphic to the mainline group of class  $j+n$  in  $\mathcal{T}(R_{i+1})$  for  $n \leq 2$ , and so it is independent of  $\gamma$ ; but if  $n > 2$  then  $K_{i+1,\gamma,j+n}$  is not isomorphic to a mainline group, since in this case  $T_{i+1,\gamma,j+n}$  is non-abelian.*

The following lemma solves the isomorphism problem for cyclic extensions by a 3-cycle.

**Lemma 7.9** *Let  $G$  and  $H$  be groups containing a normal subgroup  $M$  of index 3. Let  $x \in G \setminus M$  and  $y \in H \setminus M$ . Let  $x^3 = \zeta$  and  $y^3 = \zeta'$ , and let  $\alpha$  and  $\beta$  be the automorphisms of  $M$  induced by conjugation by  $x$  and  $y$  respectively. Then there is an isomorphism of  $G$  to  $H$  that normalises  $M$ , and maps  $x$  to an element of the coset  $My$ , if and only if there is an automorphism  $\lambda$  of  $M$ , and an element  $h$  of  $M$ , such that  $\zeta\lambda = \zeta'h^{\beta^2}h^\beta h$  and  $\lambda^{-1}\alpha\lambda = \beta\nu$ , where  $\nu$  is conjugation by  $h$ .*

PROOF: If such  $\lambda$  and  $h$  exist, then an isomorphism  $\phi$  from  $G$  to  $H$  may be defined by taking the restriction of  $\phi$  to  $M$  to be  $\lambda$ , and defining  $x\phi = yh$ ; and conversely. •

As a final preliminary, we require two number-theoretic results.

**Lemma 7.10**

- (a) *For every integer  $k$  the endomorphism  $f$  of  $\mathfrak{p}^k$  defined by  $y \mapsto \sigma_4(y) - y$  covers  $\mathfrak{p}^l/\mathfrak{p}^{l+1}$  if and only if  $l \geq k + 2$  and  $l \not\equiv 2 \pmod{3}$ .*
- (b) *For every integer  $k$  the image of the endomorphism  $1 + \sigma_4 + \sigma_7$  of  $\mathfrak{p}^k$  is  $3\mathfrak{p}^k \cap \mathcal{O}_{(3)}$ . In particular, the image does not cover  $\mathfrak{p}^j/\mathfrak{p}^{j+1}$  unless  $j \equiv 0 \pmod{3}$ .*

PROOF:

- (a) If  $l \equiv 0 \pmod{3}$  and  $\alpha_l = (\theta^3 - 1)^{l/3}$ , then  $f(\alpha_l) = 0$ . If  $l \not\equiv 0 \pmod{3}$  and  $\alpha_l \in \mathfrak{p}^l \setminus \mathfrak{p}^{l+1}$ , then  $\alpha_l \equiv \pm(\theta - 1)^l \pmod{\mathfrak{p}^{l+1}}$ ; also
 
$$(\theta^4 - 1)^l - (\theta - 1)^l = (\theta - 1)^l((\theta^3 + \theta^2 + \theta + 1)^l - 1) \equiv l(\theta - 1)^{l+2} \pmod{\mathfrak{p}^{l+2}},$$
 since  $\theta^3 + \theta^2 + \theta + 1 \equiv 1 + (\theta - 1)^2 \pmod{\mathfrak{p}^3}$ . Hence  $f(\alpha_l) \in \mathfrak{p}^{l+2} \setminus \mathfrak{p}^{l+3}$ .
- (b) Observe that  $\theta^i + \sigma_4(\theta^i) + \sigma_7(\theta^i) = \theta^i(1 + \theta^{3i} + \theta^{6i})$ . But  $1 + \theta^{3i} + \theta^{6i} = 0$  if  $i \not\equiv 0 \pmod{3}$ , and  $1 + \theta^{3i} + \theta^{6i} = 3$  otherwise. •

#### 7.4.2 Determining the branch of a skeleton group

The following theorem determines the precise maximal mainline quotient of a skeleton group  $G_{i+1,\gamma,m}$ .

**Theorem 7.11** *Let  $j \geq 7$ , let  $i \equiv j \pmod{3}$ , and let  $0 \leq n \leq j - \chi_j$ , where  $\chi_j = 0$  if 3 divides  $j$ , and  $\chi_j = 1$  otherwise. If  $\gamma$  is a surjection in  $\text{Hom}_D(T_i \wedge T_i, T_{i+j})$ , then  $G_{i+1,\gamma,j+n}$  is isomorphic to a mainline group if and only one of the following holds:*

- (a)  $n = 0$  for  $i \equiv 0 \pmod{3}$ ;
- (b)  $n \leq 2$  for  $i \equiv 1 \pmod{3}$ ;

(c)  $n \leq 1$  for  $i \equiv 2 \pmod 3$ .

PROOF: If  $n = 0$  then  $G_{i+1,\gamma,j+n}$  is a mainline group by construction. If  $n > 2$ , then  $G_{i+1,\gamma,j+n}$  is not a mainline group by Remark 7.8. We consider the three cases in turn.

(a) We assume, without loss of generality, that  $i = 0$ , and show that  $G_{1,\gamma,j+1}$  is not isomorphic to the mainline group of the same order. Recall that  $G_{1,\gamma,j+1}$  contains a maximal subgroup  $K_{1,\gamma,j+1}$  that is isomorphic to the mainline group  $C \times T_1/T_{j+1}$  for  $R_1$  by Remark 7.8, and is independent of  $\gamma$ . To shorten notation, we denote it by  $M_{1,j}$ . (The first suffix indicates that  $T_1$  is its lattice and the second suffix reflects the order).

Suppose that  $\iota : G_{1,\gamma,j+1} \rightarrow G_{1,0,j+1}$  is an isomorphism. Then  $\iota$  maps  $M_{1,j}$  onto one of the subgroups  $\langle a(bt_0)^k, t_1 \rangle$  for  $k = 0, 1, 2$  of  $G_{1,0,j+1}$ . Since the automorphism group of  $G_{1,0,j+1}$  acts transitively on these subgroups by Lemma 7.7, we may assume  $k = 0$ . Equating  $T_0$  with  $\mathcal{O}$ , the generator  $bt_0$  of  $G_{1,\gamma,j+1}$  lying outside  $K_{1,\gamma,j+1}$  becomes  $b1$ ; since  $\theta - 1 \in \mathfrak{p}$ , which lies in the Frattini subgroup of  $G_{1,\gamma,j+1}$ , we may replace  $b1$  by  $b\theta$ . Since  $(b\theta)^3 = 1$ , the group  $G_{1,\gamma,j+1}$  is a split extension of  $M_{1,j}$  by  $\langle b\theta \rangle$ . Clearly the mainline group  $G_{1,0,j+1}$  is also a split extension of  $M_{1,j}$  by the 3-cycle  $b\theta$ .

Theorem 5.12 shows that the automorphism group of  $M_{1,j}$  is  $(\text{Aut}(C) \times \mathcal{U}/\mathcal{U}_j) \times T/T_j$ . Thus we write an element of this automorphism group as  $(\sigma, u, o)$ , where  $\sigma \in \text{Aut}(C)$ , and  $u \in \mathcal{U}/\mathcal{U}_j$ , and  $o \in \mathcal{O}/\mathfrak{p}^j$ .

The 3-cycle  $b\theta$  in  $G_{1,0,j+1}$  induces the automorphism  $(\sigma_4, 1, \theta)$  of  $M_{1,j}$ , where 1 and  $\theta$  represent the images of these elements in  $\mathcal{U}/\mathcal{U}_j$  and  $\mathcal{O}/\mathfrak{p}^j$  respectively. The corresponding 3-cycle  $b\theta$  in  $G_{1,\gamma,j+1}$  induces the automorphism  $(\sigma_4, 1 + w, \theta + \phi)$ , where  $w$  and  $\phi$  are in  $\mathfrak{p}^{j-1}/\mathfrak{p}^j$ . Note that  $w$  is non-zero, as  $\gamma$  is a surjection onto  $T_j$ , and thus the action of  $b\theta$  on  $T_1/T_{j+1}$  is not the same as the action of  $b$ , since  $\theta$ , as an element of  $T_0/T_{j+1}$ , needs no longer centralise  $T_1/T_{j+1}$ .

We use Lemma 7.9 to prove that these two automorphisms do not define isomorphic extensions. The lemma applies since every automorphism of  $D$  centralises  $b$  modulo  $\langle a \rangle$ . Note that, in the lemma,  $\zeta = \zeta' = 1$ , as our groups are split extensions, and the role of  $h$  is played by  $a^{3x}t \in M_{1,j}$  for some  $x \in \{0, 1, 2\}$ , where  $t^{1+ba^{3x}+b^2a^{6x}} = 0$ . Conjugation by this element is the automorphism  $(1, \theta^{3x}, t)$ .

