NZMRI Summer School 2018
Nelson NZ
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Flag varieties
Lecture 2

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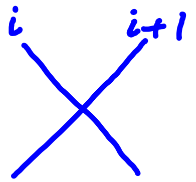
Lecture 2

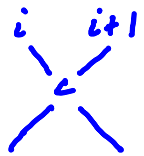
Last time:

(a) Generators and relations for S_n



$$= s_i = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & 1 & & \\ & & & 1 & 0 & & \\ & & & & & 1 & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$$

(b) Generators and relations for $GL_n(\mathbb{C})$



$$= y_i(c) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & c & 1 & & \\ & & & 1 & 0 & & \\ & & & & & 1 & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$$

Flag varieties G/B

$$G = GL_n(\mathbb{C})$$
$$\cup\!\!\mid$$
$$B = \{\text{upper triangular matrices}\}$$
$$\cup\!\!\mid$$
$$T = \{\text{diagonal matrices}\}$$

$$W = \{\text{permutation matrices}\} = S_n$$

$$G = \bigsqcup_{w \in W} BwB$$

The *flag variety* is $G/B = \{ gB \mid g \in G \}$

The points of G/B in BwB:

$$BwB = \{ y_{i_1}(c_1) y_{i_2}(c_2) \cdots y_{i_\ell}(c_\ell) B \mid c_1, c_2, \ldots, c_\ell \in \mathbb{C} \}$$

if $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ is a reduced word.

$n=2$, $\mathbb{P}^1 = GL_2(\mathbb{C})/B$

$W = \{ \,||,\ X\, \} = \{ 1, s_1 \}$

$G = B \sqcup Bs_1B$

B is a point in G/B

$Bs_1B = \{ y_1(c)B \mid c \in \mathbb{C} \}$

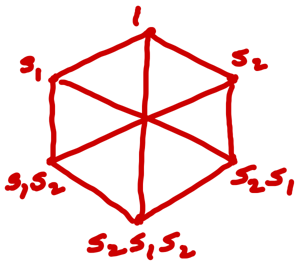
If $\mathrm{Card}(\mathbb{C}) = q$ then $\mathrm{Card}(G/B) = 1+q$.

$$\mathrm{Card}(G/B)\Big|_{q=1} = \mathrm{Card}(W).$$

$n=3$: $GL_3(\mathbb{C})/B$

$W = \{ \,|||,\ X|,\ |X,\ XX,\ XX,\ * \,\}$

$= \{ 1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1 \}$



B is a point in G/B

$Bs_1B = \{ y_1(c)B \mid c \in \mathbb{C} \}$

$Bs_2B = \{ y_2(c)B \mid c \in \mathbb{C} \}$

$Bs_1s_2B = \{ y_1(c_1)y_2(c_2)B \mid c_1, c_2 \in \mathbb{C} \}$

$Bs_2s_1B = \{ y_2(c_1)y_1(c_2)B \mid c_1, c_2 \in \mathbb{C} \}$

$Bs_2s_1s_2B = \{ y_2(c_1)y_1(c_2)y_2(c_3)B \mid c_1, c_2, c_3 \in \mathbb{C} \}$

$GL_3(\mathbb{C}) = B \sqcup Bs_1B \sqcup Bs_2B \sqcup Bs_1s_2B \sqcup Bs_2s_1B \sqcup Bs_2s_1s_2B$

If $\mathrm{Card}(\mathbb{C}) = q$ then $\mathrm{Card}(G/B) = 1 + 2q + 2q^2 + q^3 = (1+q)(1+q+q^2)$

Schubert Cells and Schubert varieties

The Schubert cells are BwB in G/B

The Schubert varieties are $\overline{BwB}$ in G/B

Proposition (Bott-Samelson resolution and Bruhat-Chevalley order)

Let $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ be a reduced word.

(a) $BwB = Bs_{i_1}B \cdot Bs_{i_2}B \cdot \cdots \cdot Bs_{i_\ell}B$

(b) $\overline{BwB} = \overline{Bs_{i_1}B} \cdot \overline{Bs_{i_2}B} \cdot \cdots \cdot \overline{Bs_{i_\ell}B}$

(c) $\overline{BwB} = \bigsqcup_{v \leq w} BvB$, where $v \leq w$ means v is a subword of w.

