The Funk Transform as a Penrose Transform

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Abstract

The Funk transform is the integral transform from the space of smooth even functions on the unit sphere $S^2 \subset \mathbb{R}^3$ to itself defined by integration over great circles. One can regard this transform as a limit in a certain sense of the Penrose transform from $\mathbb{CP}_2$ to $\mathbb{CP}_2^*$. We exploit this viewpoint by developing a new proof of the bijectivity of the Funk transform which proceeds by considering the cohomology of a certain involutive (or formally integrable) structure on an intermediate space. This is the simplest example of what we hope will prove to be a general method of obtaining results in real integral geometry by means of complex holomorphic methods derived from the Penrose transform.

1 Introduction

In 1917 Radon [R] introduced a transform $f \mapsto \mathcal{R}f$ from a space of suitable real-valued functions on $\mathbb{R}^2$ to the space of functions on the manifold of straight lines in $\mathbb{R}^2$, given by integration:

$$(\mathcal{R}f)(L) = \int_L f.$$

See, for example, [G] for a review. There are many generalisations of this in real integral geometry with lines being replaced by, possibly higher dimensional, 'cycles' (see e.g. [GGG]).

On the other hand, there is transform in complex integral geometry introduced independently in about 1967 by Penrose [P] and Schmid [Sc]. (A version of the transform was discovered in 1904 by Bateman [B].) See, for example, [E] for a review of this 'Penrose transform'. Again, there are many developments of the original

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theme. There are analogies between the transforms on several different levels. Many cases of the Penrose transform can be realised by integrating differential forms over certain cycles (generally complex submanifolds of a complex manifold) to arrive at a function on (or section of a vector bundle over) the parameter space of such submanifolds. In this sense, the Penrose transform is a generalised Radon-type transform, and indeed the term "Radon-Penrose transform" has been used to describe it.

A different viewpoint on the relationship is the observation that in some sense some real integral transforms (such as the original Radon transform) are a "limit" of appropriate Penrose transforms when one restricts the target manifold to a real submanifold. This suggests a programme whereby one obtains results on real integral transforms using the complex holomorphic geometry methods of the Penrose transform.

In this paper, we begin this programme by showing that in the simplest possible case, one can indeed use these methods to obtain a new proof of the bijectivity of the Funk transform, a close relative of the Radon transform which we will describe in the next section. We expect these methods to generalise considerably. Indeed, our investigations originally concentrated on the X-ray transform (see, for example [GS]), a real analogue of the classical Penrose transform. Much of this article goes through in this more general case. The X-ray transform has also been investigated from this point of view by Sparling [Sp]. Rather than embark on any general discussion, however, we prefer, in this article, to concentrate on what we believe to be the simplest non-trivial example.

The plan of this paper is as follows. In §2 we introduce the Funk transform in its most "invariant" formulation on projective space and in §3 we give a brief account of the Penrose transform to which it is related. In §4 we introduce one of the main mathematical ideas, that of involutive structures on manifolds and their cohomology. The proof we will give of the bijectivity of the Funk transform is by showing that the space of functions which constitutes the domain is isomorphic to a cohomology group on an intermediate space. We introduce the intermediate space in §5 and prove the isomorphism in §6. In §7 we construct an isomorphism from this cohomology group to the target space of the Funk transform. Finally, in §8 we give several more examples of transforms related to the Funk transform which can be immediately computed using this method.

It is possible to extract from our methods an explicit inverse transform which involves solving a $\overline{\partial}$-problem on $\mathbb{CP}_2 \setminus \mathbb{RP}_2$ (and indeed one can see how this leads to an alternative regularisation of the usual integral for the inverse transform as in, e.g., [GGG]). Given this explicit inverse, one can prove bijectivity of the Funk transform more directly, and we will explain this elsewhere [BEGM]. The proof we give here however is in the generality which we will need to extend this method to other real integral transforms.

We will use the standard index notation of differential geometry with Greek indices running over the range $1, 2, 3$ and the Einstein summation convention. Except where stated otherwise, we are working throughout in the category of smooth (by which we mean C^∞) manifolds, functions and maps.

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2 The Funk transform

Following Funk [F], consider the integral transform defined on functions on the unit sphere $S^2 \subset \mathbb{R}^3$ by integration over great circles:

$$(\mathcal{F}f)(U) = \int_{X\cdot U=0} f(X)\mathrm{d}s,$$

where $\mathrm{d}s$ is the usual arc-length measure on the great circle $X \cdot U = 0$. It is clear that the kernel of this transform contains the odd functions, and its range is contained in the even functions. In fact, it is an isomorphism from the space of smooth even functions to itself. The main object of this paper is to provide a new proof of this well-known fact using methods derived from the Penrose transform.

As stated above, the Funk transform is SO(3)-equivariant. We now present, essentially as in [GGG, §2.2], an $\mathrm{SL}(3,\mathbb{R})$-equivariant version. Let $\epsilon_{\alpha\beta\gamma}$ denote the components of the usual volume form on $\mathbb{R}^3$.

Consider a function f (necessarily even) on $\mathbb{R}^3 \setminus \{0\}$ satisfying

$$f(tX) = t^{-2}f(X), \quad \text{for } t \in \mathbb{R}^\times,$$

where we write $\mathbb{R}^\times$ for $\mathbb{R} \setminus \{0\}$. Given non-zero $U \in \mathbb{R}^{3*}$, choose $X, Y \in \mathbb{R}^3$ such that

$$U_\alpha = \epsilon_{\alpha\beta\gamma}X^\beta Y^\gamma$$

and consider

$$\tfrac{1}{2\pi}\oint f(uX + vY)\,(u\,\mathrm{d}v - v\,\mathrm{d}u) \tag{1}$$

where the integral is taken around any smooth closed curve in the (u, v)-plane of winding number one about the origin. The homogeneity of f implies that the integrand is closed and so the integral is independent of the choice of curve. If X, Y are replaced by $aX + bY, cX + dY$, then a change of variables shows that the integral is multiplied by $|ad - bc|^{-1}$. Hence the integral defines a function ϕ on $\mathbb{R}^{3*} \setminus \{0\}$ satisfying

$$\phi(tU) = (\operatorname{sign} t)t^{-1}\phi(U).$$

Thus we obtain an integral transform equivariant under $\mathrm{SL}(3,\mathbb{R})$ which on restriction to the unit sphere becomes the Funk transform as previously described. For convenience in what follows we will extend the transform to act on complex-valued functions.