Taking  $\lambda$  in the lemma to be  $(\sigma_m, u, o)$ , we are led to

$$(\sigma_4, 1, \theta)(\sigma_m, u, o) = (\sigma_m, u, o)(\sigma_4, 1 + w, \theta + \phi)(1, \theta^{3x}, t),$$

as an equation in  $(m, u, o, t, x)$ , where  $t + t^{ba^{3x}} + t^{b^2a^{6x}} = 0$ . This is equivalent to the simultaneous equations:

$$u = \theta^{3x}(1 + w)\sigma_4(u) \tag{*1}$$

$$u\sigma_m(\theta) + o = \theta^{3x}(1 + w)\sigma_4(o) + \theta^{3x}(\theta + \phi) + t. \tag{†1}$$

We now show that (\*1) has no solution in  $\mathcal{U}/\mathcal{U}_j$ . Multiplying  $u$  by  $\theta^{-x}$  we may take  $x = 0$ . So now (\*1) has become

$$u = (1 + w)\sigma_4(u). \tag{*2}$$

Suppose first that  $u \in \mathcal{U}_4$  (modulo  $\mathcal{U}_j$ ). Since  $\mathcal{U}_4$  is naturally isomorphic to  $\mathfrak{p}^4$  by Lemma 5.5(c), we must consider the image of the endomorphism  $f$  of  $\mathfrak{p}^4$  defined by  $f(y) = \sigma_4(y) - y$ , which, by Lemma 7.10(a), covers  $\mathfrak{p}^l/\mathfrak{p}^{l+1}$  if and only if  $l \not\equiv 2 \pmod{3}$  and  $l \geq 6$ . For arbitrary  $u$  and  $l \in \{1, 2\}$ , note that  $\sigma_4(u)u^{-1} \in \mathcal{U}_{l+2} \setminus \mathcal{U}_{l+3}$  if  $u \in \mathcal{U}_l \setminus \mathcal{U}_{l+1}$ , and  $\sigma_4(u) = u$  if  $u = \pm\theta^3$ ; so since  $j > 6$  the assumption that  $u \in \mathcal{U}_4$  (modulo  $\mathcal{U}_j$ ) is harmless. Thus we can solve (\*2) if and only if  $j \not\equiv 0 \pmod{3}$ . But  $j \equiv 0 \pmod{3}$  in our considered case, so (\*2) cannot be solved.

Thus an isomorphism  $\iota$  cannot exist, so  $G_{1,\gamma,j+1}$  is not isomorphic to a mainline group.

(b) We may assume that  $i = 1$ , and show that  $G_{2,\gamma,j+2}$  is isomorphic to the mainline group  $G_{2,0,j+2}$ . The parameter  $j + 2$  reflects the fact that exceptional isomorphisms cause the branch in question to leave the mainline two steps lower. Recall that  $G_{2,\gamma,j+2}$  contains the maximal subgroup  $K_{2,\gamma,j+2}$ . By Remark 7.8, the latter is isomorphic to  $C \rtimes T_2/T_{j+3}$ , a mainline group for  $R_2$ . Again, this group is independent of  $\gamma$ , and we denote it by  $M_{2,j+1}$ .

We consider  $G_{2,\gamma,j+2}$ , and the corresponding mainline group, as extensions of  $M_{2,j+1}$  by  $b(\theta^2 - \theta)$ . To adapt the argument of case (a) we require both extensions to split. For both groups we observe that  $b(\theta^2 - \theta)$  has order 3. This is equivalent to the statement that  $(\theta^2 - \theta)^{b^2+b+1} = 0$ , and since  $\theta^b = \theta^4$ , this is easily checked for the mainline group. Indeed,  $(\theta^2 - \theta)^{b^2+b+1} = \theta^5 - \theta^7 + \theta^8 - \theta^4 + \theta^2 - \theta = 0$ .

Repeating this calculation in  $G_{2,\gamma,j+2}$  is a little more complicated. We need to evaluate  $(\theta^2 - \theta)^{b^2+b+1}$ , or rather  $(\theta^2 - \theta)^{b^2} \cdot (\theta^2 - \theta)^b \cdot (\theta^2 - \theta)$ , where the operation  $\cdot$  is addition twisted by  $\gamma$ . Writing  $x$  for  $\theta^2 - \theta$  we find  $x^{b^2} \cdot x^b \cdot x = x^{b^2} + x^b + x + y$ , where  $y = \frac{1}{2}\gamma(x^{b^2} \wedge x^b + x^{b^2} \wedge x + x^b \wedge x) = \frac{1}{2}\gamma(x^b \wedge x)^{b-b^2+1}$ . Now  $x^{b^2} + x^b + x = 0$ ; and  $\gamma = c\vartheta$ , so  $y = \frac{1}{2}c\vartheta(\theta^8 - \theta^4 \wedge \theta^2 - \theta)^{b-b^2+1}$ . But  $\vartheta(\theta^i \wedge \theta^j) = 0$  if  $i \equiv j \pmod{3}$ , so  $y = -\frac{1}{2}c\vartheta(\theta^8 \wedge \theta + \theta^4 \wedge \theta^2)^{b-b^2+1} = -\frac{3}{2}c(\sigma_4 - \sigma_7 + 1)\theta^6 = -\frac{3}{2}c\theta^6$ , and hence vanishes, as required, in  $G_{2,\gamma,j+2}$ .

We now have to solve the equation

$$(\sigma_4, 1, \theta^2 - \theta)(\sigma_m, u, o) = (\sigma_m, u, o)(\sigma_4, 1 + w, \theta^2 - \theta + \phi)(1, \theta^{3x}, t),$$

where  $w \in \mathfrak{p}^{j-1}/\mathfrak{p}^{j+1}$  and  $\phi \in \mathfrak{p}^j/\mathfrak{p}^{j+2}$  are given.

We find a solution with  $m = 1$  and  $x = o = 0$  by solving the simultaneous equations

$$\begin{aligned} u &= (1 + w)\sigma_4(u) & (*3) \\ u(\theta^2 - \theta) &= \theta^2 - \theta + \phi + t. & (\dagger 3) \end{aligned}$$

Since  $j \equiv 1 \pmod{3}$ , the cases that are not covered by the image of  $f$  (see Lemma 7.10) are avoided, and (\*3) can be solved, with  $u \in \mathcal{U}_{j-3}$ . (Note that (\*3) only determines  $u$  modulo  $\mathcal{U}_{j-1}$ .) It remains to choose  $t$  so that ( $\dagger 3$ ) is satisfied. Note that  $t$  is subject to the condition that  $t^{1+b+b^2} = 0$ . Lemma 7.10(b) implies that  $t \mapsto t^{1+b+b^2}$  maps  $\mathfrak{p}^i$  into  $3\mathfrak{p}^i$  for all  $i$ , so this condition is satisfied if  $3t \in \mathfrak{p}^{j+2}$ . Since  $(u - 1)(\theta^2 - \theta) \in \mathfrak{p}^{j-2}/\mathfrak{p}^{j+2}$ , and  $\phi \in \mathfrak{p}^j/\mathfrak{p}^{j+2}$ , it follows that  $t$  can be chosen so that ( $\dagger 3$ ) is satisfied, and  $3t \in \mathfrak{p}^{j+2}$ .

This defines the desired isomorphism between  $G_{2,\gamma,j+2}$  and  $G_{2,0,j+2}$ .

(c) We may assume that  $i = 2$ , and show that  $G_{3,\gamma,j+1}$  is isomorphic to  $G_{3,0,j+1}$ , but  $G_{3,\gamma,j+2}$  is not isomorphic to  $G_{3,0,j+2}$ .

First we show that  $G_{3,\gamma,j+1}$  is isomorphic to  $G_{3,0,j+1}$ . Both are non-split extensions of  $K_{3,\gamma,j+1}$  by a 3-cycle. Now  $K_{3,\gamma,j+1} \cong C \rtimes T_3/T_{j+3}$  and we denote it by  $M_{3,j}$ .

The method of proof is as follows. The groups that we need to prove isomorphic are both of the form  $G_{3,\gamma,j+1}$ , where in one case  $\gamma$  maps onto  $\mathfrak{p}^{j+1}/\mathfrak{p}^{j+2}$ , and in the other case is zero. Both have index 3 in a group that is a split extension of its Fitting subgroup by  $D$ . In the first the Fitting subgroup is twisted by  $\gamma$ , and in the second it is abelian. However, the Fitting subgroups consist of the same set of elements in both cases, with the same action of the Galois group. It is the group multiplication that is different. We also take the split extensions to have the same underlying set, and express  $G_{3,\gamma,j+1}$  in each case as a non-split extension of the same maximal subgroup  $M_{3,j}$  by the same element, namely  $b(1-\theta)^2$ . We prove that the cube of this element is the same element  $\zeta$  of  $M_{3,j}$  in each case. We then solve the appropriate analogues of  $(*)$  and  $(\dagger)$ , and observe that the automorphism of  $M_{3,j}$  we have constructed centralises  $\zeta$ , and, in the notation of Lemma 7.9,  $h^{\beta^2}h^\beta h = 1$ . This will complete the proof.