Example of (a) for $n=3$

$$Bs_1B \cdot Bs_2B = \{y_1(c)B \mid c \in \mathbb{C}\} \cdot Bs_2B$$

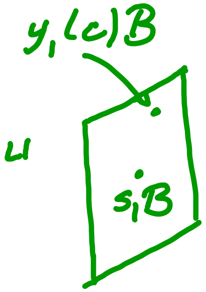
$$= \{y_1(c)B \cdot Bs_2B \mid c \in \mathbb{C}\} = \{y_1(c)Bs_2B \mid c \in \mathbb{C}\}$$

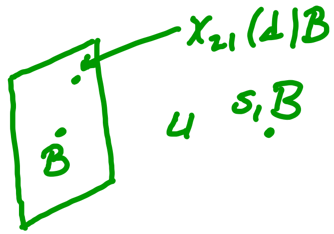
$$= \{y_1(c_1)\, y_2(c_2) B \mid c_1, c_2 \in \mathbb{C}\} = Bs_1s_2B$$

$\mathbb{P}^1 = GL_2(\mathbb{C})/B$

Key computation: $x_{21}(d) = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} = \begin{pmatrix} d^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d & 1 \\ 0 & -d^{-1} \end{pmatrix}$

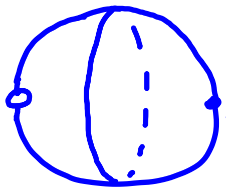
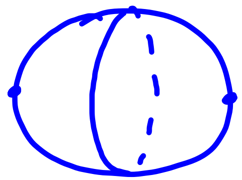
So $x_{21}(d)B = y_1(d^{-1})B$ if $d \neq 0$.

$\mathbb{P}^1 = B \cup Bs_1B = B \cup \{y_1(c)B \mid c \in \mathbb{C}\} = \overset{B}{\cdot} \cup$ 

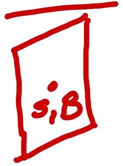
$= \{x_{21}(d)B \mid d \in \mathbb{C}\} \cup s_1B =$ 

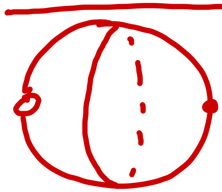
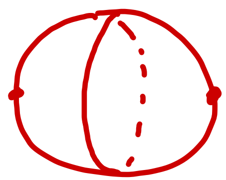
$\cup \ \overset{s_1B}{\cdot}$

So

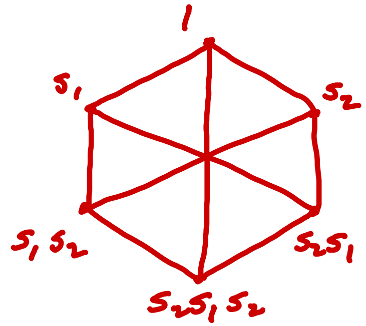
$\mathbb{P}^1 = \overset{B}{\cdot} \cup$  s_1B $= {}^{B}$  $\cup \ \overset{s_1B}{\cdot}$ $= {}^{B}$  s_1B

So

$\overline{Bs_1B} = \overline{}$ 

$= \overline{}$  s_1B $= {}^{B}$  s_1B $= B \cup Bs_1B$

Closures in $Gl_3(\mathbb{C})/B$

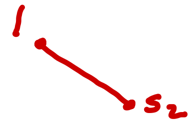


$$Gl_3(\mathbb{C}) = B \sqcup Bs_1B \sqcup Bs_2B \sqcup Bs_1s_2B \sqcup Bs_2s_1B \sqcup Bs_2s_1s_2B$$

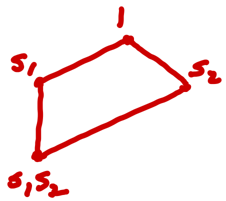
$\overline{B} = B$ is a point in G/B

$\overline{Bs_1B} = B \sqcup Bs_1B$

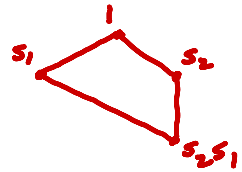
$\overline{Bs_2B} = B \sqcup Bs_2B$



$\overline{Bs_1s_2B} = B \sqcup Bs_1B \sqcup Bs_2B \sqcup Bs_1s_2B$



$\overline{Bs_2s_1B} = B \sqcup Bs_1B \sqcup Bs_2B \sqcup Bs_2s_1B$



$$\overline{Bs_1s_2B} = \overline{Bs_1B \cdot Bs_2B} = (B \sqcup Bs_1B) \cdot (B \sqcup Bs_2B)$$

Bruhat-Chevalley-Ehresmann order In G/B

$$\overline{BwB} = \bigsqcup_{v \leq w} BvB,$$

where $v \leq w$ means v is a subword of $w = s_{i_1} \cdots s_{i_\ell}$ (a reduced word for w).