Definition 2.1 *Let $\mathbb{RP}_2$ be the projective space of $\mathbb{R}^3$ and let $L_{\mathbb{RP}_2}$ denote the complexification of the hyperplane bundle over $\mathbb{RP}_2$. For $p \in \mathbb{Z}$ we write $\mathcal{E}_{\mathbb{RP}_2}(p)$ for the sheaf of smooth sections of $L^p_{\mathbb{RP}_2}$. Sections of $\mathcal{E}_{\mathbb{RP}_2}(p)$ can be identified with complex-valued functions on the corresponding subset of $\mathbb{R}^3 \setminus \{0\}$ satisfying*

$$f(tX) = t^p f(X), \qquad t \in \mathbb{R}^\times .$$

We write $\widetilde{\mathcal{E}}_{\mathbb{RP}_2}(p)$ for the sheaf of sections of the complex line bundle whose smooth sections can be identified with smooth complex-valued functions satisfying

$$f(tX) = (\operatorname{sign} t) t^p f(X), \qquad t \in \mathbb{R}^\times .$$

We will sometimes refer to sections of these bundles as being "twisted". We adopt a similar notation for $\mathbb{RP}^*_2$. We drop the subscripts on the sheaves where there is no risk of confusion.

(Note that the above are all different $\mathrm{SL}(3,\mathbb{R})$-homogeneous bundles.)

The above construction defines the Funk transform as an $\mathrm{SL}(3,\mathbb{R})$-equivariant map

$$\mathcal{F} : \Gamma(\mathbb{RP}_2, \mathcal{E}(-2)) \longrightarrow \Gamma(\mathbb{RP}^*_2, \widetilde{\mathcal{E}}(-1)).$$

If one restricts to an an affine plane $\mathbb{R}^2 \subset \mathbb{RP}_2$, one obtains the Radon transform.

3 The Penrose transform

In this section, we work entirely in the holomorphic category. The analogous holomorphic Penrose transform is from $\mathbb{CP}_2$ to $\mathbb{CP}^*_2$. We present this using the methods originally applied in [EPW]. A very general description of the method can be found in [BE]. Let $\mathbb{CP}_2$ denote the projective space of $\mathbb{C}^3$, and $\mathbb{CP}^*_2$ the dual projective space. Let $\mathbb{F}$ denote the space of full flags in $\mathbb{C}^3$. Consider the correspondence:

$$\mathbb{CP}_2 \xleftarrow{\mu} \mathbb{F} \xrightarrow{\nu} \mathbb{CP}^*_2$$

where μ, ν are the obvious forgetful maps.

On $\mathbb{CP}_2$, let $\mathcal{O}(k)$ denote the sheaf of holomorphic local sections of the k^{th} power $L^k_{\mathbb{CP}_2}$ of the hyperplane bundle $L_{\mathbb{CP}_2}$, and similarly on $\mathbb{CP}^*_2$. On $\mathbb{F}$, for $p, q \in \mathbb{Z}$, we write $\mathcal{O}(p,q)$ for the sheaf of local holomorphic sections of $\mu^* L^p_{\mathbb{CP}_2} \otimes \nu^* L^q_{\mathbb{CP}^*_2}$.

Let $\mathcal{X} \subset \mathbb{CP}^*_2$ be Stein, and sufficiently convex that defining $\mathcal{F} = \nu^{-1}\mathcal{X}$ and $\mathcal{Z} = \mu\mathcal{F}$, the fibres of $\mu : \mathcal{F} \to \mathcal{Z}$ are contractible. We work with the restricted correspondence

$$\mathcal{Z} \xleftarrow{\mu} \mathcal{F} \xrightarrow{\nu} \mathcal{X}.$$

One can show that $H^1(\mathcal{Z}, \mathcal{O}(V)) = H^1(\mathcal{F}, \mu^{-1}\mathcal{O}(V))$ for any holomorphic vector bundle V. We will concentrate on the case where V is a holomorphic line bundle.

The "holomorphic relative de Rham resolution" in this case is the short exact sequence of sheaves over $\mathcal{F}$:

$$0 \to \mu^{-1}\mathcal{O}(p) \to \mathcal{O}(p,0) \stackrel{\mathrm{d}_\mu}{\to} \mathcal{O}(p+1,-2) \to 0,$$

where d_μ is essentially the holomorphic exterior derivative acting in the fibre directions.

The fact that $\mathcal{X}$ is Stein allows one to compute the cohomology $H^*(\mathcal{F}, \mathcal{O}(p,q))$. In fact,

$$\begin{aligned} H^0(\mathcal{F}, \mathcal{O}(p,q)) &= \begin{cases} 0, & p \leq -1 \\ \Gamma(\mathcal{X}, \odot^p \mathcal{O}^1_{\mathbb{CP}_2^*}(q+2p)), & p \geq 0 \end{cases} \\ H^1(\mathcal{F}, \mathcal{O}(p,q)) &= \begin{cases} \Gamma(\mathcal{X}, \odot^{-p-2} \mathcal{O}^1_{\mathbb{CP}_2^*}(q-p-3)), & p \leq -2 \\ 0, & p \geq -1 \end{cases} \end{aligned} \tag{2}$$

where $\mathcal{O}^1_{\mathbb{CP}_2^*}$ denotes the sheaf whose sections are holomorphic 1-forms, and higher cohomology vanishes. Here we are adopting a notation that we will use throughout of writing $\odot^p \mathcal{O}^1_{\mathbb{CP}_2^*}(q+2p)$ for $(\odot^p \mathcal{O}^1_{\mathbb{CP}_2^*}) \otimes \mathcal{O}(q+2p)$, etc.

Thus the long exact cohomology sequence of the preceding short exact sequence of sheaves allows us to compute $H^*(\mathcal{F}, \mu^{-1}\mathcal{O}(p))$ and hence $H^*(\mathcal{Z}, \mathcal{O}(p))$ in terms of sections of holomorphic bundles on $\mathcal{X}$. In particular, it is easy to check that the resulting *Penrose transform*

$$H^1(\mathcal{Z}, \mathcal{O}(-2)) \to \Gamma(\mathcal{X}, \mathcal{O}(-1))$$

is an isomorphism.

If one takes $\mathcal{X}$ to be a suitable neighbourhood in $\mathbb{CP}_2^*$ of some small convex set in $\mathbb{RP}_2^*$, and represents the cohomology class in $H^1(\mathcal{Z}, \mathcal{O}(-2))$ by means of a suitably chosen Čech cocycle, then one can see that the integral one does to evaluate the Penrose transform at a point in $\mathbb{RP}_2$ can be taken to be the same as the integral involved in the Funk transform. It is this connection that we are exploiting in what follows.