The first step is to prove that if  $r := b(1-\theta)^2$  in either group then  $r^3 = 3$ . Thus we need to calculate  $r^3$  modulo  $\mathfrak{p}^{j+3}$ ; but, for later use, we calculate  $r^3$  modulo  $\mathfrak{p}^{j+4}$ . If  $\gamma = 0$  it is easy to see that  $r^3 = 3$ . If  $\gamma \neq 0$ , then, since  $b^3 = 1$ , it follows that  $r^3 = (1-\theta)^{2b^2} \cdot (1-\theta)^{2b} \cdot (1-\theta)^2$ , so we must evaluate the expression

$$(1 - 2\theta^7 + \theta^5) \cdot (1 - 2\theta^4 + \theta^8) \cdot (1 - 2\theta + \theta^2)$$

in  $\mathfrak{p}^3/\mathfrak{p}^{j+4}$ . Since  $(1 - 2\theta^7 + \theta^5) + (1 - 2\theta^4 + \theta^8) + (1 - 2\theta + \theta^2) = 3$  this evaluates to

$$\begin{aligned} & 3 + \frac{1}{2}\gamma((1-\theta)^{2b^2} \wedge (1-\theta)^{2b}) + \frac{1}{2}\gamma((1-\theta)^{2b^2} \wedge (1-\theta)^2) + \frac{1}{2}\gamma((1-\theta)^{2b} \wedge (1-\theta)^2) = \\ & 3 + \frac{1}{2}((\gamma(1-\theta)^{2b} \wedge (1-\theta)^2)^b + \gamma((1-\theta)^2 \wedge (1-\theta)^{2b})^{b^2} + \gamma((1-\theta)^{2b} \wedge (1-\theta)^2)) = \\ & 3 + \frac{1}{2}(\sigma_4 - \sigma_7 + 1)(\gamma((1-\theta)^{2b} \wedge (1-\theta)^2)), \end{aligned}$$

where  $b$  acts as  $\sigma_4$ . Now  $\gamma = c\vartheta$ , where  $c \in \mathfrak{p}^{j-5}$ , so we calculate  $\vartheta((1-\theta^4)^2 \wedge (1-\theta)^2)$  modulo  $\mathfrak{p}^9$ :

$$\begin{aligned} \vartheta((1-\theta^4)^2 \wedge (1-\theta)^2) &= (1-\theta^8)^2(1-\theta^{-1})^2 - (1-\theta^{-4})^2(1-\theta^2)^2 \\ &= (1-\theta^4)^2(1-\theta)^2((1+\theta^4)^2\theta^{-2} - \theta^{-8}(1+\theta)^2), \end{aligned}$$

and this evaluates to  $-(1-\theta)^8$  modulo  $\mathfrak{p}^9$ . Thus, if  $c = c_0(1-\theta)^{j-5}$  for some unit  $c_0$ , then

$$r^3 \equiv 3 - (\sigma_4 - \sigma_7 + 1)(\frac{1}{2}c_0(1-\theta)^{j+3}) \equiv 3 + c_0(1-\theta)^{j+3} \pmod{\mathfrak{p}^{j+4}},$$

and  $r^3 \equiv 3 \pmod{\mathfrak{p}^{j+3}}$ , as required.

We now turn to the analogues of (\*) and (†). Again we will solve them with  $m = 1$ , and  $x = o = 0$ , by solving the simultaneous equations

$$u = (1 + w)\sigma_4(u) \quad (*4)$$

$$(u - 1)(1 - \theta)^2 = \phi + t. \quad (\dagger 4)$$

Equation (\*4) is to be solved in  $\mathcal{U}/\mathcal{U}_j$ , with  $w \in \mathfrak{p}^{j-1} \setminus \mathfrak{p}^j$ , which is possible since  $j - 1 \equiv 1 \pmod{3}$ ; and (†4) is to be solved in  $\mathfrak{p}^2/\mathfrak{p}^{j+2}$ , with  $\phi \in \mathfrak{p}^{j+1}/\mathfrak{p}^{j+2}$ , which is possible as before.

Now  $u \in \mathcal{U}_{j-3}$ ; so  $u$  centralises  $r^3$ , as required, and we deduce that  $G_{3,\gamma,j+1} \cong G_{3,0,j+1}$ .

To complete case (c), we show that  $G_{3,\gamma,j+2}$  is not isomorphic to  $G_{3,0,j+2}$ . We need to consider the equation

$$u = \theta^{3x}(1 + w)\sigma_4(u) \quad (*5)$$

in  $\mathcal{U}/\mathcal{U}_{j+1}$ . Here  $w \in \mathfrak{p}^{j-1}/\mathfrak{p}^{j+1}$ . We may again assume that  $x = 0$ . A careful calculation shows that this equation has a solution; but as we are proving non-isomorphism this is more than we need. However, we need more information about  $w$ . Since  $1 + w$  is the unit by which one multiplies an element of  $\mathfrak{p}^3/\mathfrak{p}^{j+4}$  on conjugating by  $(1 - \theta)^2$  in the twisted group,  $w(1 - \theta)^3 = \gamma((1 - \theta)^3 \wedge (1 - \theta)^2)$ . One checks that  $\vartheta(1 \wedge (1 - \theta)^2) \equiv -(1 - \theta)^4 \pmod{\mathfrak{p}^5}$ , so if  $\gamma = c_0(1 - \theta)^{j-5}\vartheta$  then  $\gamma(1 \wedge (1 - \theta)^2) \equiv -c_0(1 - \theta)^{j-1} \pmod{\mathfrak{p}^j}$ . Also

$$\begin{aligned} \gamma((1 - \theta)^3 \wedge (1 - \theta)^2) &\equiv \gamma((1 - \theta^3) \wedge (1 - \theta)^2) \\ &\equiv (1 - \theta^6)\gamma(1 \wedge (1 - \theta)^2) \\ &\equiv (1 - \theta^3)(1 + \theta^3)(-c_0(1 - \theta)^{j-1}) \\ &\equiv c_0(1 - \theta)^{j+2} \pmod{\mathfrak{p}^{j+3}}. \end{aligned}$$

Thus  $w \equiv c_0(1 - \theta)^{j-1} \pmod{\mathfrak{p}^j}$ . Now  $u = 1 + u_0(1 - \theta)^{j-3}$  for some unit  $u_0$ , and since  $j \equiv 2 \pmod{3}$  it follows from (\*5) that  $u - \sigma_4(u) \equiv u_0(1 - \theta)^{j-1} \pmod{\mathfrak{p}^j}$ . So  $u_0 \equiv c_0 \pmod{\mathfrak{p}}$ .

Following the notation of Lemma 7.9, we have  $\zeta = 3$  and  $\zeta' = 3 + c_0(1 - \theta)^{j+3}$  in  $(\mathfrak{p}^6 \setminus \mathfrak{p}^7)/\mathfrak{p}^{j+4}$ ; but

$$\zeta\lambda = \sigma_m(\zeta)u = \zeta u = \zeta(1 + u_0(1 - \theta)^{j-3}) = 3 + 3c_0(1 - \theta)^{j-3} = 3 - c_0(1 - \theta)^{j+3}.$$

This clashes with the above expression for  $\zeta'$ . Thus the groups are not isomorphic: no suitable  $h$  can be found, since Lemma 7.10 shows that  $1 + \sigma_4 + \sigma_7$  does not cover  $\mathfrak{p}^{j+3}/\mathfrak{p}^{j+4}$ . •

We now know that a branch in  $\mathcal{T}(G_{i+1})$  with non-trivial skeleton has a root of class  $h$  where  $3|h$ . It remains to determine the shape of each skeleton.

### 7.4.3 Determining the shape of $\mathcal{S}_{i+1,h,10}$

By Theorem 7.6 it suffices now to determine the precise shape of  $\mathcal{S}_{i+1,h,10}$  for  $h \geq 9$ . By Theorem 7.5, if  $j \geq 7$  and  $i = j - 3$ , a skeleton group  $G_{i+1,\gamma,m}$ , not on the mainline, is defined by a homomorphism

$$\gamma = (-1)^{u_0} \lambda_3^{u_2} \vartheta,$$

where  $\lambda_3 = 1 + (\omega - 1)^3$ ; and if  $j$  is odd we may take both  $u_0$  and  $u_2$  in  $\{0, 1\}$ ; and if  $j$  is even we may take  $u_0 = 0$  and  $u_2 \in \{0, 1, 2\}$ . Recall that we assume that  $\gamma : \mathfrak{p}^i \wedge \mathfrak{p}^i \rightarrow \mathfrak{p}^{i+j}$  where  $i = j - 3$ .

