PROBLEMS FOR GUS LEHRER'S NZMRI LECTURES, 2010

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The problems range in difficulty from routine to unsolved research problems. The latter will be marked with a *.

Problem 1. Let $a_1, a_2, \ldots, a_n$ be indeterminates over the complex numbers $\mathbb{C}$, and define polynomials $b_1, b_2, \ldots, b_n$ in the a_i by $t^n + b_1 t^{n-1} + \cdots + b_n$. The b_i are the "elementary symmetric functions" in the a_i. Let $Q(a_1, a_2, a_3) = (a_1 - a_2)^2(a_2 - a_3)^2(a_3 - a_1)^2$. Note that Q is symmetric in the a_i. Given that there is a unique polynomial Δ such that $\Delta(b_1, b_2, b_3) = Q(a_1, a_2, a_3)$, show that $\Delta(0, b_2, b_3) = 4b_2^2 + 27b_3^2$. Deduce that the polynomial $t^3 + \alpha t + \beta = 0$ (where $\alpha, \beta \in \mathbb{C}$) has distinct roots if and only if $4\alpha^2 + 27\beta^3 \neq 0$.

Problem 2. Suppose P is a skein invariant of oriented links with parameters x, y, z. Show that if $P(O) = 1$, where O is the unknotted circle, then
(i) $P(O \amalg O) = -\frac{x+y}{z}$, where $O \amalg O$ denotes two unlinked circles.
(ii) $P(\ell_c) = \frac{xy+y^2-z^2}{xz}$, where ℓ_c is two linked unknotted circles.
(iii) $P(\text{trefoil}) = -\frac{y^2+2xy-z^2}{x^2}$.

Problem 3. For $w \in \mathrm{Sym}_n$ let $\ell(w)$ denote its minimal length as a product of simple transpositions s_i. If $w \in \mathrm{Sym}_n$ is equal to $s_{i_1} \ldots s_{i_\ell}$ with $\ell = \ell(w)$, call the given expression for w *reduced*.

Let $H_n(q)$ be the Hecke algebra as defined in the lectures. Assume (as stated in lectures) that if $w = s_{i_1} \ldots s_{i_\ell}$ is a reduced expression for w, then $T_w := T_{i_1} \ldots T_{i_\ell}$ is a well defined element of $H_n(q)$ (i.e. is independent of the reduced expression). Show by induction on $\ell(w)$ that any product of the T_i is a linear combination of $T_w's$.

Problem 4. Show that (i) If a link invariant P factors through $R[B_n]/\langle(\sigma_i - X)(\sigma_i - Y)\rangle$ (for each n) and $XY \neq 0$ then P factors through $H_n(q)$, where

$$q = -i\left(\frac{X}{Y}\right)^{\frac{1}{2}}$$

(ii) If P satisfies a skein relation $xP(L^+) + yP(L^-) + zP(L^0) = 0$, then P factors through $R[B_n]/\langle(x\sigma_i^2 + z\sigma_i + y)\rangle$.

(iii) If P satisfies a skein relation with parameters $x = \ell, y = \ell^{-1}, z = m$, then P factors through $H_n(q)$, where

$$q = -i\left(\frac{m^2}{2} - 1 + \frac{1}{2}m\sqrt{(}m^2 - 4)\right)^{\frac{1}{2}}.$$

(iv) Deduce that the Jones-Homfly invariant factors through $H_n(q)$, in the notation of the lectures.

Problem 5.*(Unsolved) Determine whether the group homomorphism $B_n \longrightarrow H_n(q)$, where B_n is the n-string braid group, is injective. This is true for $n=3$, but unsolved even for $n=4$.

Problem 6. Determine for which values of q the Temperley-Lieb algebra $TL_3(q)$ is semisimple (a sum of matrix algebras). What is its structure when it (a) is, and (b) is not, semisimple?

Problem 7. (Degeneration of the trace on $H_n(q)$ to $TL_n(q)$). Let τ be the Jones-Ocneanu trace : $H_n(q) \longrightarrow \mathbb{C}(q,z)$.

(i) Let $A_q = \sum_{w\in \mathrm{Sym}_3}(-q)^{-\ell(w)}T_w$ be the 'quantum alternator' introduced in lectures. Show that

$$\tau(A_q) = 1-(q^{-1}+2q)z+(q^{-2}+q^{-4})z^2.$$

(ii) Deduce that $\tau(A_q)=0$ if and only if $z=q$ or $z=\frac{q^2}{q+q^{-1}}$.

(iii) Show that if $z=\frac{q^2}{q+q^{-1}}$, the resulting form $\bar\tau$ on $TL_n(q) := H_n(q)/\langle A_q\rangle$ satisfies $\bar\tau(hC_n) = -\bar\tau(h)(q+q^{-1})^{-1}$, where $h\in TL_n(q)$ and C_n is the 'Kazhdan-Lusztig' generator of $TL_n(q)$.

(iv) Show that the Jones-Homfly link invariant descends, via the composite

$$B_n \longrightarrow H_n(q) \longrightarrow TL_n(q)$$

to the invariant V, where

$$V(\beta) = (q+q^{-1})^{n(\beta)-1}q^{-3e(\beta)}\bar\tau(\beta).$$

(v) Show that V satisfies a skein relation with $x=iq, y=-iq^{-1}, z=i(q-q^{-1})$.

Problem 8. Let $q\in\mathbb{C}$, and consider $A := H_2(q)$ as a (two-dimensional) $\mathbb{C}$-algebra.

(i) Show that the space of all traces (not necessarily normalised) on A may be identified with $\mathbb{C}^2$.

(ii) Show that the space of non-degenerate forms is $\mathbb{C}^2$ minus two lines if A is semisimple, and is $\mathbb{C}^2$ minus one line if not.

Problem 9. (For this problem, you may need to read some extra material on cellular algebras, if the lectures do not cover all terms used in the question)

(i) Compute the displacements of all diagrams in TL_3 and verify that the displacement is equal to the minimal length of a diagram in the generators f_i.

(ii) For the same case, write down the matrix representations of all diagrams in the two cell representations W_1 and W_3 of TL_3.

(iii) Show that if there is a non-zero module homomorphism $W_3 \longrightarrow W_1$ then q is a root of unity.

(iv) Show that if q is not a root of unity then W_1 and W_3 are simple modules.

Problem 10. Let q^2 be a primitive fourth root of unity (e.g. $q^2=i$). Compute the element $E=\sum_{d\in M_3}h_d(q)\cdot d\in TL_3(q)$. Prove that $f_iE=Ef_i=0$ for all i, and deduce that $E^2=E$.

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