4 Involutive structures

A basic tool will be the notion of an "involutive structure" or "formally integrable structure" on a manifold. A detailed discussion (except for the idea of "compatible vector bundle") can be found in [T]. We will write $\mathcal{E}^1$ for the sheaf whose sections are smooth complex-valued 1-forms. We write $\mathcal{E}^p$ for the p-fold exterior power of $\mathcal{E}^1$, etc. We let $\mathcal{E}$ denote the sheaf whose sections are smooth complex-valued functions.

Definition 4.1 *An* <u>*involutive structure*</u> *on a manifold M is a complex sub-bundle of the complexified cotangent bundle of M which is formally integrable in the sense that if we write $\mathcal{A}^{1,0}$ for the sheaf of smooth sections of the sub-bundle then*

$$\mathrm{d}\mathcal{A}^{1,0} \subset \mathcal{A}^{1,0} \wedge \mathcal{E}^1.$$

Definition 4.2 *We define $\mathcal{A}^{0,1}$ by the exactness of*

$$0 \to \mathcal{A}^{1,0} \to \mathcal{E}^1 \to \mathcal{A}^{0,1} \to 0,$$

and denote by $\mathcal{A}^{0,q}$ its q^{th} exterior power.

Lemma 4.3 *The quotient of the de Rham complex $(\Gamma(M, \mathcal{E}^\bullet), \mathrm{d})$ by the ideal generated by $\Gamma(M, \mathcal{A}^{1,0})$ is a differential complex with differential operator $\overline{\mathrm{d}}$ induced by d. We refer to $\overline{\mathrm{d}}$ as the partial connection of the involutive structure. The complex can be identified with*

$$\Gamma(M, \mathcal{E}) \stackrel{\overline{\mathrm{d}}}{\to} \Gamma(M, \mathcal{A}^{0,1}) \stackrel{\overline{\mathrm{d}}}{\to} \Gamma(M, \mathcal{A}^{0,2}) \stackrel{\overline{\mathrm{d}}}{\to} \ldots$$

Definition 4.4 *Given a complex vector bundle $V \to M$, we write $\mathcal{E}(V)$ for its sheaf of smooth sections and, for example, $\mathcal{E}^1(V)$ for the sheaf whose sections are smooth V-valued 1-forms. A complex vector bundle $V \to M$ is* <u>*compatible*</u> *with the involutive structure $\mathcal{A}^{1,0}$ if it is equipped with a $\overline{\mathrm{d}}$-operator*

$$\overline{\mathrm{d}} : \Gamma(M, \mathcal{E}(V)) \to \Gamma(M, \mathcal{A}^{0,1}(V))$$

obeying the usual linearity and Leibniz rules and such that if one extends in the usual way so that $\overline{\mathrm{d}} : \Gamma(M, \mathcal{A}^{0,q}(V)) \to \Gamma(M, \mathcal{A}^{0,q+1}(V))$ we have $\overline{\mathrm{d}}^2 = 0$. We define $H^*_{\overline{\mathrm{d}}}(M, V)$ to be the homology of $(\Gamma(M, \mathcal{A}^{0,\bullet}(V)), \overline{\mathrm{d}})$.

To provide bearings for the reader unfamiliar with these structures, we remark that if also $\mathcal{A}^{1,0} \cap \overline{\mathcal{A}^{1,0}} = 0$, then the involutive structure is a CR-structure, and the cohomology is the well-known $\overline{\partial}_b$ cohomology. If $\mathcal{E}^1 = \mathcal{A}^{1,0} \oplus \overline{\mathcal{A}^{1,0}}$, then the involutive structure is a complex structure, compatibility of a vector bundle is equivalent to its being holomorphic and the resulting cohomology is Dolbeault cohomology.

5 The correspondence space

Definition 5.1 *Let $F = \{(L, P)\}$, where $L \subset \mathbb{C}^3$ is a complex 1-dimensional linear subspace, $P \subset \mathbb{R}^3$ is a real 2-dimensional linear subspace and L is contained in the complexification $P + iP$ of P.*

In coordinates, we can identify F as follows:

$$F = \{(Z^\alpha, U_\alpha) \,|\, Z^\alpha \in \mathbb{C}^3, U_\alpha \in \mathbb{R}^{3\,*}, Z^\alpha U_\alpha = 0\}/\sim,$$

where $(Z^\alpha, U_\alpha) \sim (\lambda Z^\alpha, rU_\alpha)$, for $\lambda \in \mathbb{C}^\times, r \in \mathbb{R}^\times$.

We define $F_R \subset F$ to be the set of all pairs (L, P) as above where L is the complexification of a real 1-dimensional subspace of $\mathbb{R}^3$.

Consider

$$\mathbb{CP}_2 \xleftarrow{\eta} F \xrightarrow{\tau} \mathbb{RP}_2^*,$$

where η and τ are the obvious forgetful maps. The inverse image under η of $\mathbb{RP}_2 \subset \mathbb{CP}_2$ is F_R. It is easy to check (see the coordinate calculations which follow) that F is a smooth manifold, that τ is a fibrebundle projection with fibre $\mathbb{CP}_1$, and that η restricted to F_R is a fibre bundle projection to $\mathbb{RP}_2 \subset \mathbb{CP}_2$ with fibre S^1, but that the fibres of η over non-real points of $\mathbb{CP}_2$ are single points. In fact, one can identify F with the "real blow-up" of $\mathbb{RP}_2 \subset \mathbb{CP}_2$. (In the notation of §3, note that F is the subset $\nu^{-1}\mathbb{RP}_2^*$ of $\mathbb{F}$.)