We now decide when groups defined by different values of these parameters are isomorphic under exceptional isomorphisms.

**Theorem 7.12** *Let  $\gamma$  and  $\gamma'$  be surjections from  $T_i \wedge T_i$  to  $T_{i+j}$ . If  $n \leq 9$  then  $G_{i+1,\gamma,j+n} \cong G_{i+1,\gamma',j+n}$  if and only if they are orbit isomorphic, or they are both isomorphic to a mainline group.*

PROOF: (a) Consider first the case  $i \equiv 0 \pmod{3}$  and  $n = 1$ , as in (a) of Theorem 7.11. We have seen that the two groups obtained by taking  $u_0 = 0$  and  $u_0 = 1$  are not isomorphic to the mainline group of the same order. It remains to prove that, if  $j$  is odd, then these two groups are not isomorphic to each other *via* an exceptional isomorphism. We saw in Theorem 7.11(a) that the critical parameter for  $G_{1,\gamma,j+1}$  was  $w \in \mathfrak{p}^{j-1}/\mathfrak{p}^j$ . We now have two such parameters, say  $w_1$  and  $w_2 = -w_1$ . These determine  $\gamma$  and  $\gamma'$ . Then Equation (\*) is replaced by

$$(1 + \sigma_m(w_1))u = \sigma_4(u)(1 - w_1).$$

or equivalently  $\sigma_4(u) - u = \sigma_m(w_1) + w_1$ . Since, as we saw, this has no solution when the right hand side is a non-zero element of  $\mathfrak{p}^{j-1}/\mathfrak{p}^j$ , it remains to prove that the right hand side cannot lie in  $\mathfrak{p}^j$ . We may take  $w_1 = (\theta - 1)^{j-1} + \mathfrak{p}^j$ . But  $\sigma_m(\theta - 1)^{j-1} \equiv m^{j-1}(\theta - 1)^{j-1} \pmod{\mathfrak{p}^j}$ , and  $m^{j-1}$  cannot be congruent to  $-1$  modulo 3 if  $j$  is odd.

(b) Now consider the case  $i \equiv 1 \pmod{3}$ . We must now take  $n = 3$ , and prove that, if  $j$  is odd, then the two parameters  $w_1$  and  $-w_1$  (taking  $u_0 \in \{0, 1\}$ ) in  $(\mathfrak{p}^{j-1} \setminus \mathfrak{p}^j)/\mathfrak{p}^{j+2}$  do not define isomorphic groups. But this is clearly the case, since an exceptional isomorphism between two groups  $G_{i+1,\gamma_k,m}$  (for  $k = 1, 2$ ) must induce orbit isomorphisms between the groups  $G_{i+1,\gamma_k,m-2}$ . For, if  $G_{i+1,\gamma_1,m}$  is isomorphic to  $G_{i+1,\gamma_2,m}$  then since  $G_{i+1}$  is an extension of  $T_{i+1}$  by  $D$  it follows that, by applying an orbit automorphism, we may assume that  $\gamma_1$  and  $\gamma_2$  induce the same homomorphisms from  $T_{i+1} \wedge T_{i+1}$  to  $T_{i+j}/T_{i+m}$ . Then  $\gamma_2 = \gamma_1 + \gamma + \gamma_3$ , where  $\gamma_3$  maps  $T_{i+1} \wedge T_{i+1}$  into  $T_{i+m}$ . It follows that  $\gamma_3$  maps  $T_i \wedge T_i$  into  $T_{i+m-2}$ ; so  $\gamma_1$  and  $\gamma_2$  induce the same homomorphisms from  $T_{i+1} \wedge T_{i+1}$  to  $T_{i+j}/T_{i+m-2}$ , as required.

(c) Now consider the case  $i \equiv 2 \pmod{3}$ . We must now take  $n = 2$ , and prove that, if  $j$  is odd, then the two parameters  $\gamma$  and  $\gamma'$  taking values  $\pm(1 - \theta^3)^{(j-5)/3}$  respectively do not

define isomorphic groups. Following the argument of case (c) of Theorem 7.11, one sees that a necessary condition for the groups to be isomorphic is that  $\sigma_m(3 + (1 - \theta)^{j+3}) \equiv 3 - (1 - \theta)^{j+3} \pmod{\mathfrak{p}^{j+4}}$  for some  $m$ . This is equivalent to  $m^{j+3} \equiv -1 \pmod{3}$ , which is impossible if  $j$  is odd.  $\bullet$

Now consider the case where  $n \geq 10$ . It remains to decide when two different values of  $u_2$ , as above, can give rise to isomorphic groups.

(i) Suppose that  $i \equiv 0 \pmod{3}$ , so we may assume that  $i = 0$ . Taking  $n = 10$ , there are either two or three groups of the form  $G_{1,\gamma,j+10}$ , for fixed  $j \equiv i \pmod{3}$ , according to whether  $j$  is respectively odd or even, that are not orbit isomorphic, but where the corresponding quotients  $G_{1,\gamma,j+9}$  are orbit isomorphic, and we need to prove that the groups  $G_{1,\gamma,j+10}$  are not isomorphic. Since  $\vartheta(\theta \wedge \theta^4) = \vartheta(\theta \wedge \theta^7) = \vartheta(\theta^4 \wedge \theta^7) = 0$  it follows that  $b\theta$ , as an element of  $G_{1,\gamma,j+10}$ , has order 3 in all cases, so we may again use Lemma 7.9 with  $\zeta = \zeta' = 1$ . The role of  $M$  in that lemma is now played by  $R_{1,\delta,j+9}$ , where  $\delta$  is the restriction of  $\gamma$  to  $\mathfrak{p} \wedge \mathfrak{p}$ . The automorphism group of this group is described in Theorem 5.12. The role of  $\gamma'$  in that theorem is played by  $\gamma$  here. So the automorphism group of  $R_{1,\delta,j+9}$  is a subgroup of  $(C_6 \times \mathcal{U}/\mathcal{U}_{j+9}) \times T_{0,\gamma,j+9}$ . The automorphism induced by  $b\theta$  will then be of the form  $(\sigma_4, 1 + w_0, \theta \cdot \phi_0)$ , where  $w_0, \phi_0 \in \mathfrak{p}^{j-1}/\mathfrak{p}^{j+9}$  for one value of  $\gamma$ , and similarly with  $w_0$  and  $\phi_0$  replaced by  $w_1$  and  $\phi_1$  for another value of  $\gamma$ . Since these two values of  $\gamma$  differ by a factor of  $\lambda_3$  (or  $\lambda_3^2$ ), it follows that  $w_0$  and  $w_1$  differ modulo  $\mathfrak{p}^j$ . It now follows, as in case (a) of Theorem 7.5, that we must consider the equation

$$(\sigma_4, 1 + w_0, \theta \cdot \phi_0)(\sigma_m, u, o) = (\sigma_m, u, o)(\sigma_4, 1 + w_1, \theta \cdot \phi_1)(1, \theta^{3x}, t)$$

in  $(C_6 \times \mathcal{U}/\mathcal{U}_{j+9}) \times \mathcal{O}/\mathfrak{p}^{i+j+9}$  for  $(m, u, o, x, t)$ , the other parameters being given, where the operation ‘ $\cdot$ ’ on  $\mathcal{O}/\mathfrak{p}^{i+j+9}$  is obtained by twisting with  $\gamma$ . As before, by considering the analogue of Equation (\*1) that arises from the displayed equation, one sees that two groups  $G_{1,\gamma,j+10}$ , for different values of  $\gamma$ , that are not orbit isomorphic are not isomorphic.

(ii) Suppose that  $i \equiv 1 \pmod{3}$ , so we may assume that  $i = 1$ . Taking  $n = 11$ , we need to prove that if  $G_{2,\gamma_1,j+9}$  and  $G_{2,\gamma_2,j+9}$  are orbit isomorphic then  $G_{2,\gamma_1,j+11}$  and  $G_{2,\gamma_2,j+11}$  are isomorphic. An additional complication arises here, since  $b(\theta^2 - \theta)$  no longer has order 3. As observed above, in  $G_{2,\gamma_k,j+11}$ , one finds that  $(b(\theta^2 - \theta))^3 = 3c_k\theta^6$ , where  $\gamma_k = c_k\vartheta$  for  $k = 1, 2$ . Here  $c_k \in \mathfrak{p}^{j-2}$ , and  $c_1 - c_2 \in \mathfrak{p}^{j+7}$ , so  $(b(\theta^2 - \theta))^3$  is the same element in the two groups; so we may apply Lemma 7.9 with  $\zeta = \zeta' \in \mathfrak{p}^{j+4}/\mathfrak{p}^{j+11}$ . Postponing further consideration of  $\zeta$  and  $\zeta'$ , we need to solve essentially the same equations as before (with  $i = 1$ ). In particular, we need a conjugating automorphism  $(\sigma_m, u, o)$ . Repeating the earlier argument, we find a solution with  $m = 1$ ,  $u \in \mathcal{U}_7$ , and  $o = 0$ . One then checks, from Theorem 5.12, that  $(1, u, 0)$  is an automorphism of  $R_{2,\gamma,j+11}$  that centralises  $\zeta$ . Thus the equations required to construct the isomorphism may be constructed as before.