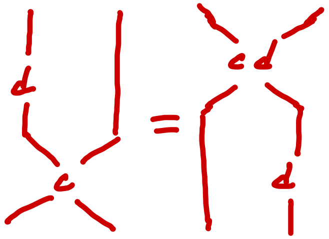
The moment graph of G/B

The **moment graph** of G/B has

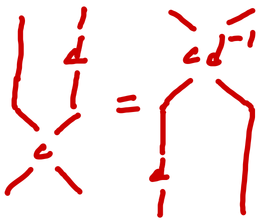
vertices: the T-fixed points of G/B,

labeled edges: the 1-dimensional T-orbits in G/B

Key computations:



$$\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} cd & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \qquad h_j(d)y_j(c) = y_j(cd)h_{j+1}(d)$$



$$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} cd^{-1} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \qquad h_{j+1}(d)y_j(c) = y_j(cd^{-1})h_j(d)$$

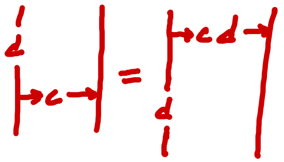
T-fixed points: Let $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ be a reduced word. Then

$$wB = s_{i_1}s_{i_2}\cdots s_{i_\ell}B = y_{i_1}(0)y_{i_2}(0)\cdots y_{i_\ell}(0)B \qquad \text{for } w \in W$$

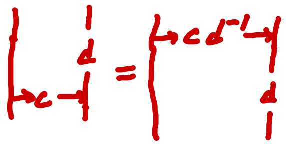
are the T-fixed points in G/B.

One dimensional T-orbits

Key computation:



$$\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & cd \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \qquad h_i(d)\,x_{ij}(c) = x_{ij}(cd)\,h_i(d)$$



$$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & cd^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \qquad h_j(d)\,x_{ij}(c) = x_{ij}(cd^{-1})\,h_j(d)$$

$$h_i(d)\big(x_{ij}(c)wB\big) = x_{ij}(cd)h_i(d)wB = x_{ij}(cd)\,w\,h_{w(i)}(d)B = x_{ij}(cd)wB$$

$$h_j(d)\big(x_{ij}(c)wB\big) = x_{ij}(cd^{-1})h_j(d)wB = x_{ij}(cd^{-1})\,w\,h_{w(j)}(d)B = x_{ij}(cd^{-1})wB$$

$$h_k(d)\big(x_{ij}(c)wB\big) = x_{ij}(c)h_k(d)wB = x_{ij}(c)\,w\,h_{w(j)}(d)B = x_{ij}(c)wB$$

<u>1-dimensional T-orbits</u>: Let $i,j \in \{1,\dots,n\}$ with $i<j$.

$$\{x_{ij}(c)wB \mid c \in \mathbb{C}^\times\} \text{ is a 1-dimensional } T\text{-orbit}$$

and these are <u>all</u> the 1-dimensional T-orbits.

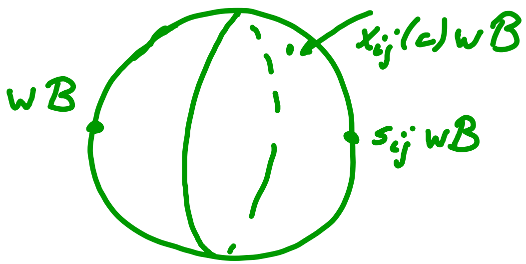
Since $x_{ij}(c)wB = y_{ij}(c^{-1})wB$ for $c \neq 0$, then

(where $y_{ij}(c^{-1}) = x_{ij}(c^{-1})s_{ij}$, with s_{ij} the transposition switching i and j.)

$$\lim_{c \to 0} x_{ij}(c)wB = wB \quad \text{and} \quad \lim_{c \to \infty} x_{ij}(c)wB = \lim_{c^{-1} \to 0} y_{ij}(c^{-1})wB = s_{ij}wB$$