Definition 5.2 *For $p, q, r \in \mathbb{Z}$, we denote by $\mathcal{E}(p, q, r)$ the sheaf of smooth sections of the complex line bundle over F*

$$\eta^*(L^p_{\mathbb{CP}_2} \otimes \overline{L^q_{\mathbb{CP}_2}}) \otimes \tau^* L^r_{\mathbb{RP}_2^*},$$

where the bar denotes complex conjugate. Local sections of $\mathcal{E}(p, q, r)$ can be identified with smooth complex-valued functions on the corresponding region of $(\mathbb{C}^3 \setminus \{0\}) \times (\mathbb{R}^{3\,*} \setminus \{0\})$ satisfying

$$f(\lambda Z^\alpha, tU_\alpha) = \lambda^p \bar{\lambda}^q t^r f(\lambda Z^\alpha, tU_\alpha), \quad \textit{for } \lambda \in \mathbb{C}^\times, t \in \mathbb{R}^\times.$$

Similarly, $\widetilde{\mathcal{E}}(p, q, r)$ is defined so that its sections correspond to functions satisfying

$$f(\lambda Z^\alpha, tU_\alpha) = (\operatorname{sign} t) \lambda^p \bar{\lambda}^q t^r f(\lambda Z^\alpha, tU_\alpha), \quad \textit{for } \lambda \in \mathbb{C}^\times, t \in \mathbb{R}^\times.$$

Note that $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}(0,0,0)$ is the pull-back by τ of the line bundle on $\mathbb{RP}_2^*$ whose sections are odd functions on the double cover of $\mathbb{RP}_2^*$. Hence it has a canonical flat connection.

We use the "tilde" throughout to denote tensoring with $\widetilde{\mathcal{E}}$ or its underlying complex line bundle.

We identify $\mathbb{C}^2$ with the affine chart $\mathbb{C}^2 \subset \mathbb{CP}_2$ given by $(z, w) \mapsto [1 : z : w]$. We define $F^0 = \eta^{-1}\mathbb{C}^2$ and $F^0_R = F^0 \cap F_R$. We do coordinate calculations on F^0 as follows. The map from $\mathbb{R}^4$ to F^0 given by

$$(x, u, a, \theta) \mapsto \left(\begin{bmatrix} 1 \\ x + ia\cos\theta \\ u + ia\sin\theta \end{bmatrix}, \quad [u\cos\theta - x\sin\theta : \sin\theta : -\cos\theta] \right)$$

identifies (x,u,a,θ) and $(x,u,-a,\theta+\pi)$. Thus we can identify smooth functions on F^0 with smooth functions f on $\mathbb{R}^4$ satisfying

$$f(x,u,a,\theta)=f(x,u,-a,\theta+\pi).$$

We will refer to such functions as being *periodic*. The map η restricted to F^0 is given by

$$(x,u,a,\theta)\mapsto\begin{pmatrix}x+ia\cos\theta\\ u+ia\sin\theta\end{pmatrix}$$

and F^0_R is defined by $a=0$.

Lemma 5.3 *The map $\eta^*\mathcal{E}^{1,0}_{\mathbb{CP}_2}\to\mathcal{E}^1$ is injective. We denote its image by $\mathcal{A}^{1,0}$. This defines an involutive structure on F whose partial connection we denote by $\overline{\mathrm{d}}$.*

Proof. In local coordinates $\eta^*\mathrm{d}z$ and $\eta^*\mathrm{d}w$ are

$$\mathrm{d}x+i\cos\theta\mathrm{d}a-ia\sin\theta\mathrm{d}\theta,\qquad \mathrm{d}u+i\sin\theta\mathrm{d}a+ia\cos\theta\mathrm{d}\theta,$$

which are everywhere linearly independent. The formal integrability is an immediate consequence of properties of pull-back. □

In local coordinates we can, and will, choose to identify $\mathcal{A}^{0,1}$ with the span of the 1-forms $\mathrm{d}a$, $\mathrm{d}\theta$.

Lemma 5.4 *The pull back to F of any holomorphic vector bundle over $\mathbb{CP}_2$ is compatible with the involutive structure $\mathcal{A}^{1,0}$, with $\overline{\mathrm{d}}$ operator induced from the usual $\overline{\partial}$ operator on $\mathbb{CP}_2$. These vector bundles are also compatible when tensored with $\widetilde{\mathcal{E}}$.*

Proof. The $\overline{\mathrm{d}}$-operator on the pull-back η^*V of a holomorphic vector bundle $V\to\mathbb{CP}_2$ is characterised by setting

$$\overline{\mathrm{d}}(f\eta^*s)=f\eta^*(\overline{\partial}s)+(\overline{\mathrm{d}}f)\eta^*s,$$

where $s\in\Gamma(\mathbb{CP}_2,\mathcal{E}(V))$ and f is a smooth function on F. Since $\widetilde{\mathcal{E}}$ has a canonical flat connection, there is a canonical $\overline{\mathrm{d}}$ operator defined on it too. □

6 The pull-back mechanism

In the following, V is a holomorphic vector bundle over $\mathbb{CP}_2$. This section is devoted to proving:

Theorem 6.1 *There is a canonical exact sequence*

$$0\to\Gamma(\mathbb{CP}_2,\mathcal{O}(V))\xrightarrow{\rho}\Gamma(\mathbb{RP}_2,\mathcal{E}(V))\xrightarrow{\phi}H^1_{\overline{\mathrm{d}}}(F,\widetilde{\eta^*V})\xrightarrow{\psi}H^1(\mathbb{CP}_2,\mathcal{O}(V))\to 0,$$

where the map ρ is restriction of functions.

Lemma 6.2 *There is a canonical constant section of $\widetilde{\mathcal{E}}$ over $F \setminus F_R$ which we will denote by H.*

Proof. Given Z^α, U_α satisfying $Z^\alpha U_\alpha = 0$, one has

$$\tfrac{-i}{2}\epsilon_{\alpha\beta\gamma} Z^\beta \overline{Z}^\gamma = \sigma U_\alpha,$$

for some $\sigma \in \mathbb{R}$, which is zero if, and only if, $(Z^\alpha, U_\alpha) \in F_R$. The section H is defined to be $\mathrm{sign}(\sigma)$. □

A section of $\widetilde{\mathcal{E}}$ over F^0 can be represented by a function on $\mathbb{R}^4$ satisfying

$$f(x,u,a,\theta) = -f(x,u,-a,\theta+\pi).$$

We will refer to such functions as *antiperiodic*. The section H is then given by $H = \mathrm{sign}(a)$.