Taking  $i = 1$  and  $n = 12$ , we observe, as before, that if  $G_{2,\gamma_1,j+10}$  and  $G_{2,\gamma_2,j+10}$  are not orbit isomorphic, then  $G_{2,\gamma_1,j+12}$  and  $G_{2,\gamma_2,j+12}$  are not isomorphic.

(iii) Suppose that  $i \equiv 2 \pmod{3}$ , so we may assume that  $i = 2$ . Taking  $n = 10$ , we need to prove that if  $G_{3,\gamma_1,j+9}$  and  $G_{3,\gamma_2,j+9}$  are orbit isomorphic then  $G_{3,\gamma_1,j+10}$  and  $G_{3,\gamma_2,j+10}$  are isomorphic. This can be carried out as with case (ii), as the cubes  $\zeta$  and  $\zeta'$  are equal.

Finally, we need to prove that if  $G_{3,\gamma_1,j+9}$  and  $G_{3,\gamma_2,j+9}$  are orbit isomorphic, but  $G_{3,\gamma_1,j+10}$  and  $G_{3,\gamma_2,j+10}$  are not, then  $G_{3,\gamma_1,j+11}$  and  $G_{3,\gamma_2,j+11}$  are not isomorphic. Now the cubes  $\zeta$  and  $\zeta'$  are not equal, and the argument is similar to case (c) of Theorem 7.12 with  $n = 2$ .

## 7.5 Identifying the skeleton groups in $\mathcal{T}(G_{i+1})$

In this section we describe the skeleton groups in  $\mathcal{T}(G_{i+1})$ . These are  $G_{j-2,\gamma,m}$  with  $j \equiv i \pmod{3}$  and  $\gamma \in \{\pm c\vartheta \mid c = 1 \text{ or } c = 1 + 3(\theta^3 - 1)\}$ . They are constructed as subgroups of small index in  $H_{j-3,\gamma,m}$ . Explicit descriptions for  $H_{j-3,\gamma,m}$  are given in Section 6.5 for  $\gamma = c\vartheta$  with  $c = 1$  or  $c = 1 + 3(\theta^3 - 1)$ . The others are obtained using the following lemma.

**Lemma 7.13**  $G_{i+1,-\gamma,m} \cong \langle a, bt_i^{-1} \rangle \leq H_{i,\gamma,m}$ .

PROOF: Multiplication by  $-1 \in \mathcal{U}(3)$  induces an isomorphism between  $H_{i,\gamma,m}$  and  $H_{i,-\gamma,m}$ . This acts by inversion on  $T_{i,\gamma,m}$  and maps  $G_{i+1,\gamma,m}$  to  $G_{i+1,-\gamma,m}$ .  $\bullet$

We now equate skeleton groups in  $\mathcal{T}(G_{j-2})$  with subgroups of low index in skeleton groups in  $\mathcal{T}(H_{j-3})$ . This implicitly defines presentations for these groups. Let  $H_{j-3,\gamma,m} = \langle a, b, t \rangle$  be a skeleton group for  $H_{j-3}$  of depth  $n := m - j$  in the skeleton with root of class  $j$ . Let  $\gamma = c\vartheta$  with  $c = 1$  or  $c = 1 + 3(\theta^3 - 1)$ .

- (a) The two subgroups  $\langle a, bt^{\pm 1} \rangle$  of index 3 in  $H_{j-3,\gamma,m}$  are skeleton groups for  $G_1$ . They have depth  $n$  in the skeleton with root of class  $j$ . They are isomorphic if and only if  $\gamma = \vartheta$  and  $6 \mid j$ .
- (b) The two subgroups  $\langle a, b[t, a]^{\pm 1} \rangle$  of index 9 in  $H_{j-3,\gamma,m}$  are skeleton groups for  $G_2$ . They have depth  $n - 4$  in the skeleton with root of class  $j + 3$ . They are isomorphic if and only if  $\gamma = \vartheta$  and  $6 \mid j$ .
- (c) The two subgroups  $\langle a, b[t, a, a]^{\pm 1} \rangle$  of index 27 in  $H_{j-3,\gamma,m}$  are skeleton groups for  $G_3$ . They have depth  $n - 5$  in the skeleton with root of class  $j + 3$ . They are isomorphic if and only if  $\gamma = \vartheta$  and  $6 \mid j - 3$ .

## 8 Twigs for $\mathcal{T}(R)$ and $\mathcal{T}(G_i)$

Recall that all coclass trees in  $\mathcal{G}(3, 2)$  except  $\mathcal{T}(R)$  and  $\mathcal{T}(G_i)$  for  $i \in \{1, 2, 3\}$  can be constructed from a finite subtree using the periodicity exhibited in Theorem 2.1. In this section we provide evidence that these exceptions can also be constructed from a finite subtree using Theorem 2.1 and a second periodic pattern that we now describe.

## 8.1 A second periodic pattern

Let  $G \in \{R, G_1, G_2, G_3\}$  and choose  $k_G$  so that the skeleton  $\mathcal{S}_j(G)$  of the branch  $\mathcal{B}_j(G)$  for  $j \geq 7$  has no branching at depth  $k_G$  or larger. Let  $t(G)$  be the absolute bound to the depth of twigs in  $\mathcal{T}(G)$  identified in Theorem 3.4.

Let  $f_G = f(k_G + 6 + t(G))$ , where  $f$  is the defect function of Theorem 2.1. If  $j \geq f_G$ , then the isomorphism  $\pi_j$  of Theorem 2.1 maps the skeleton groups in  $\mathcal{B}_{j, k_G+6}(G)$  to the skeleton groups in  $\mathcal{B}_{j+6, k_G+6}(G)$ , and it induces a graph isomorphism between the twigs of these skeleton groups.

Here we investigate the twigs for the skeleton groups of depth at least  $k_G$ . The explicit descriptions of the skeleton groups in  $\mathcal{T}(R)$  and  $\mathcal{T}(G_i)$  allow us to compute the twigs of these groups using the  $p$ -group generation algorithm [19]. Let  $\mathcal{W}_{j,n}(G, \gamma)$  denote the twig of the skeleton group  $G_{\gamma, m}$  of depth  $n = m - j$  in  $\mathcal{S}_j(G)$  defined by the homomorphism  $\gamma$ . Our computations support the following.

**Conjecture 8.1** *Let  $j \geq 7$  and  $G \in \{R, G_1, G_2, G_3\}$ .*

- (a)  $\mathcal{W}_{j,n}(G, \gamma) \cong \mathcal{W}_{j+6, n+6}(G, 3\gamma)$  for each homomorphism  $\gamma$  and integer  $n$  satisfying  $k_G \leq n \leq \text{depth}(\mathcal{S}_j(G))$ .
- (b) *There exists an integer  $u_G$  such that  $\mathcal{W}_{j,n}(G, \gamma) \cong \mathcal{W}_{j, n+6}(G, \gamma)$  for each homomorphism  $\gamma$  and integer  $n$  satisfying  $k_G \leq n < u_G - 6$ .*

Part (a) and Theorem 2.1 imply that, for  $j \geq f_G$ , every twig of a skeleton group in  $\mathcal{B}_{j+6}(G)$  can be read off from a twig in  $\mathcal{B}_j(G)$ : for the skeleton groups of depth at most  $k_G + 6$  we use Theorem 2.1, and for all others we use (a). This reduces the construction of  $\mathcal{T}(G)$  to a finite subgraph consisting of all branches  $\mathcal{B}_j(G)$  for  $j < f_G + 6$ .

Part (b) implies that the twigs of skeleton groups having depth in  $\{k_G, \dots, k_G + 5\}$  describe those of the skeleton groups having depth in  $\{k_G, \dots, u_G - 1\}$ . Hence we can describe compactly the twigs of the skeleton groups of depth at least  $k_G$ .

Table 3 summarises our choice for  $k_G$ ; informed by our computations, we conjecture values for  $u_G$  and the depth of  $\mathcal{B}_j(G)$ . Recall that  $\mathcal{S}_j(R)$  is non-trivial for every  $j \geq 7$  and  $\mathcal{S}_j(G_i)$  is non-trivial for every  $j \geq 7$  divisible by 3.