so the T-fixed points wB and $s_{ij}wB$

are limit points of the orbit $\{x_{ij}(c)wB \mid c \in \mathbb{C}^\times\}$

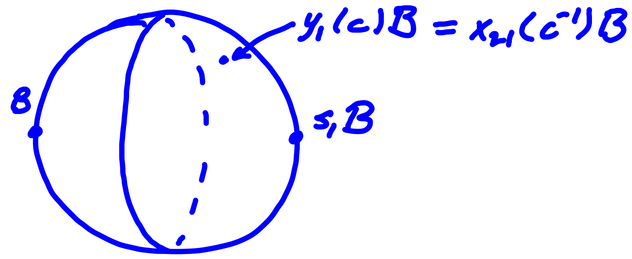


The *moment graph* of G/B has

vertices: W

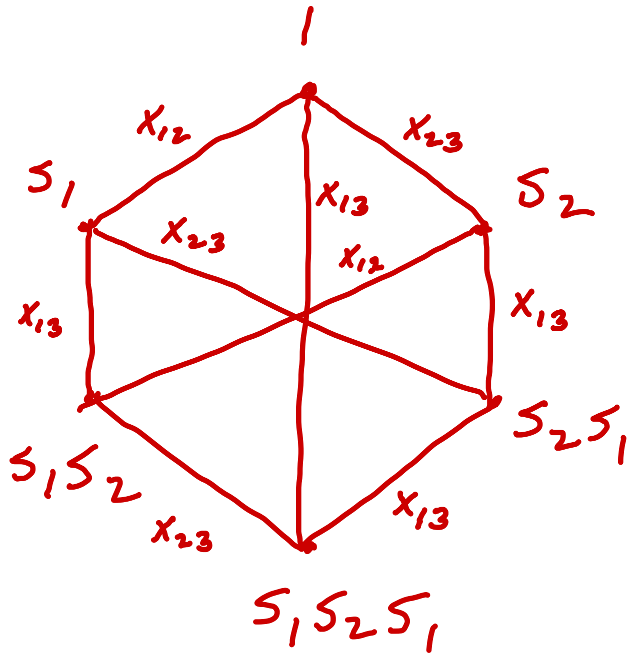
labeled edges: $w \xrightarrow{x_{ij}} s_{ij}w$

The moment graph for $\mathbb{P}^1$

$\mathbb{P}^1 = GL_2(\mathbb{C})/B = B \sqcup Bs_1B =$ 

The moment graph for $GL_3(\mathbb{C})/B$

$$GL_3(\mathbb{C})/B = \begin{array}{cc} \multicolumn{2}{c}{B} \\ Bs_1B & Bs_2B \\ Bs_1s_2B & Bs_2s_1B \\ \multicolumn{2}{c}{Bs_2s_1s_2B} \end{array}$$



Moment graphs of Schubert varieties $\overline{BwB}$ in $GL_3(\mathbb{C})/B$

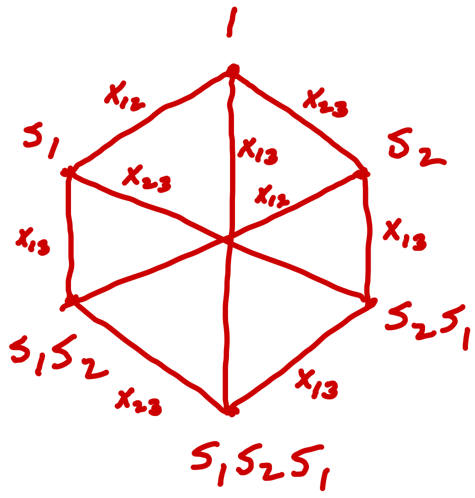
$\overline{Bs_1s_2s_1B} = GL_3(\mathbb{C})/B =$

B

$Bs_1B \quad Bs_2B$

$Bs_1s_2B \quad Bs_2s_1B$

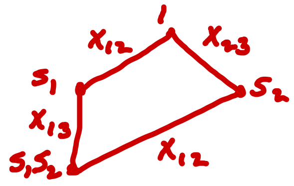
$Bs_2s_1s_2B$



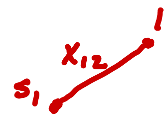
B

$\overline{Bs_1s_2B} = \quad Bs_1B \quad Bs_2B$

Bs_1s_2B



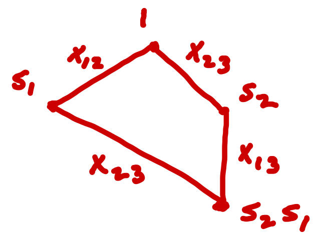
$\overline{Bs_1B} = B \cup Bs_1B$



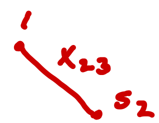
B

$\overline{Bs_2s_1B} = \quad Bs_1B \quad Bs_2B$

Bs_2s_1B



$\overline{Bs_2B} = B \cup Bs_2B$



$\overline{B} = B$