Lemma 6.3 *Let g be a smooth function on $\mathbb{R}^2$. Then there exists a smooth function G on F^0 such that G and $\eta^* g$ agree on F^0_R and such that $\bar{\mathrm{d}}G$ vanishes to infinite order at all points $p \in F^0_R$. If G_1 is another such extension, then $G - G_1$ vanishes to infinite order on F^0_R. If g has compact support in $\mathbb{R}^2$, then G can be chosen to have compact support in F^0.*

Proof. We define vector fields

$$\alpha = \frac{\partial}{\partial a} - i\cos\theta\frac{\partial}{\partial x} - i\sin\theta\frac{\partial}{\partial u}, \qquad \beta = \frac{\partial}{\partial \theta} + ia\sin\theta\frac{\partial}{\partial x} - ia\cos\theta\frac{\partial}{\partial u}.$$

Then it is easy to check that for a smooth function G on F^0,

$$\bar{\mathrm{d}}G = \alpha(G)\mathrm{d}a + \beta(G)\mathrm{d}\theta.$$

We consider the infinite formal power series

$$G^0 + G^1 a + G^2 a^2 + \dots,$$

where the coefficients are smooth functions on F^0_R. Then $\alpha(G) = 0$ to infinite order is equivalent to

$$\begin{gathered} G^1 - i\cos\theta G^0_x - i\sin\theta G^0_u = 0 \\ \vdots \\ kG^k - i\cos\theta G^{k-1}_x - i\sin\theta G^{k-1}_u = 0 \\ \vdots \end{gathered}$$

Setting $G^0 = \eta^* g$, these equations determine all the G^k uniquely.

We claim that $\beta(G)$ now automatically vanishes to all orders. This is equivalent to

$$\begin{gathered} G_\theta^0 = 0 \\ G_\theta^1 + i\sin\theta G_x^0 - i\cos\theta G_u^0 = 0 \\ \vdots \\ G_\theta^k + i\sin\theta G_x^{k-1} - i\cos\theta G_u^{k-1} = 0 \\ \vdots \end{gathered}$$

The first of these is satisfied since G^0 is a pull-back. It is easy to check inductively that the other equations are all satisfied. Now choose a smooth function G with the given jet. This is always possible (see e.g. [H, Thm. 1.2.6]).

Finally, assume that g has compact support in $\mathbb{R}^2$. Let ψ be a smooth function of compact support on F^0 such that $\psi = 1$ in a neighbourhood of the support of $\eta^* g \subset F_R^0$. Constructing G as above, we note that the jet of G on F_R vanishes identically outside the support of $\eta^* g$. Hence, defining $G_1 = \psi G$, we have that $\overline{\mathrm{d}} G_1$ is also zero to infinite order on F_R^0 and hence provides a compactly supported extension. $\square$

Lemma 6.4 *Let g be a smooth section of V over $\mathbb{RP}_2$. Then the section $\eta^* g$ of $\eta^* V$ over F_R can be extended to a section G over F such that $G = \eta^* g$ on F_R and such that $\overline{\mathrm{d}} G$ vanishes to all orders on F_R. If G_1 is another such extension, then $G - G_1$ vanishes to infinite order on F_R^0.*

Proof. Choose a partition of unity $\{\rho_i\}$ subordinate to a finite covering of $\mathbb{CP}_2$ by standard affine charts. The bundles involved are trivial over the pull-back of such a chart. Apply the previous Lemma to $\rho_i g$ in each chart, taking the extension to be compactly supported, and add. $\square$

Lemma 6.5 *Let $\omega \in \Gamma(F^0, \widetilde{\mathcal{A}}^{0,1})$ with $\overline{\mathrm{d}}\omega = 0$. Then there exists $f \in \Gamma(F^0, \widetilde{\mathcal{E}})$ such that $\omega - \overline{\mathrm{d}} f$ vanishes to infinite order on F_R^0. If f_1 is another such section, then $f - f_1$ vanishes to infinite order on F_R^0. If ω is compactly supported, then one can take f to be compactly supported also.*

Proof. With our conventions, such a $(0,1)$-form can be written

$$\omega = A\mathrm{d}a + \Theta \mathrm{d}\theta,$$

where A, Θ are smooth functions, with A periodic and Θ antiperiodic. We consider the Taylor developments in a about F_R^0:

$$A \sim A^0 + A^1 a + A^2 a^2 + \dots, \quad \Theta \sim \Theta^0 + \Theta^1 a + \Theta^2 a^2 + \dots.$$

Now, $\overline{\mathrm{d}}\omega = 0$ if, and only if, $\alpha(\Theta) = \beta(A)$ (where α, β are as in the proof of Lemma 6.3): i.e.

$$\begin{aligned}\Theta^1 - i\cos\theta\Theta^0_x - i\sin\theta\Theta^0_u &= A^0_\theta \\ 2\Theta^2 - i\cos\theta\Theta^1_x - i\sin\theta\Theta^1_u &= A^1_\theta + i\sin\theta A^0_x - i\cos\theta A^0_u \\ &\vdots \\ (k+1)\Theta^{k+1} - i\cos\theta\Theta^k_x - i\sin\theta\Theta^k_u &= A^k_\theta + i\sin\theta A^{k-1}_x - i\cos\theta A^{k-1}_u \\ &\vdots\end{aligned} \tag{3}$$

Now let $f \sim f^0 + f^1 a + f^2 a^2 + \ldots$. Then $\overline{\mathrm{d}}f = \alpha(f)\mathrm{d}a + \beta(f)\mathrm{d}\theta$, so we need $\alpha(f) = A$ and $\beta(f) = \Theta$ to infinite order. Calculating, we find this is equivalent to:

$$\begin{aligned}A^0 &= f^1 - i\cos\theta f^0_x - i\sin\theta f^0_u \\ &\vdots \\ A^k &= (k+1)f^{k+1} - i\cos\theta f^k_x - i\sin\theta f^k_u \\ &\vdots\end{aligned} \tag{4}$$

and

$$\begin{aligned}\Theta^0 &= f^0_\theta \\ \Theta^1 &= f^1_\theta + i\sin\theta f^0_x - i\cos\theta f^0_u \\ &\vdots \\ \Theta^k &= f^k_\theta + i\sin\theta f^{k-1}_x - i\cos\theta f^{k-1}_u \\ &\vdots\end{aligned} \tag{5}$$

The first of equations (5) determines f^0 uniquely—noting that both Θ^0 and f^0 are *antiperiodic*, this unique solution is given by

$$f^0(x,u,\theta) = \int_{\nu=0}^{\theta} \Theta^0(x,u,\nu)\mathrm{d}\nu - \tfrac{1}{2}\int_{\nu=0}^{\pi} \Theta^0(x,u,\nu)\mathrm{d}\nu.$$

Equations (4) then determine all the remaining f^k uniquely. It is easy to check then that the remaining equations (5) are satisfied as a consequence of (3). Now choose f with the given jet.

