Background Reading

- M. Culler and P. Shalen, Varieties of group representations and splitting of 3-manifolds, Annals of Math 117 (1983).
- P. Shalen, Representations of 3-manifold groups, Handbook of Geometric Topology. (2002)
- M. Culler, C. McA. Gordon, J. Luecke and P. Shalen, Dehn surgery on knots.
- C. Maclachlan and A. W. Reid, Arithmetic of Hyperbolic Manifolds, Springer.
- T. Chinburg, A. W. Reid and M. Stover, Azumaya algebras and canonical curves, arXiv 1706.00952.

From Lecture 2

Examples:(1) The figure-eight knot complement.

$\Gamma = \pi_1(S^3 \setminus K) =< a, b | waw^{-1} = b, w = ab^{-1}a^{-1}b >$

Interested in components containing irreducible representations. So normalize (i.e. conjugate):

$$\rho(a) = \begin{pmatrix} x & 1 \\ 0 & 1/x \end{pmatrix} \text{ and } \rho(b) = \begin{pmatrix} x & 0 \\ r & 1/x \end{pmatrix}.$$

Evaluate ρ on the relation (i.e. $wa - bw = 0$) and we obtain:

$$\begin{pmatrix} 0 & p(x,r)/x^2 \\ -rp(x,r)/x^2 & 0 \end{pmatrix}$$

where

$p(x, r) = rx^4 - x^4 + r^2x^2 - 3rx^2 + 3x^2 + r - 1.$

Converting to traces

Set $z = x + x^{-1} = \chi_\rho(a) = \mathrm{tr}(\rho(a)) = \chi_\rho(b)$ and
$T = \chi_\rho(ab^{-1}) = 2 - r$.

Converting $p(x, r)$ into a polynomial in z and T:

$P(z, T) = z^2(2-T) - z^2 + (2-T)^2 - 5(2-T) + 5 = (1-T)z^2 + T^2 + T - 1$

So we have an algebraic set cut out by:

$$z^2(T-1) = T^2 + T - 1$$

In particular, this determines a cubic curve.

A further change of co-ordinates makes it clearer what this curve is, and indeed that it is irreducible.

Multiply both sides by $(T-1)$ and set $Y=(z-1)T$ gives:

$$Y^2=(T-1)(T^2+T-1)$$

Note the RHS is cubic with distinct roots and so we deduce that there is a unique component containing the character of an irreducible representation. In this case is a genus 1 curve.

From Lecture 3

(2) The knot 5_2

$\Gamma = \pi_1(S^3 \setminus K) =< a, b | waw^{-1} = b, w = a^{-1}ba^{-1}b^{-1}ab^{-1} >$

As above set:

$$\rho(a) = \begin{pmatrix} x & 1 \\ 0 & 1/x \end{pmatrix} \text{ and } \rho(b) = \begin{pmatrix} x & 0 \\ r & 1/x \end{pmatrix}.$$

Evaluate ρ on the relation $wa - bw = 0$ and simplifying produces a unique component contain the character of an irreducible representation described as the vanishing set of:

$$p(z, T) = z^2(T - T^2) + T^3 + T^2 - 2T - 1.$$

This can be put in the form of a hyperelliptic curve:

$$Y^2 = (T^2 - T)(T^3 + T^2 - 2T + 1).$$

(3) The knot 7_4

$\Gamma = \pi_1(S^3 \setminus K) =< a, b | aw^2 = w^2 b, w = ab^{-1}ab^{-1}a^{-1}ba^{-1}b >$
Repeating the above we get 2 curves containing the characters of irreducible representations described as the vanishing set of.

$$(-1+2T^2+T^3-T^2z^2)(1+4T-4T^2-T^3+T^4-2Tz^2+3T^2z^2-T^3z^2).$$

Using Snap, the component containing the character of the faithful discrete representation can shown to be cut out by the first factor.

(4) $K = (-2, 3, 7)$-Pretzel knot

$\Gamma = \pi_1(S^3 \setminus K) = < a, b | aab^{-1}aabbabb >$

As above conjugate:

$$\rho(a) = \begin{pmatrix} x & 1 \\ 0 & 1/x \end{pmatrix} \text{ and } \rho(b) = \begin{pmatrix} y & 0 \\ r & 1/y \end{pmatrix}.$$

Evaluate ρ on the relation and convert to traces (with co-ordinates):

$$P = \chi_\rho(a), \;\; Q = \chi_\rho(b) \text{ and } R = \chi_\rho(ab).$$

We find:

$$P = \frac{Q}{(Q^2 - 1)} \text{ and } R = \frac{(1 - 2Q^2)}{Q^2(Q^2 - 1)}$$

i.e. P, R are rational functions of Q, so X_0 (actually the smooth projective model) in this case is $\mathbb{CP}^1$.

From Lecture 4

Theorem 1 (Chinburg-R-Stover)

Let K be a hyperbolic knot and suppose that $\Delta_K(t)$ satisfies:

$(\star)$ *for any root z of $\Delta_K(t)$ and w a square root of z, we have an equality of fields: $\mathbb{Q}(w) = \mathbb{Q}(w + w^{-1})$.*

Then there exists a finite set S of rational primes p so that if some prime $\mathcal{P}$ of k_r ramifies B_r then $\mathcal{P}|p$ for some $p \in S$.

Remarks:(1) When $\Delta_K(t) = 1$ then $S = \emptyset$ and so B_r as above is unramified at all finite places.

(2) The figure-eight knot

$\Delta_K(t) = t^2 - 3t + 1$, and so has roots

$$z = \frac{3 \pm \sqrt{5}}{2},$$

and $z = (\pm w)^2$, where

$$w = \frac{1 \pm \sqrt{5}}{2}.$$

Then $w + 1/w = \pm\sqrt{5}$,

$\mathbb{Q}(w) = \mathbb{Q}(w + w^{-1})$ in this case. Hence $(\star)$ holds.

In this case $S = \{2\}$.

(3) The knot 7_4

Exercise: $\Delta_K(t) = 4t^2 - 7t + 4$ does not satisfy $(*)$.

(4) $(-2, 3, 7)$-Pretzel knot

$$\Delta_K(t) = t^{10} - t^9 + t^7 - t^6 + t^5 - t^4 + t^3 - t + 1$$

If z is a root and $w^2 = z$, can check:

w satisfies an irreducible polynomial of degree 20

but $T = w + 1/w$ satisfies a degree 10 polynomial:

$$1 + 12 * T^2 + 31 * T^4 + 27 * T^6 + 9 * T^8 + T^{10}.$$

Hence condition $(\star)$ does not hold.

An intriguing connection:

Conjecture *If K is a hyperbolic L-space knot it is never Azumaya positive (i.e. $(\star)$ does not hold.)*

e.g. The $(-2, 3, 7)$-Pretzel knot is an L-space knot.

Some positive evidence:

Theorem 1

Suppose that K is a hyperbolic L-space knot for which the canonical component C is the unique component of $X(K)$ containing the character of an irreducible representation. Then $(\star)$ does not hold.