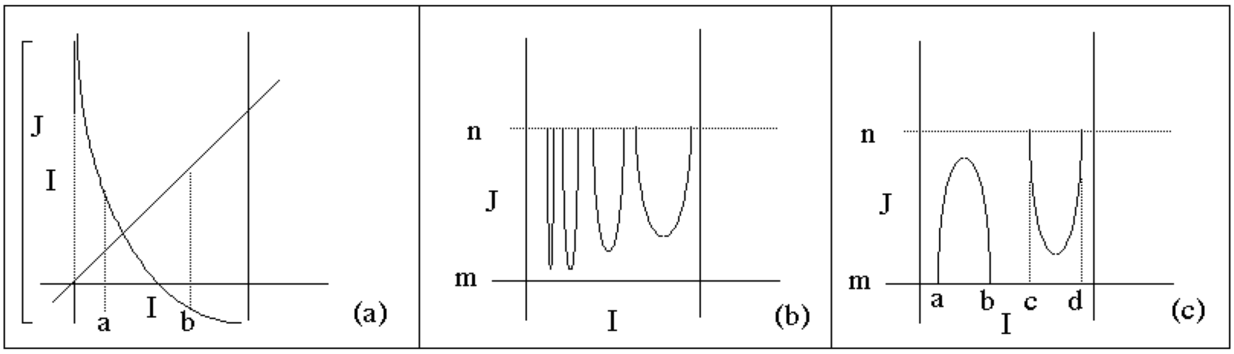
26.745 The Compleat Sarkovski's Theorem

A diagrammatic sketch



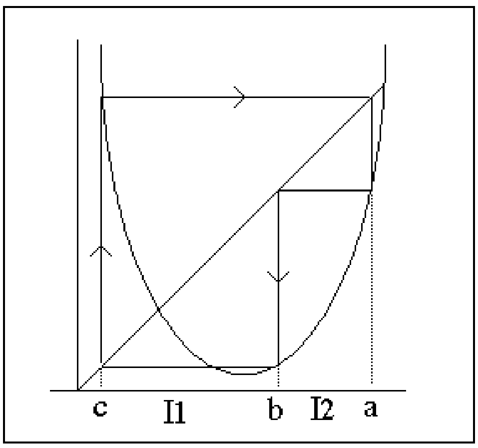
Sarkovski's theorem : Let $f : \mathbb{R} \to \mathbb{R}$ be continuous with a periodic point of principal period k.
If $k>l$ in the ordering $3 > 5 > 7 > ... > 3.2^n > 5.2^n > ... > 2^n > 2^{n-1} > ... > 4 > 2$
then f also has a periodic point of period l.

A : Assumptions : I have sketched proofs of Devaneys assumptions for completeness of discussion.
My second proof is sphagetti!! I bet you can make it much more concise.

[1] Fig 1(a). I, J closed intervals : $I \subseteq J$ but $f(I) \supseteq J$ then f has a fixed point in I.
Since $f(I) \supseteq J \supseteq I$, we can pick a, b $\in$ I : $f(a) \leq \min (i \in I)$, $f(b) \geq \max (i \in I)$.
Then $g(x) = f(x) - x$ has $g(a) < 0 < g(b)$, so by the intermediate value theorem $\exists$ c : $g(c) = f(c) - c = 0$.

[2] Fig 1(b,c). I, J closed intervals : $f(I) \supseteq J$ then there is a subinterval H of I : $f(H) = J$.
Let $O = f^{-1}(\mathrm{int}(J))$. O is a countable union of open intervals (open set in $\mathbb{R}$). These have end points in the boundary of $O = f^{-1}(\partial(J)) = f^{-1}(\{m,n\})$. If any interval has one of each of m, n we choose this one.
If not we end in contradiction. All intervals are *cups* or *caps*, as in (c). We must have both types.
If we have only cups, their inf must be m or $f(I) \not\supseteq J$, but then we have a (countable) set of cups, each with min > m having inf m (b). These contradict continuity because they determine a subsequence in compact I whose limit is a discontinuity, since the cups range remains extensive. If we do have both types, there must be a pair equivalent to fig 1(b), with no other cups and caps in between (i.e. f outside J). But by the Intermediate value theorem, between b & c there must be every intermediate value between m and n for f to be continuous.

B : The simple case period 3 has periods of all orders
Proof : Pick the case a > b > c and $f(a) = b$, $f(b) = c$, $f(c) = a$ as shown below.



From the way a, b, c permute, we have $f(I_2) \supseteq I_1, f(I_1) \supseteq I_2 \cup I_1$. (The minimum makes the second <u>not</u> equality).

We can say I_2 covers I_1 and I_1 covers both I_1 and I_2. I.e. in diagramatic form $I1 \rightleftarrows I2$.

Note : while the original points were period 3 this gives the intervals minimum period 2 !!!