$G$	$k_G$	$u_G$	$\text{depth}(\mathcal{B}_j(G))$	case
$R$	12	$j - 2$	$j + 3$	$3 \mid j$
	12	$j - 2$	$j + 1$	$3 \nmid j$
$G_1$	10	$j$	$j + 4$	$3 \mid j$
$G_2$	10	$j - 6$	$j$	$3 \mid j$
$G_3$	10	$j - 6$	$j + 2$	$3 \mid j$

Table 3: Summary data for conjecture on  $\mathcal{G}(3, 2)$

While the depth of  $\mathcal{S}_j(G_3)$  is  $j - 3$ , our descriptions of the skeleton groups allow us to investigate these only to depth  $j - 6$ , so  $u_{G_3}$  is conjectured for this shorter skeleton.

## 8.2 Twigs for $\mathcal{T}(R)$

### Conjecture 8.2

- (a) *There are 45 isomorphism types of twigs for the skeleton groups of depth at least 13 in a branch of  $\mathcal{T}(R)$ .*
- (b) *Let  $j \geq 13$ , let  $12 \leq n \leq \text{depth}(\mathcal{S}_j(R))$ , and let  $\gamma_1, \dots, \gamma_{17}$  define the 17 isomorphism types of groups in  $\mathcal{S}_j(R)$  of depth  $n$  that are summarised in Table 1. Let  $\Sigma_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ ,  $\Sigma_2 = \{10, 14, 15\}$ ,  $\Sigma_3 = \{11, 12, 13, 16\}$ , and  $\Sigma_4 = \{17\}$ . Then  $\mathcal{W}_{j,n}(R, \gamma_e) \cong \mathcal{W}_{j,n}(R, \gamma_f)$  if  $e$  and  $f$  are in the same set  $\Sigma_k$ .*
- (c) *Table 4 describes the twigs of the skeleton groups of depth  $n \geq 12$  in  $\mathcal{B}_j(R)$ ; it lists two invariants  $(a, b)$  for each twig – its depth  $a$  and number of vertices  $b$ . The top part of a table describes these twigs for  $n \in \{12, \dots, j-3\}$ ; we write  $n = l + 6x$  with  $l \in \{12, \dots, 17\}$  and  $x \geq 0$ . The bottom part of a table describes the twigs for  $n \in \{u_R, \dots, \text{depth}(\mathcal{S}_j(R))\}$ .*

We verified this conjecture for  $13 \leq j \leq 26$ .

We exhibit 3 of the 45 twigs in Figure 4. These are drawn compactly: if a subtree occurs  $b$  times in the tree, then the subtree is drawn only once and its root has  $b$  attached to it. Observe  $A$  has invariants  $(3, 20169)$ , both  $B$  and  $C$  have invariants  $(3, 7317)$ ; so our invariants do not distinguish all twigs. We conjecture that all twigs with invariants  $(3, 20169)$  in  $\mathcal{B}_j(R)$  are isomorphic to Tree  $A$ ; all twigs with invariants  $(3, 7317)$  in  $\mathcal{B}_j(R)$  where  $3 \mid j$  are isomorphic to Tree  $B$ ; all twigs with invariants  $(3, 7317)$  in  $\mathcal{B}_j(R)$  where  $3 \nmid j$  are isomorphic to Tree  $C$ .

## 8.3 Twigs for $\mathcal{T}(G_i)$

Figure 3 shows that there are either 3 or 4 pairwise non-isomorphic skeleton groups at depth  $n \geq 10$ .

**Conjecture 8.3** *The number of isomorphism types of twigs for skeleton groups of depth at least 10 in a branch of  $\mathcal{T}(G_i)$  is 11, 13, and 11 for  $i \in \{1, 2, 3\}$  respectively. The twigs are identified uniquely by their invariants. Tables 5–7 describe the invariants of the twigs of the skeleton groups of depth  $n \geq 10$  in  $\mathcal{B}_j(G_i)$ .*

We verified the description for  $G_1$  for  $j \in \{12, 15, 18, 21, 24\}$ , for  $G_2$  for  $j \in \{15, 18, 21, 24, 27\}$ , and for  $G_3$  for  $j \in \{18, 21, 24, 27, 30\}$ .

$n$	$\Sigma_1$	$\Sigma_2$	$\Sigma_3$	$\Sigma_4$	$n$	$\Sigma_1$	$\Sigma_2$	$\Sigma_3$	$\Sigma_4$
$12 + 6x$	(3,20169)	(3,20169)	(3,6993)	(3,3647)	$12 + 6x$	(3,20169)	(3,20169)	(3,6993)	(3,3636)
$13 + 6x$	(3,20169)	(3,20169)	(3,7317)	(3,3659)	$13 + 6x$	(3,20169)	(3,20169)	(3,7317)	(3,3690)
$14 + 6x$	(3,20169)	(3,20169)	(3,7209)	(3,3767)	$14 + 6x$	(3,20169)	(3,20169)	(3,7209)	(3,3744)
$15 + 6x$	(3,20169)	(3,20169)	(3,6993)	(3,3497)	$15 + 6x$	(3,20169)	(3,20169)	(3,6993)	(3,3522)
$16 + 6x$	(3,20169)	(3,20169)	(3,7317)	(3,3827)	$16 + 6x$	(3,20169)	(3,20169)	(3,7317)	(3,3798)
$17 + 6x$	(3,20169)	(3,20169)	(3,7209)	(3,3605)	$17 + 6x$	(3,20169)	(3,20169)	(3,7209)	(3,3630)
$j - 2$	(3,20331)	(3,20331)	(3,7209)	(3,3767)	$j - 2$	(3,20331)	(3,20331)	(3,7209)	(3,3633)
$j - 1$	(2,775)	(2,775)	(2,277)	(2,144)	$j - 1$	(2,775)	(2,775)	(2,277)	(2,151)

$j \equiv 2 \pmod 6$   $j \equiv 5 \pmod 6$

$n$	$\Sigma_1$	$\Sigma_2$	$\Sigma_3$	$\Sigma_4$	$n$	$\Sigma_1$	$\Sigma_2$	$\Sigma_3$	$\Sigma_4$
$12 + 6x$	(3,20169)	(3,20169)	(3,20169)	(3,10434)	$12 + 6x$	(3,20169)	(3,20169)	(3,20169)	(3,10483)
$13 + 6x$	(3,20169)	(3,20169)	(3,7317)	(3,3696)	$13 + 6x$	(3,20169)	(3,20169)	(3,7317)	(3,3659)
$14 + 6x$	(3,20169)	(3,20169)	(3,7317)	(3,3792)	$14 + 6x$	(3,20169)	(3,20169)	(3,7317)	(3,3811)
$15 + 6x$	(3,20169)	(3,20169)	(3,20169)	(3,10140)	$15 + 6x$	(3,20169)	(3,20169)	(3,20169)	(3,10085)
$16 + 6x$	(3,20169)	(3,20169)	(3,7317)	(3,3792)	$16 + 6x$	(3,20169)	(3,20169)	(3,7317)	(3,3811)
$17 + 6x$	(3,20169)	(3,20169)	(3,7317)	(3,3696)	$17 + 6x$	(3,20169)	(3,20169)	(3,7317)	(3,3659)
$j - 2$	(3,19845)	(3,19845)	(3,7317)	(3,3696)	$j - 2$	(3,19845)	(3,19845)	(3,7317)	(3,3811)
$j - 1$	(3,19701)	(3,19701)	(3,7065)	(3,3666)	$j - 1$	(3,19701)	(3,19701)	(3,7065)	(3,3533)
$j$	(2,757)	(3,20440)	(3,20440)	(3,10303)	$j$	(2,757)	(3,20440)	(3,20440)	(3,10603)

$j \equiv 3 \pmod 6$   $j \equiv 0 \pmod 6$

$n$	$\Sigma_1$	$\Sigma_2$	$\Sigma_3$	$\Sigma_4$	$n$	$\Sigma_1$	$\Sigma_2$	$\Sigma_3$	$\Sigma_4$
$12 + 6x$	(3,20169)	(3,20169)	(3,6993)	(3,3647)	$12 + 6x$	(3,20169)	(3,20169)	(3,6993)	(3,3636)
$13 + 6x$	(3,20169)	(3,20169)	(3,7209)	(3,3605)	$13 + 6x$	(3,20169)	(3,20169)	(3,7209)	(3,3630)
$14 + 6x$	(3,20169)	(3,20169)	(3,7317)	(3,3827)	$14 + 6x$	(3,20169)	(3,20169)	(3,7317)	(3,3798)
$15 + 6x$	(3,20169)	(3,20169)	(3,6993)	(3,3497)	$15 + 6x$	(3,20169)	(3,20169)	(3,6993)	(3,3522)
$16 + 6x$	(3,20169)	(3,20169)	(3,7209)	(3,3767)	$16 + 6x$	(3,20169)	(3,20169)	(3,7209)	(3,3744)
$17 + 6x$	(3,20169)	(3,20169)	(3,7317)	(3,3659)	$17 + 6x$	(3,20169)	(3,20169)	(3,7317)	(3,3690)
$j - 2$	(3,19845)	(3,19845)	(3,7317)	(3,3827)	$j - 2$	(3,19845)	(3,19845)	(3,7317)	(3,3690)
$j - 1$	(2,775)	(2,775)	(2,331)	(2,174)	$j - 1$	(2,775)	(2,775)	(2,331)	(2,181)

