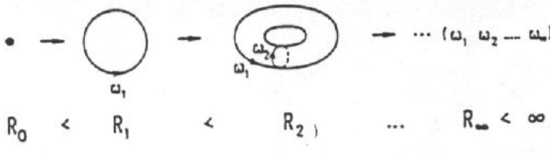
# The Quasi-periodic Route to Chaos and the Circle Map

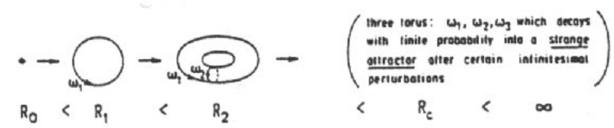
This handout is to complement the two reference handouts 'Maps of the Circle' from Devaney (D) and 'The transition from quasi-periodicity' from Schuster (S) and to make a very wooly introduction by me more easily accessible. Note that my paper on Nervous Systems also has a parallel brief introduction (p 14 - 16).

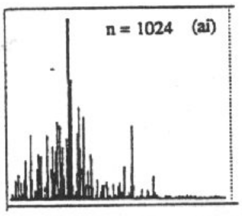
One of the most common and dynamically important ways in which chaos can develop is via the buildup of many linked oscillations. This is seen clearly in eddies in a swift river, the weather, in variations of the heart rhythm, and many experimental systems such as the Benard experiment (convection cells in a fluid heated from below), a viscous fluid between two rolling cylinders, electric conductivity in crystals etc.

The original Landau route to turbulence (S147) involved the buildup of an infinite sequence of independent periodicities by Hopf bifurcations, much in the manner of the period doubling route.

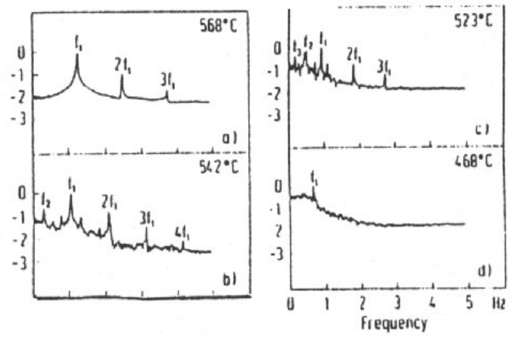


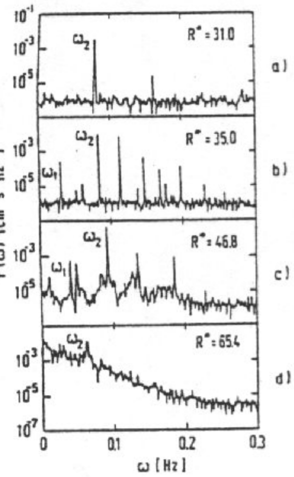

Subsequently Ruelle, Takens & Newhouse discovered that a more direct route could exist after only three bifurcations, via the collapse of the three torus so formed. In (S 148 and 151) are illustrated Fourier transforms of two experimental systems illustrating the formation of 2 and 3 independent periodicities before chaos sets in.




Fast Fourier Trans of Lorentz flow (King P7)




Various model systems can picture something of this behaviour even for a flow on a 2-torus. One example is the simplified model system of a sinusoidally-kicked rotator under constant torque determined by $\Omega$:

$$\theta_{n+1} = \theta_n + \Omega - \frac{K}{2\pi}\sin(2\pi\theta_n) + br_n \mod 1$$

$$r_{n+1} = br_n - \frac{K}{2\pi}\sin(2\pi\theta_n)$$

This can be modelled on computer by plotting in Cartesian coordinates for a shifted r i.e.

$$x_n = (1 + 4r_n)\cos(\theta_n),\quad y_n = (1 + 4r_n)\sin(\theta_n)$$

The computer listing included shows this iteration carried out for varying K, and in (S156) the result is illustrated.* The discrete iteration models the Poincare section of a continuous system on the torus, and hence the phase portrait is a circle. As the parameter K is increased, we see the break up of this circle to form a putative strange attractor. This can be nicely lined up with experimental results such as those from a rotating viscous fluid (S154) and the Benard. exprt (S153) shown.

*

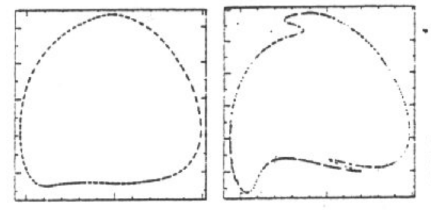
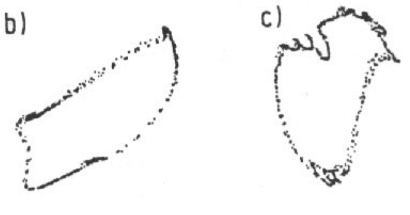
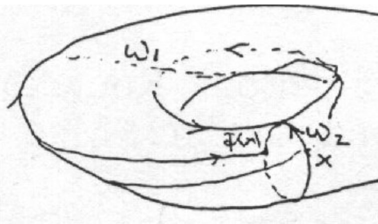


Fig. 104: Break up of a torus as described by the dissipative circle map (eqs. 6.12a, b) (after Hauer, priv. comm.).