Now by [2], $f(I_1) \supseteq I_1 \supseteq A_i$ we can make a sequence of sets

$$A_{n-2} \subseteq \dots A_1 \subseteq A_0 = I_1 \quad : f(A_{n-2}) = A_{n-3},\ f(A_1) = A_0 = I_1.$$

$$\text{i.e. } An\text{-}2 \xrightarrow{f} An\text{-}3 \xrightarrow{f} \dots \rightarrow A1 \xrightarrow{f} A0$$

Now since $f^{n-2}(A_{n-2}) = I_1$ and $f(I_1) \supseteq I_2$, we have $f^{n-1}(A_{n-2}) \supseteq I_2$ so there is $A_{n-1} : f^{n-1}(A_{n-1}) = I_2$.

But $f(I_2) \supseteq I_1$, so $f^n(A_{n-1}) \supseteq I_1 \supseteq A_{n-1}$. Hence by [1] A_{n-1} has a fixed point.

Finally this has true principal period n since $I1 \rightarrow I1 \rightarrow \dots \rightarrow I1 \rightarrow I0$

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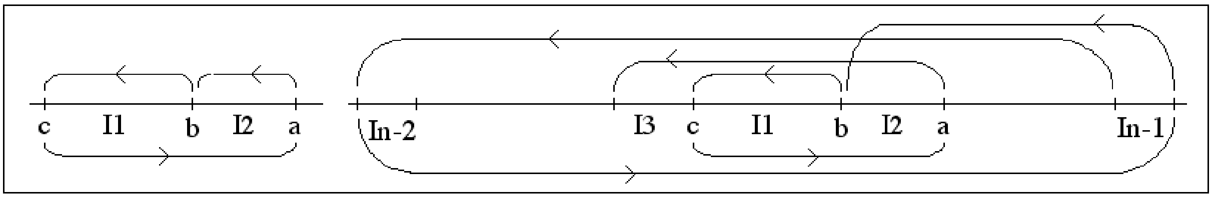
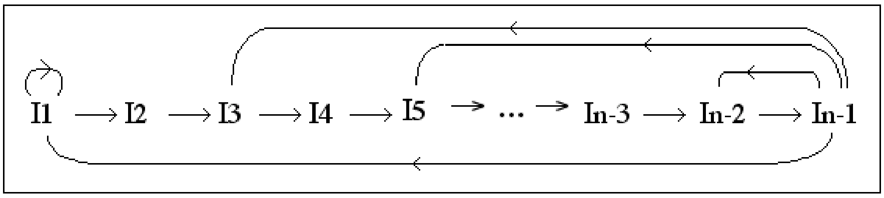
C : A sketch of the full theorem.



(a) Suppose n is an odd number. Then a careful analysis of the structure of the mapping confirms that it must have the above elaboration of the pattern for period 3, or its mirror image. This arises from the fact that some must change sides of I_1 and some must not (see Devaney).

Discussion on paragraph 2 p 64.
Suppose I1-I2-...-Ik-I1 is the shortest path from I1 to I1. Then one of I1-I2-...-Ik-I1 and I1-I2-...-Ik-I1-I1 has an ODD number of steps. Moreover the first has k steps and the second k+1.
If k < n-1 then these have < n steps and one is odd. Using the fact that f^m I1 contains I1, we thus have a fixed point x of f^m x where m=k or k+1 and m odd. But if k < n-1, we have m<n and odd. I.e. we have an odd period m point m<n. The prime period of x must be either m, an odd factor of m or 1. The first two cases will contradict the hypothesis that there are no odd periods of prime period less than n. Also x cannot be prime period 1 because f x is in I2 and the only point of intersection of I1 and I2 is the end point, which has prime period n > m. Hence k cannot be less than n-1, so k IS n-1.

Examination of this pattern of coverings gives the following diagram :



(i) All periods larger than n are gained as before going round the lower loop and adding multiple circuits of I_1.

(ii) Even periods 2.1, 2.3, 2.5 etc. up to n-1 arise from going round the upper loops. These include all cases.

(b) n is even. Then if some points swap sides of I_1 there is a period 2 point by the same agruments as before using $I_{n-1} \rightleftharpoons I_{n-2}$. If all swap sides the same is true by inspection.

(i) If $n = 2^m$. Let $k = 2^h$ $h < m$, then if $g = f^{k/2} = f^{2^{h-1}}$ but f has period 2^m so g has period $\dfrac{2^m}{2^{h-1}} = 2^{m-h+1}$ even hence by (b) g has period 2, so f has period $2 \cdot 2^{h-1} = k$

(ii) If $n = o.2^m$ then it can be reduced to the above cases. Let $g = f^{2^m}$ then g has odd period so g has every odd period $p > o$ giving f every period $p.2^m > o.2^m$. f also has every power of 2 (exercise).