$j \equiv 4 \pmod 6$   $j \equiv 1 \pmod 6$

Table 4: Conjectured twigs for branch  $\mathcal{B}_j$  in  $\mathcal{T}(R)$  where  $j \geq 13$

$n$	$k = 1, 2, 3$
$10 + 6x$	(2,2187)
$11 + 6x$	(2,1215)
$12 + 6x$	(3,19683)
$13 + 6x$	(2,2187)
$14 + 6x$	(2,1215)
$15 + 6x$	(3,19683)
$j$	(4,22681)

$j \equiv 0 \pmod{6}$

$n$	$k = 1, 3$	$k = 2, 4$
$10 + 6x$	(2,1119)	(2,2187)
$11 + 6x$	(2,636)	(2,1215)
$12 + 6x$	(3,9948)	(3,19683)
$13 + 6x$	(2,1122)	(2,2187)
$14 + 6x$	(2,627)	(2,1215)
$15 + 6x$	(3,9897)	(3,19683)
$j$	(4,11437)	(4,22681)

$j \equiv 3 \pmod{6}$

Table 5: Conjectured twigs for branch  $\mathcal{B}_j$  in  $\mathcal{T}(G_1)$  for  $j \geq 12$

$n$	$k = 1, 2, 3$
$10 + 6x$	(4,1863)
$11 + 6x$	(4,2673)
$12 + 6x$	(4,1215)
$13 + 6x$	(4,1863)
$14 + 6x$	(4,2673)
$15 + 6x$	(4,1215)
$j - 6$	(6,5265)
$j - 5$	(1,28)

$j \equiv 0 \pmod{6}$

depth	$k = 1, 3$	$k = 2, 4$
$10 + 6x$	(4,957)	(4,1863)
$11 + 6x$	(4,1350)	(4,2673)
$12 + 6x$	(4,645)	(4,1215)
$13 + 6x$	(4,942)	(4,1863)
$14 + 6x$	(4,1380)	(4,2673)
$15 + 6x$	(4,618)	(4,1215)
$j - 6$	(6,2700)	(6,5265)
$j - 5$	(1,19)	(1,28)

$j \equiv 3 \pmod{6}$

Table 6: Conjectured twigs for branch  $\mathcal{B}_j$  in  $\mathcal{T}(G_2)$  for  $j \geq 15$

$n$	$k = 1, 3$	$k = 2, 4$
$10 + 6x$	(3,97)	(3,171)
$11 + 6x$	(3,60)	(3,111)
$12 + 6x$	(6,629)	(6,1257)
$13 + 6x$	(3,90)	(3,171)
$14 + 6x$	(3,75)	(3,111)
$15 + 6x$	(6,681)	(6,1257)
$j - 6$	(8,912)	(8,1717)

$j \equiv 0 \pmod{6}$

$n$	$k = 1, 2, 3$
$10 + 6x$	(3,171)
$11 + 6x$	(3,111)
$12 + 6x$	(6,1257)
$13 + 6x$	(3,171)
$14 + 6x$	(3,111)
$15 + 6x$	(6,1257)
$j - 6$	(8,1717)

$j \equiv 3 \pmod{6}$

Table 7: Conjectured twigs for branch  $\mathcal{B}_j$  in  $\mathcal{T}(G_3)$  for  $j \geq 18$

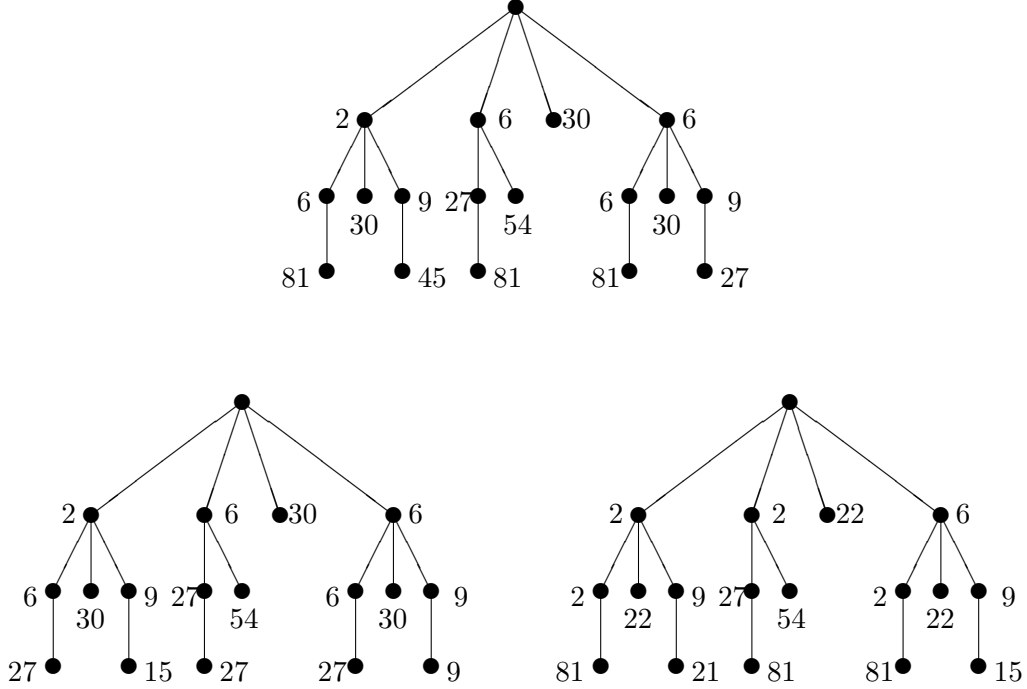


Figure 4: Trees  $A$ ,  $B$ , and  $C$

## 9 A general conjecture

Let  $G$  be an infinite pro- $p$ -group of coclass  $r$  and dimension  $d$ . We now conjecture how a sequence of branches of unbounded depth in  $\mathcal{T}(G)$  can be constructed from a finite subgraph, and demonstrate how this conjecture allows us to construct  $\mathcal{T}(G)$  by a finite calculation.

Before stating the conjecture formally, we first consider the significance of the work of [11]. For given  $k > 0$ , Theorem 2.1 implies that for  $\ell \geq f(k)$ , where  $f$  is the defect function, the graphs  $\mathcal{B}_{\ell+id,k}$ , for  $i = 0, 1, \dots$  are all isomorphic.

Every skeleton group in  $\mathcal{T}(G)$  is defined *via* the minimal split supergroup  $H$  of  $G$  and a homomorphism  $\gamma : T \wedge T \rightarrow T$  defining a skeleton group in  $\mathcal{T}(H)$ . Consider the set  $\Gamma_{\ell,k}$  of the skeleton groups of depth  $k$  in the branch  $\mathcal{B}_\ell$  of  $\mathcal{T}(G)$  up to isomorphism and assume that  $\ell$  is large enough with respect to  $k$ . The construction in [11, Theorem 9] ensures that, if  $\gamma$  defines a skeleton group in  $\Gamma_{\ell,k}$ , then  $p^i \gamma$  defines a skeleton group in  $\Gamma_{\ell+id,k}$ . Further, this correspondence induces a bijection between  $\Gamma_{\ell,k}$  and  $\Gamma_{\ell+id,k}$  for each  $i \geq 0$ . If  $P \in \Gamma_{\ell+id,k}$ , then  $\bar{P}$  is its image under this bijection in  $\Gamma_{\ell,k}$ .

Following Theorem 2.1, we arrange the infinitely many branches of  $\mathcal{T}(G)$  into  $d$  sequences. The theorem completely describes a sequence of branches of bounded depth. We now state the conjecture for a sequence of branches of unbounded depth, and illustrate it in Figure 5.

### Conjecture W

Let  $G$  be an infinite pro- $p$ -group of finite coclass  $r$  and dimension  $d$ . Let  $\mathcal{B}_{\ell_0+id}$  for  $i = 0, 1, \dots$  be a sequence of branches in  $\mathcal{T}(G)$  of unbounded depth. There exist integers  $k \geq d$  and  $\ell = \ell_0 + ed$  for some  $e \geq 0$  and a map  $\nu : \Gamma_{\ell,k} \rightarrow \Gamma_{\ell,k-d}$  that satisfy the following: if  $P$  is in  $\Gamma_{\ell+id,k}$ , and  $Q$  is in  $\Gamma_{\ell+(i-1)d,k-d}$  and  $\nu(\overline{P}) = \overline{Q}$ , then the descendant trees of  $P$  and  $Q$  are isomorphic.