Finally, suppose ω is compactly supported. Let ψ be a compactly supported smooth function on F^0 such that $\psi = 1$ in a neighbourhood of $D = \eta^{-1}(\eta \operatorname{supp}\omega)$. Observe that the jet of f as constructed above vanishes identically on $F_R \backslash D$. Hence ψf satisfies the required conditions and is compactly supported. $\square$

Lemma 6.6 *Let $\omega \in \Gamma(F, \mathcal{A}^{0,1}(\widetilde{\eta^* V}))$ with $\overline{\mathrm{d}}\omega = 0$. Then there exists $f \in \Gamma(F, \mathcal{E}(\widetilde{\eta^* V}))$ such that $\omega - \overline{\mathrm{d}}f$ vanishes to infinite order on F_R. If f_1 is another such section, then $f - f_1$ vanishes to infinite order on F^0_R.*

Proof. Partition of unity and the previous Lemma, essentially as for Lemma 6.4. □

The map ϕ in Theorem 6.1 is obtained as follows. Let $g \in \Gamma(\mathbb{RP}_2, \mathcal{E}(V))$. Choose an extension G of $\eta^* g$ as in Lemma 6.4. Then $H\overline{\mathrm{d}}G$ is a smooth section over F of $\mathcal{A}^{0,1}(\widetilde{\eta^* V})$. Set

$$\phi(g) = [H\overline{\mathrm{d}}G] \in H^1_{\overline{\mathrm{d}}}(F, \widetilde{\eta^* V}).$$

To see that ϕ is well-defined, observe first that $H\overline{\mathrm{d}}G$ is smooth despite the jump discontinuity in H since $\overline{\mathrm{d}}G$ vanishes to infinite order on F_R. It is also clear that $H\overline{\mathrm{d}}G$ is $\overline{\mathrm{d}}$-closed, and hence defines a class in $H^1_{\overline{\mathrm{d}}}(F, \widetilde{\eta^* V})$. To show that the construction is independent of the choice of G, suppose G_1 is another extension. Then $H(G - G_1)$ is a smooth section of $\widetilde{\eta^* V}$, and

$$H\overline{\mathrm{d}}G - H\overline{\mathrm{d}}G_1 = \overline{\mathrm{d}}(H(G - G_1)).$$

The map ψ in Theorem 6.1 is obtained similarly. Suppose $\omega \in \Gamma(F, \mathcal{A}^{0,1}(\widetilde{\eta^* V}))$ represents a class in $H^1_{\overline{\mathrm{d}}}(F, \widetilde{\eta^* V})$. Let f be as in Lemma 6.6 and set $\gamma = H(\omega - \overline{\mathrm{d}}f)$. Thus γ is a smooth section over F of $\mathcal{A}^{0,1}(\eta^* V)$ and it is easy to check that $\overline{\mathrm{d}}\gamma = 0$. Since γ vanishes to infinite order on F_R, it is the pull-back of a unique $\overline{\partial}$-closed $(0,1)$-form on $\mathbb{CP}_2$ with values in V. The Dolbeault class of this form defines an element of $H^1(\mathbb{CP}_2, \mathcal{O}(V))$. It is easy to check that this class is independent of the choice of f.

Proof of Theorem 6.1. We start with exactness at $\Gamma(\mathbb{RP}_2, \mathcal{E}(V))$. If $g \in \Gamma(\mathbb{RP}_2, \mathcal{E}(V))$ is the restriction of some $h \in \Gamma(\mathbb{CP}_2, \mathcal{O}(V))$ then one can take the extension G in the construction of ϕg to be $\eta^* h$, and so $\overline{\mathrm{d}}G$ is identically zero on F and thus $\phi g = 0$. Now suppose that $\phi g = 0$, so that

$$H\overline{\mathrm{d}}G = \overline{\mathrm{d}}K, \qquad K \in \Gamma(F, \mathcal{E}(\widetilde{\eta^* V})).$$

Since $\overline{\mathrm{d}}K$ vanishes to infinite order on F_R, then so does K by Lemma 6.6 (with ω in the Lemma taken to be zero). Then $G - HK$ is $\overline{\mathrm{d}}$-closed and agrees with $\eta^* g$ on F_R. Consequently it is the pull-back of a continuous section of V over $\mathbb{CP}_2$ which agrees with g on $\mathbb{RP}_2$ and is holomorphic on $\mathbb{CP}_2 \setminus \mathbb{RP}_2$. Such a section is necessarily holomorphic on all of $\mathbb{CP}_2$.

We now turn to exactness at $H^1_{\overline{\mathrm{d}}}(F, \widetilde{\eta^* V})$. If ω represents a class in the image of ϕ so that $\omega = H\overline{\mathrm{d}}G$ then one can take $f = 0$ in the construction of $\psi\omega$ and so $\psi\omega = 0$ in cohomology. Now suppose that $\psi\omega = 0$ so that

$$H(\omega - \overline{\mathrm{d}}f) = \overline{\mathrm{d}}k,$$

where k is the pull-back of a smooth section of V over $\mathbb{CP}_2$ and $\overline{\mathrm{d}}k$ vanishes to infinite order on F_R. Thus we have

$$\omega - \overline{\mathrm{d}}f = H\overline{\mathrm{d}}k,$$

and thus ω is equivalent in $H^1_{\overline{\mathrm{d}}}(F, \widetilde{\eta^* V})$ to $H\overline{\mathrm{d}}k$ which is in the image of ϕ.

The injectivity of ρ is trivial and thus it remains only to see that ψ is surjective. This follows by similar arguments to the above, using the fact that a cohomology class in $H^1(\mathbb{CP}_2, \mathcal{O}(V))$ can always be represented by a $(0,1)$-form which vanishes to infinite order on $\mathbb{RP}_2$. We do not need this part of the Theorem later and so we will omit the details. $\square$

7 Pushing down

Lemma 7.1 *The map η holomorphically embeds the fibres of τ as complex projective lines in $\mathbb{CP}_2$. Thus the fibres of τ come equipped with a complex structure. Let $\mathcal{C}^{1,0} \subset \mathcal{E}^1$ be the sheaf of smooth sections of the complex sub-bundle which is the annihilator of the type-$(0,1)$ tangent vectors to the fibres of τ. Then $\mathcal{C}^{1,0}$ defines an involutive structure on F, whose partial connection we will denote by $\overline{\partial}_\tau$. The pull-back of a holomorphic vector bundle on $\mathbb{CP}_2$ is compatible with this involutive structure, as is the pull back of any vector bundle on $\mathbb{RP}^*_2$.*

Proof. A 1-dimensional complex distribution is necessarily integrable in the sense of Lie brackets, and dually the sub-bundle of the 1-forms annihilating it is formally integrable. The results on compatibility are immediate from the construction. $\square$