If we return for a minute to the flow on the torus, we can see that the Poincare map has a natural winding number represented by $\Omega = \frac{\omega_2}{\omega_1}$, i.e. $\theta_{n+1} = f(\theta_n) = \theta_n + \Omega \bmod 1$ (where $S^1$ is identified with $\mathbb{R}$ wrapped mod 1).
Note that $\Omega$ is rational whenever the two frequencies $\omega_1$ and $\omega_2$ are rationally related ( resonant ).
We can form a simplified model of resonant linkage between these two frequencies (and the breakup of the previous rotator system) by adding a sinusoidal perturbation to the above equation :

$$\theta_{n+1} = f(\theta_n) = \theta_n + \Omega - \frac{K}{2\pi}\sin(2\pi\theta_n) \bmod 1$$

Now we can define an iterative winding number for this map by taking repeated iterations and measuring the average amount of twist. Schuster p 155 puts the definition simply, but in a way which is ambiguous viz :

$$\rho = \lim_{n \to \infty} \frac{f^n(\theta_0) - \theta_0}{n}$$

We will shortly see that Devaney goes about this in a more rigorous way, but for now lets see what the general results look like. As K increases and crosses 1 the form of $f(\theta)$ makes the following transition from 1-1 to non 1-1 form. The dynamics is general to functions making this type of transition. Similar dynamics can be seen in complex julia sets of indifferent points (bifurcating systems) and in the heart rhythm (King fig 14(b)).

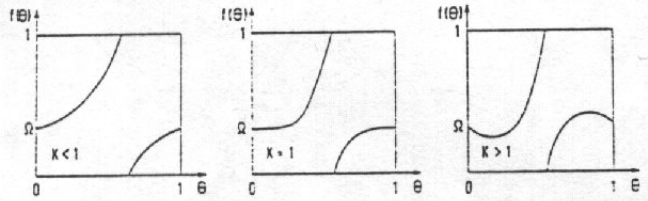
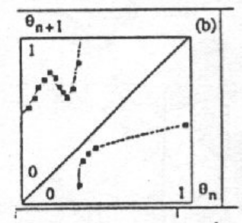
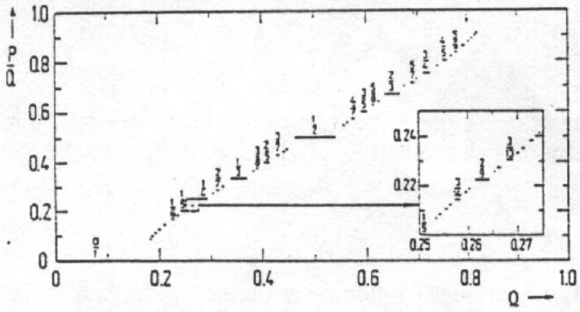



Fig. 105: Variation of the map $f(\theta)$ with the parameter $K$. Note, that for $K > 1$, the map becomes noninvertible.

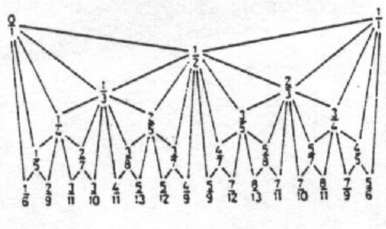



F(θ) for heart rhythm.

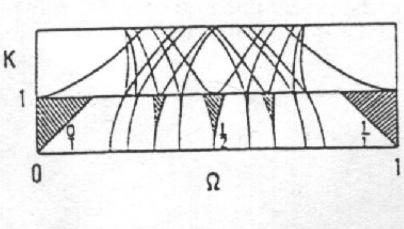
In the range to 0<K<1, the relationship between the actual winding number $\rho$ and the parameter $\Omega$ displays *mode-locking* i.e. near any rational value of $\Omega$ there is a whole interval of values of $\Omega$ for which the dynamics is rationally mode-locked and $\rho$ is fixed at the rational value. We can see this by plotting $\rho$ against $\Omega$ as shown below for the value K=1. This is called the devil's staircase (S160) and consists of an interval for each rational, and a single value for each irrational. At K = 1, the rational values remove all but a measure zero irrational Cantor set.



These numbers are ordered into the Farey tree of rationals (S160). Finally we can plot K against $\Omega$ highlighting the rational intervals, and we see these form spreading tongues. For K > 1 chaos appears. There are two rays retaining the rational mode-locking and chaotic dynamics is densely interwoven with quasi-periodic motion elsewhere (quasi-periodic motion is simple non-periodic motion such as the irrational flow).