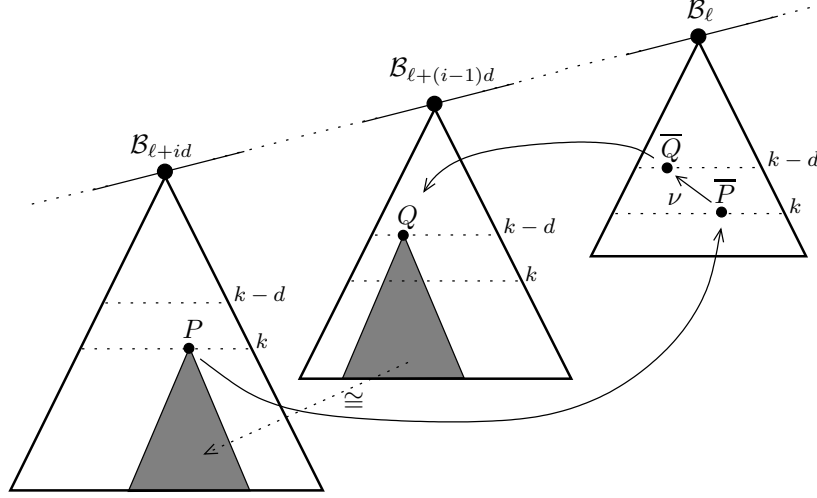


Figure 5: An illustration of Conjecture W

The data of Section 8 supports the conjecture. For each sequence of branches of unbounded depth in  $\mathcal{T}(R)$ , we can choose  $k = 18$  and  $\ell = 19, 20, 21, 22, 23, 18$  for  $\ell_0 = 7, \dots, 12$  respectively. For those in  $\mathcal{T}(G_i)$ , we can choose  $k = 16$ ; for  $\ell_0 \equiv 0 \pmod{6}$  we can take  $\ell = 18, 24, 24$  respectively; for  $\ell_0 \equiv 3 \pmod{6}$  we can take  $\ell = 21, 27, 27$  respectively.

If the conjecture is true, then we can construct the infinite sequence  $\mathcal{B}_{\ell+id}$  for  $i = 0, 1, \dots$  by a finite calculation.

To do so, we must first choose an explicit value of  $k$ . Theorem 2.1 implies, that for  $\ell \geq f(k)$ , the graphs  $\mathcal{B}_{\ell+id,k}$  are all isomorphic, and hence may be constructed. The graph isomorphism maps a skeleton group  $P$  of depth  $k$  to a skeleton group  $\overline{P}$  of the same depth.

The conjecture posits a map  $\nu_\ell := \nu$  from  $\Gamma_{\ell,k}$  into  $\Gamma_{\ell,k-d}$  such that the graph of descendants of  $P \in \Gamma_{\ell+id,k}$  is isomorphic to the graph of descendants of  $\nu_\ell(\overline{P}) \in \Gamma_{\ell,k-d}$ . Thus the subgraph of  $\mathcal{B}_{\ell+id}$  containing both the groups of depth at most  $k$  and those descendants of skeleton groups of depth  $k$  may be constructed, together with a map,  $\nu_{\ell+id}$ , from  $\Gamma_{\ell+id,k}$  into  $\Gamma_{\ell+id,k-d}$  that corresponds to  $\nu_\ell$  under the isomorphism between  $\mathcal{B}_{\ell,k}$  and  $\mathcal{B}_{\ell+id,k}$ .

Now the subgraph of  $\mathcal{B}_{\ell+2d}$  containing both the groups of depth at most  $k$  and those descendants of skeleton groups of depth  $k$  can be constructed from the corresponding

subgraph of  $\mathcal{B}_{\ell+d}$  and the map  $\nu_{\ell+d}$ . The corresponding subgraphs of  $\mathcal{B}_{\ell+id}$  for all  $i > 0$  can be constructed recursively in the same way.

Finally, the complete graphs  $\mathcal{B}_{\ell+id}$  may be constructed by a finite calculation, again using Theorem 2.1. All that needs to be added are the descendants of the groups of depth  $k$  that are not skeleton groups. Since the twigs have depth at most  $t(G)$ , the subgraphs  $\mathcal{B}_{l+id, k+t(G)}$  are isomorphic for  $l + id \geq f(k + t(G))$ .

The central difficulty in proving Conjecture W is finding a description for the map  $\nu_\ell$ . The investigations of  $\mathcal{G}(3, 2)$  and  $\mathcal{G}(5, 1)$  suggest that  $\nu_\ell$  can be defined as taking the  $d$ -step ancestor of a given group. However [6, Remark 4] suggests that this is not true for  $\mathcal{G}(p, 1)$  for  $p \geq 7$ .

## References

- [1] Judith Ascione, George Havas and C.R. Leedham-Green. A computer aided classification of certain groups of prime power order. *Bull. Austral. Math. Soc* **17**, 257–274, 1977.
- [2] Judith A. Ascione. On 3-groups of second maximal class. PhD thesis, Australian National University, 1979.
- [3] N. Blackburn. On a special class of  $p$ -groups. *Acta Math.*, **100**, 45–92, 1958.
- [4] Heiko Dietrich, Bettina Eick, and Dörte Feichtenschlager. Investigating  $p$ -groups by coclass with GAP. *Contemp. Math. AMS* **470**, 45–61, 2008.
- [5] Heiko Dietrich. Periodic patterns in the graph of  $p$ -groups of maximal class. *J. Group Theory* **13**, 851–871, 2010.
- [6] Heiko Dietrich. A new periodic pattern in the graph of  $p$ -groups of maximal class. *Bull. London Math. Soc.* **42**, 1073–1088, 2010.
- [7] Marcus du Sautoy. Counting  $p$ -groups and nilpotent groups. *Inst. Hautes Études Sci. Publ. Math.* **92**, 63–112, 2001.
- [8] Bettina Eick. Determination of the uniserial space groups with a given coclass. *J. London Math. Soc.* **71**, 622–642, 2005.
- [9] Bettina Eick. Automorphism groups of 2-groups. *J. Algebra* **300**, 91–101, 2006.
- [10] Bettina Eick and Dörte Feichtenschlager. Symbolic computation with infinite coclass families of  $p$ -groups. To appear *International Journal Algebra and Computation*.
- [11] Bettina Eick and Charles Leedham-Green. Classification of prime-power groups by coclass. *Bull. London Math. Soc.* **40**, 274–288, 2008.

- [12] Helmut Hasse. Über die Klassenzahl abelscher Zahlkörper. Akademie-Verlag, Berlin, 1952.
- [13] C.R. Leedham-Green. The structure of finite  $p$ -groups, *J. London Math. Soc.* (2) **50**, 49–67, 1994.
- [14] C.R. Leedham-Green and S. McKay. The structure of groups of prime power order. London Mathematical Society Monographs. New Series, **27**. Oxford University Press, Oxford, 2002.
- [15] C.R. Leedham-Green and Susan McKay. On the classification of  $p$ -groups of maximal class. *Quart. J. Math. Oxford Ser. (2)* **35**, 293–304, 1984.
- [16] C.R. Leedham-Green and M.F. Newman. Space groups and groups of prime-power order I. *Archiv der Mathematik* **35**, 193–203, 1980.
- [17] M.F. Newman. Groups of prime-power order. Groups—Canberra 1989, 49–62, Lecture Notes in Math. **1456**, Springer, Berlin, 1990.
- [18] M.F. Newman and E.A. O’Brien. Classifying 2-groups by coclass. *Trans. Amer. Math. Soc.* **351**, 131–169, 1999.
- [19] E.A. O’Brien. The  $p$ -group generation algorithm. *J. Symbolic Comput.*, **9**, 677–698, 1990.
- [20] E.A. O’Brien. Isomorphism testing for  $p$ -groups. *J. Symbolic Comput.* **17**, 133–147, 1994.

Institut Computational Mathematics  
 Universität Braunschweig  
 38016 Braunschweig  
 Germany  
 eick@tu-bs.de

School of Mathematical Sciences  
 Queen Mary, University of London  
 Mile End Road London E1 4NS  
 UK  
 c.r.leedham-green@qmul.ac.uk

Mathematical Sciences Institute  
 Australian National University  
 ACT 0200  
 Australia  
 newman@maths.anu.edu.au

Department of Mathematics  
 University of Auckland  
 Auckland  
 New Zealand  
 obrien@math.auckland.ac.nz