Lemma 7.2

$$\begin{aligned} H^0_{\overline{\partial}_\tau}(F, \widetilde{\mathcal{E}}(p,0,q)) &= \begin{cases} 0, & p \leq -1 \\ \Gamma(\mathbb{RP}^*_2, \widetilde{\odot^p \mathcal{E}^1_{\mathbb{RP}^*_2}}(q+2p)), & p \geq 0 \end{cases} \\ H^1_{\overline{\partial}_\tau}(F, \widetilde{\mathcal{E}}(p,0,q)) &= \begin{cases} \Gamma(\mathbb{RP}^*_2, \widetilde{\odot^{-p-2} \mathcal{E}^1_{\mathbb{RP}^*_2}}(q-p-3)), & p \leq -2 \\ 0, & p \geq -1 \end{cases} \end{aligned}$$

Proof. One can compute this directly, observing that the cohomology of the $\mathcal{C}^{1,0}$ involutive structure is essentially "fibrewise Dolbeault cohomology". Alternatively, observe that the result is also, up to tensoring with $\widetilde{\mathcal{E}}$, essentially a restriction of (2). $\square$

Lemma 7.3 *The underlying vector bundle of $\mathcal{A}^{1,0}$ is a sub-bundle of that of $\mathcal{C}^{1,0}$. Define $\mathcal{E}^1_\eta$ by the exactness of*

$$0 \to \mathcal{A}^{1,0} \to \mathcal{C}^{1,0} \to \mathcal{E}^1_\eta \to 0.$$

Then there is a short exact sequence

$$0 \to \mathcal{E}^1_\eta \to \mathcal{A}^{0,1} \xrightarrow{\pi} \mathcal{C}^{0,1} \to 0. \tag{6}$$

The operator $\overline{\partial}_\tau$ is obtained from $\overline{\mathrm{d}}$ by the projection π.

Proof. Elementary. $\square$

In coordinates on F^0, a non-zero type (0,1) tangent vector to the fibres of τ is given by

$$\frac{\partial}{\partial a} - i\cos\theta\frac{\partial}{\partial x} - i\sin\theta\frac{\partial}{\partial u},$$

and hence $\mathcal{C}^{1,0}$ is the span of $\mathcal{A}^{1,0}$ and $\mathrm{d}\theta$. Thus one can identify $\mathcal{C}^{0,1}$ with the span of $\mathrm{d}a$, and $\mathcal{E}^1_\eta$ with the span of $\mathrm{d}\theta$.

Lemma 7.4 *There is a canonical section σ of $\mathcal{E}(1,1,-1)$ which has a simple zero on F_R but is otherwise nonvanishing. In coordinates on F^0, it is given by a.*

Proof. If (Z^α, U_α) defines a point of F, then

$$\tfrac{-i}{2}\epsilon_{\alpha\beta\gamma}Z^\beta\overline{Z}^\gamma = \sigma U_\alpha,$$

and σ thus defines a section of $\mathcal{E}(1,1,-1)$. In an appropriate trivialisation, it is simple to compute that in coordinates on F^0 the section σ is given by a, and hence deduce that σ has a simple zero on F_R but is otherwise nonzero. $\square$

Lemma 7.5 *There is a canonical nowhere vanishing section ξ of $\mathcal{A}^{0,1}(-1,0,2)$ which vanishes on projection to $\mathcal{C}^{0,1}(-1,0,2)$. Hence, we can identify $\mathcal{E}^1_\eta = \mathcal{E}(1,0,-2)$.*

Proof. The 1-form $U_\alpha \mathrm{d}\overline{Z}^\alpha$ defines a 1-form on $\mathbb{C}^3 \times \mathbb{R}^{3\,*}$ which descends to give a section of $\mathcal{E}^1(0,1,1)$ over F. Let ξ_1 denote the projection of this into $\mathcal{A}^{0,1}$. Then one can compute that in coordinates $\xi_1 = 2ia\mathrm{d}\theta$, and thus ξ_1 has a simple zero on F_R, but is otherwise nonvanishing. Thus we can define $\xi = \sigma^{-1}\xi_1$, so that ξ is a nowhere vanishing section of $\mathcal{A}^{0,1}(-1,0,2)$. $\square$

Proposition 7.6 *The underlying vector bundle of $\widetilde{\mathcal{E}^1_\eta}$ is compatible with the $\mathcal{C}^{1,0}$ involutive structure. The following diagram is commutative, where V is a holomorphic vector bundle over $\mathbb{CP}_2$*

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathcal{E}^1_\eta(\widetilde{\eta^*V}) \arrow[r, "\overline{\partial}_\tau"] \arrow[d] & \mathcal{E}^1_\eta(\widetilde{\eta^*V}) \otimes \mathcal{C}^{0,1} \arrow[r] \arrow[d, "i"] & 0 \\
\mathcal{E}(\widetilde{\eta^*V}) \arrow[r, "\overline{\mathrm{d}}"] \arrow[d, "||"] & \mathcal{A}^{0,1}(\widetilde{\eta^*V}) \arrow[r, "\overline{\mathrm{d}}"] \arrow[d] & \mathcal{A}^{0,2}(\widetilde{\eta^*V}) \arrow[r] \arrow[d] & 0 \\
\mathcal{E}(\widetilde{\eta^*V}) \arrow[r, "\overline{\partial}_\tau"] & \mathcal{C}^{0,1}(\widetilde{\eta^*V}) \arrow[r] & 0 &
\end{tikzcd}$$

and the vertical isomorphism $i : \mathcal{E}^1_\eta(\widetilde{\eta^*V}) \otimes \mathcal{C}^{0,1} \to \mathcal{A}^{0,2}(\widetilde{\eta^*V})$ is obtained from the determinant of (6).