Farey Tree.

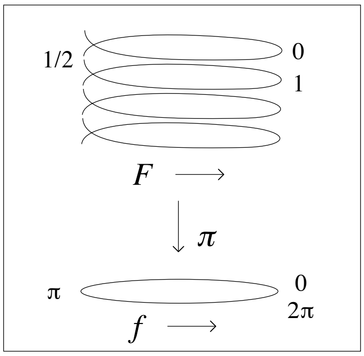


Arnold Tongues and Chaos.

# 26.457 The Circle Map : 2

The two approaches of the handouts on the circle map illustrate a deep cultural rift between the approaches of the physicist and mathematician. Schuster's treatment gives a much better feel of the real world background of the circle map in physical systems, and the link between multi-dimensional continuous dynamics and 1D iterations on the circle, but the development is obscure mathematically. Devaney tends to the other extreme, taking pages to establish that the concept of the winding number is well defined. I will try to give a different emphasis sketching the mechanical aspects, but elaborating the description of the Devil's staircase Cantor function result.

Consider any differentiable homeomorphism $f : S^1 \to S^1$ : It is natural, as Schuster assumes to assume that any such map can naturally be lifted coordinate-wise onto the real line (mod 1) by wrapping around the circle using the map $\pi(x) = e^{2\pi i x}$. The unit interval is then used as a coordinate system wrapped once around the circle (mod 1).




**(1) The Winding number exists is well-defined**

Devaney however goes at length to show how long it actually takes to establish this intuitively obvious idea.

Lift the map $f : S^1 \to S^1$ to a map $F : \mathbb{R} \to \mathbb{R}$ by using the winding function $\pi : \mathbb{R} \to S^1$ $\pi(x) = e^{2\pi i x}$.

A lift is any real map which models f when wound onto $S^1$ : i.e. $\pi \circ F = f \circ \pi$.

**(a) Uniqueness.** We consider the limit $\rho_0(F) = \lim_{n \to \infty} \frac{|F^n(x_o)|}{n}$ where $x_0$ is any starting point.

By the periodicity of the lifting $F(x+1) - (x+1) = F(x) - x$, so $F(x) - id$ has period 1.

Hence $|F^n(x) - F^n(y)| \le |F^n(x) - x - (F^n(y) - y)| + |x - y| \le 1 + |x - y|$. Hence the limit is independent of $x_0$.

Exercise : Show any two lifts differ by an integer (assignment).

We can thus define $\rho = \mathrm{fract}\left( \lim_{n \to \infty} \frac{|F^n(x_o)|}{n} \right)$ which is then independent of the choice of lifting.

This model is realistic on the circle, but it is losing all complete revolutions in the original map on the torus.

**(b) Existence**

**(i) Periodic.** We then verify that in the case of a mapping with a periodic point $f^m(\theta) = \theta$ this limit exists.

Then $F^m(x) = x + k$ for some k and we get $\lim_{j \to \infty} \frac{|F^{jm}(x)|}{jm} = \lim_{j \to \infty} \left( \frac{x}{jm} + \frac{k}{m} \right) = \frac{k}{m}$ .

But we now need to clinch the deal for <u>all</u> n.

Since $F^r$ - id is bounded by M for any r $\lim_{n \to \infty} \frac{|F^{n+r}(x) - F^n(x)|}{n} = \lim_{n \to \infty} \left( \frac{M}{n} \right) = 0$

We can then use this to interpolate all the values of n in the limit (exercise) - notice Devaney has hashed this one!

**(ii) Non-periodic.** Now we attack non-periodic mappings.

Since $F^n(x) - x$ is <u>never</u> an integer, we can squeeze it between two integers : $k_n < F^n(x) - x < k_n + 1$

and hence $k_n < F^n(0) < k_n + 1$, and so $\frac{k_n}{n} < \frac{|F^n(0)|}{n} < \frac{k_n + 1}{n}$ .

By repeating the iteration m times and adding we get $mk_n < F^{mn}(0) < m(k_n + 1)$, and $\frac{k_n}{n} < \frac{|F^{mn}(0)|}{mn} < \frac{k_n + 1}{n}$ .

$$\left| \frac{|F^n(0)|}{n} - \frac{|F^m(0)|}{m} \right| < \left| \frac{|F^n(0)|}{n} - \frac{|F^{mn}(0)|}{mn} \right| + \left| \frac{|F^{mn}(0)|}{mn} - \frac{|F^m(0)|}{m} \right| < \frac{1}{m} + \frac{1}{n}$$

Now $\frac{|F^n(0)|}{n}$ is Cauchy and hence convergent, and so $\rho$ exists for this case also.

**(2) Continuity of ρ wrt ω.** We now establish that ρ varies ε–continuously with $C^0$ δ-small perturbations of f i.e. $\forall\, \varepsilon > 0 \;\; \exists\, \delta : |\, f(\theta) - g(\theta)\,| < \delta \Rightarrow |\, \rho(f) - \rho(g)\,| < \varepsilon$.

Choose n so that $2/n < \varepsilon$, a lift so that $r - 1 < F^n(0) < r + 1$ and δ small enough so that $r - 1 < G^n(0) < r + 1$ then as before $m(r - 1) < F^{mn}(0) < m(r + 1)$, $m(r - 1) < G^{mn}(0) < m(r + 1)$, so $\left| \frac{|F^{mn}(0)|}{mn} - \frac{|G^{mn}(0)|}{mn} \right| < \frac{2}{n} < \varepsilon$ .

**(3) ρ(f) irrational ⇔ f has no periodic points.** We already know ρ is rational if f has periodic points. Suppose f has no fixed points, but ρ is rational. If $\rho(f) = p/q$ then $\rho(f^q) = pq/q = 0 \pmod 1$. So let $\rho = 0$.
Since F has no fixed points assume $F(x) > x$ for all x.

(a) $F^n(0) < 1$ for all n. Then $F^n(0)$ monotonically increasing and bounded above, so convergent. Then the limit point p is fixed since $F(\lim F^n(0)) = \lim F^{n+1}(0) = \lim F^n(0)$ contradiction.

(b) $F^n(0) > 1$ for some n. So $F^{mn}(0) > m$ making $\rho > 1/n$ again contradiction.

**(4) The devil's staircase for the standard circle map.**
We now finally come "full circle and re-examine the particular case :

$$\theta_{n+1} = f_{\omega,\varepsilon}(\theta_n) = \theta_n + 2\pi\omega + \varepsilon \sin(\theta_n)$$

with lift
$$x_{n+1} = F_{\omega,\varepsilon}(x_n) = x_n + \omega + \frac{\varepsilon}{2\pi} \sin(2\pi x_n)$$

Note that $F_{\omega,\varepsilon}(x)$ and hence $F^n{}_{\omega,\varepsilon}(x)$ monotonically increases with ω, so that $\rho_0$ and hence ρ must also for fixed ε. By the above continuity, ρ also varies continuously with ω. Let us fix $\varepsilon > 0$.

**(a) ρ rational.** We now show there is an interval of ω for which ρ remains fixed
Let $\rho = p/q$. Then $F_\omega$ has period q, i.e. we can find k : $F^q{}_{\omega_0}(x_0) = x_0 + k$ ( actually k has to be p )

Now the graph of $F^q{}_{\omega_0}(x)$ must then intersect $x + k$ at $(x_0, x_0 + k)$.

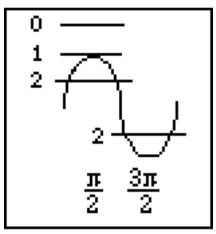
(i) $\frac{\partial}{\partial x} F_{\omega,\varepsilon}(x_o) \neq 1$. By the implicit function theorem there is an interval about ω for which x can be written explicitly as a function of ω, i.e. $x_\omega = g(\omega)$ satisfies $F^q{}_\omega(x_\omega) = x_\omega + k$, and these values of ω also have $\rho = p/q$.

(ii) $\frac{\partial}{\partial x} F_{\omega,\varepsilon}(x_o) = 1$ then since it is analytic it must have a higher order derivative non-zero. An non zero odd order derivative gives an inflection and the result still holds. An even order gives concave up or down giving a one-sided interval in each case.

(b) **ρ irrational.** The proof that each irrational value of ρ has only a single ω is too difficult for Devaney.

The resulting graph is the **Devil's staircase** Cantor function shown in the previous section, which is constant on intervals and yet everywhere continuous. At $\varepsilon = 1$ the intervals fill [0,1] leaving a measure zero Cantor set.

**(5) Dynamics on the tongues.**
Note also the development of widening **mode-locking tongues**, as ε increases as in the previous illustrations. The dynamics of this can be visualized as follows : f has a fixed point when $\sin(\theta) = \frac{-2\pi\omega}{\varepsilon}$ . But this gives 2 solutions for small values reducing to one and then none as the quotient is increased because of the shape of sin(θ) :




We can check that the fixed points have $f' \neq 1$ except at the single bifurcation point "1".
Thus as we move across the central tongue (ω increasing) a saddle-node bifurcation creates a pair of fixed points which move around the circle as ω increases, giving rational mode locking until they meet and annihilate again, thus spanning the interval $\pi/2 < \theta < 3\pi/2$ or $-\frac{\varepsilon}{2\pi} < \omega < \frac{\varepsilon}{2\pi}$ across the tongue. The tongues are not actually linear.