Proof. The $\overline{\partial}_\tau$-operator is that associated with the complex structure of the fibres of τ. Thus the pull-backs of holomorphic vector bundles from $\mathbb{CP}_2$ and of any vector bundle from $\mathbb{RP}_2^*$ are compatible. Since $\widetilde{\mathcal{E}}^1_\eta = \eta^*\mathcal{O}_{\mathbb{CP}_2}(1) \otimes \tau^*\widetilde{\mathcal{E}}_{\mathbb{RP}_2^*}(-2)$, it too is compatible. The rest of the proposition follows from noting that $\overline{\partial}_\tau$ is obtained from $\overline{\mathrm{d}}$ by means of the projection $\mathcal{A}^{0,1} \to \mathcal{C}^{0,1}$. □

Proposition 7.7 *There is a long exact sequence:*

$$\begin{aligned} \ldots \to H^0_{\overline{\partial}_\tau}(F, \widetilde{\eta^* V}) \to H^0_{\overline{\partial}_\tau}(F, \mathcal{E}^1_\eta(\widetilde{\eta^* V})) \to H^1_{\overline{\mathrm{d}}}(F, \widetilde{\eta^* V}) \to \\ H^1_{\overline{\partial}_\tau}(F, \widetilde{\eta^* V}) \to H^1_{\overline{\partial}_\tau}(F, \mathcal{E}^1_\eta(\widetilde{\eta^* V})) \to \ldots \end{aligned}$$

Proof. Taking sections of the diagram of Proposition 7.6 over F, the three rows yield a short exact sequence of complexes. The cohomology of the middle row is $H^*_{\overline{\mathrm{d}}}(F, \widetilde{\eta^* V})$. The cohomology of the top row is $H^{*-1}_{\overline{\partial}_\tau}(F, \mathcal{E}^1_\eta(\widetilde{\eta^* V}))$ and that of the bottom row is $H^*_{\overline{\partial}_\tau}(F, \widetilde{\eta^* V})$. □

Theorem 7.8 *The Funk transform*

$$\Gamma(\mathbb{RP}_2, \mathcal{E}(-2)) \to \Gamma(\mathbb{RP}_2^*, \widetilde{\mathcal{E}}(-1))$$

is an isomorphism.

Proof. The pull-back mechanism of Theorem 6.1 yields

$$\Gamma(\mathbb{RP}_2, \mathcal{E}(-2)) = H^1_{\overline{\mathrm{d}}}(F, \widetilde{\mathcal{E}}(-2,0,0)),$$

given that $H^1(\mathbb{CP}_2, \mathcal{O}(p)) = 0, \quad p \in \mathbb{Z}$. Using $\mathcal{E}^1_\eta = \mathcal{E}(1,0,-2)$, we observe that $\widetilde{\mathcal{E}}^1_\eta(\eta^*\mathcal{O}_{\mathbb{CP}_2}(-2)) = \widetilde{\mathcal{E}}(-1,0,-2)$. The long exact sequence of Proposition 7.7 with $V = \mathcal{O}(-2)$ and Lemma 7.2 yields a long exact sequence

$$\ldots \to 0 \to 0 \to H^1_{\overline{\mathrm{d}}}(F, \widetilde{\mathcal{E}}(-2,0,0)) \to \Gamma(\mathbb{RP}_2^*, \widetilde{\mathcal{E}}(-1)) \to 0 \to \ldots.$$

□

One needs, of course, to convince oneself that this isomorphism is the Funk transform, but this is essentially straightforward.

8 Further examples

To conclude, we use the Penrose transform methods on some more examples.

Theorem 8.1 *There is a generalised funk transform*

$$\Gamma(\mathbb{RP}_2, \mathcal{E}(-1)) \to \Gamma(\mathbb{RP}_2^*, \widetilde{\mathcal{E}}(-2))$$

which is an isomorphism.

Proof. Proceeding as before, we have

$$\Gamma(\mathbb{RP}_2, \mathcal{E}(-1)) = H^1_{\mathrm{d}}(F, \widetilde{\mathcal{E}}(-1,0,0))$$

and $\widetilde{\mathcal{E}}^1_\eta(\eta^* V) = \widetilde{\mathcal{E}}(0,0,-2)$. Thus we obtain

$$\ldots \to 0 \to \Gamma(\mathbb{RP}_2^*, \widetilde{\mathcal{E}}(-2)) \to H^1_{\mathrm{d}}(F, \widetilde{\mathcal{E}}(-1,0,0)) \to 0 \to 0 \to \ldots.$$

□

The transform described above is in fact like the Funk transform, except that one integrates the normal derivative of the function over the great circle instead of the function itself. This yields an isomorphism from the space of odd functions on the sphere to itself.

Theorem 8.2 *There is a generalised Funk transform*

$$\mathcal{F}_0 : \Gamma(\mathbb{RP}_2, \mathcal{E}) \to \Gamma(\mathbb{RP}_2^*, \widetilde{\mathcal{E}}(-3)).$$

Its kernel is the constant functions. Interpreting $\widetilde{\mathcal{E}}(-3)$ as the sheaf of twisted 2-forms, the range of the transform is those whose integral over $\mathbb{RP}_2^*$ vanishes.

There is also a generalised Funk transform

$$\mathcal{F}_{-3} : \Gamma(\mathbb{RP}_2, \mathcal{E}(-3)) \to \Gamma(\mathbb{RP}_2^*, \widetilde{\mathcal{E}})$$

which is an isomorphism.

Proof. In the first case, the pull-back mechanism identifies $\Gamma(\mathbb{RP}_2, \mathcal{E})$ modulo constants with $H^1_{\mathrm{d}}(F, \mathcal{E}(0,0,0))$. The long exact sequence then yields

$$\ldots \to \Gamma(\mathbb{RP}_2^*, \widetilde{\mathcal{E}}) \to \Gamma(\mathbb{RP}_2^*, \widetilde{\mathcal{E}}^1) \to H^1(F, \widetilde{\mathcal{E}}(0,0,0)) \to 0 \to 0 \to \ldots$$

which identifies $H^1_{\mathrm{d}}(F, \mathcal{E}(0,0,0))$ with twisted 1-forms on $\mathbb{RP}_2^*$ modulo the image by d of twisted functions. Since the first twisted de Rham group is zero, this in turn can be identified with its image under d in the twisted 2-forms, and the result follows.

In the second case, one obtains

$$\Gamma(\mathbb{RP}_2, \mathcal{E}(-3)) = H^1_{\mathrm{d}}(F, \widetilde{\mathcal{E}}(-3,0,0)).$$

The long exact sequence yields

$$\ldots \to 0 \to 0 \to H^1_{\mathrm{d}}(F, \widetilde{\mathcal{E}}(-3,0,0)) \to \Gamma(\mathbb{RP}_2^*, \widetilde{\mathcal{E}}^1) \to \Gamma(\mathbb{RP}_2^*, \widetilde{\mathcal{E}}(-3)) \to \ldots$$

and one obtains an isomorphism from $\Gamma(\mathbb{RP}_2, \mathcal{E}(-3))$ to closed twisted 1-forms on $\mathbb{RP}_2^*$. Again, the latter may be identified with twisted functions, given that there are no twisted constants. □

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