TRACTOR BUNDLES FOR IRREDUCIBLE PARABOLIC GEOMETRIES

by

Andreas Čap & A. Rod Gover

Abstract. — We use the general results on tractor calculi for parabolic geometries obtained in [**3**] to give a simple and effective characterisation of arbitrary normal tractor bundles on manifolds equipped with an irreducible parabolic geometry (also called almost Hermitian symmetric– or AHS–structure in the literature). Moreover, we also construct the corresponding normal adjoint tractor bundle and give explicit formulae for the normal tractor connections as well as the fundamental D–operators on such bundles. For such structures, part of this information is equivalent to giving the canonical Cartan connection. However it also provides all the information necessary for building up the invariant tractor calculus. As an application, we give a new simple construction of the standard tractor bundle in conformal geometry, which immediately leads to several elements of tractor calculus.

1. Tractor bundles and normal tractor connections

Riemannian and pseudo-Riemannian geometries are equipped with a canonical metric and the metric (or Levi-Civita) connection that it determines. For this reason, in the setting of these geometries, it is natural to calculate directly with the tangent bundle, its dual and the tensor bundles. On the other hand for many other interesting structures such as conformal geometries, CR geometries, projective geometries and quaternionic structures the situation is not so fortunate. These structures are among the broad class of so-called parabolic geometries and for the geometries within this class there is no canonical connection or metric on the tangent bundle or the tensor bundles. Nevertheless for these structures there is a class of natural vector bundles which do have a canonical connection. These are the tractor bundles and the calculus based around these bundles is a natural analog of the tensor bundle and Levi-Civita connection calculus of Riemannian geometry.

1991 ***Mathematics Subject Classification.*** — primary: 53B15, 53C05, 53C07, 53C15 secondary: 53A20, 53A30, 53A40, 53A55.
Key words and phrases. — Parabolic geometry, Cartan connection, tractor bundle, tractor calculus, invariant differential operator, invariant calculus.

Tractor calculus has its origins in the work of T.Y. Thomas [**11**] who developed key elements of the theory for conformal and projective geometries. Far more recently this was rediscovered and extended in [**1**]. Since this last work tractor calculus has been further extended and developed and the structures treated explicitly include CR and the almost Grassmannian/quaternionic geometries (see for example [**6, 7, 8, 9**] and references therein). Included in these works are many applications to the construction of invariant operators and polynomial invariants of the structures.

In our recent paper [**3**] we have introduced the concepts of tractor bundles and normal tractor connections for all parabolic geometries. Besides showing that from these bundles one can recover the Cartan bundle and the normal Cartan connection of such a geometry, we have also developed an invariant calculus based on adjoint tractor bundles and the so–called fundamental D–operators for all these geometries. Moreover, in that paper a general construction of the normal adjoint tractor bundle in the case of irreducible parabolic geometries is presented. While this approach, based on the adjoint representation of the underlying Lie–algebra, has the advantage of working for all irreducible parabolic geometries simultaneously, there are actually simpler tractor bundles available for each concrete choice of the structure. In fact, all previously known examples of tractor calculi as mentioned above are of the latter type. It is thus important to be able to recognise general normal tractor bundles for a parabolic geomtry and to find the corresponding normal tractor connections.

The main result of this paper is theorem 1.3 which offers a complete solution for the case of irreducible parabolic geometries. For a given structure and representation of the underlying Lie algebra, this gives a characterisation of the normal tractor bundle, as well as a univsersal formula for the normal tractor connection. On the one hand this may be used to identify a bundle as the normal tractor bundle and then compute the normal tractor connection. On the other hand the theorem specifies the necessary ingredients for the construction of such a bundle. It should be pointed out, that the results obtained here are independent of the construction of the normal adjoint tractor bundles for irreducible parabolic geometries given in [**3**]. From that source we only use the technical background on these structures.

We will show the power of this approach in section 2 and 3 by giving an alternative construction of the most well known example of a normal tractor bundle, namely the standard tractors in conformal geometry. Besides providing a short and simple route to all the basic elements of conformal tractor calculus, this new construction also immediately encodes some more advanced elements of tractor calculus.

1.1. Background on irreducible parabolic geometries. — Parabolic geometries may be viewed as curved analogs of homogeneous spaces of the form G/P, where G is a real or complex simple Lie group and $P \subset G$ is a parabolic subgroup. In general, a parabolic geometry of type (G, P) on a smooth manifold M is defined as a principal P–bundle over M, which is endowed with a Cartan connection, whose curvature satisfies a certain normalization condition. This kind of definition is however very unsatisfactory for our purposes. The point about this is that these normal Cartan connections usually are obtained from underlying structures via fairly complicated prolongation procedures, see e.g. [**4**]. Tractor bundles and connections are an alternative approach to these structures, which do not require knowledge of the Cartan connection but may be constructed directly from underlying structures in many cases. Hence, in this paper we will rather focus on the underlying structures and avoid the general point of view via Cartan connections.

Fortunately, these underlying structures are particularly easy to understand for the subclass of irreducible parabolic geometries, which correspond to certain maximal parabolics. The point is that for these structures, one always has a (classical first order) G_0–structure (for a certain subgroup $G_0 \subset G$) on M, as well as a class of preferred connections on the tangent bundle TM. While both these are there for any irreducible parabolic geometry, their role in describing the structure may vary a lot, as can be seen from two important examples, namely conformal and classical projective structures.

In the conformal case, the G_0–structure just is the conformal structure, i.e. the reduction of the frame bundle to the conformal group, so this contains all the information. The preferred connections are then simply all torsion free connections respecting the conformal structure, i.e. all Weyl connections. On the other hand, in the projective case, the group G_0 turns out to be a full general linear group, so the first order G_0–structure contains no information at all, while the projective structure is given by the choice of a class of preferred torsion free connections.

The basic input to specify an irreducible parabolic geometry is a simple real Lie group G together with a so–called $|1|$–grading on its Lie algebra $\mathfrak{g}$, i.e. a grading of the form $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. It is then known in general (see e.g. [**12**, section 3]) that $\mathfrak{g}_0$ is a reductive Lie algebra with one dimensional centre and the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is irreducible (which is the reason for the name "irreducible parabolic geometries"). Moreover, any $\mathfrak{g}$–invariant bilinear form (for example the Killing form) induces a duality of $\mathfrak{g}_0$–modules between $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$. Next, there is a canonical generator E, called the *grading element*, of the centre of $\mathfrak{g}_0$, which is characterised by the fact that its adjoint action on $\mathfrak{g}_j$ is given by multiplication by j for $j = -1, 0, 1$.

Having given these data, we define subgroups $G_0 \subset P \subset G$ by

$$G_0 = \{g \in G : \mathrm{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \text{ for all } i\}$$
$$P = \{g \in G : \mathrm{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \oplus \mathfrak{g}_{i+1} \text{ for } i = 0, 1\},$$

where Ad denotes the adjoint action and we agree that $\mathfrak{g}_i = \{0\}$ for $|i| > 1$. It is easy to see that G_0 has Lie algebra $\mathfrak{g}_0$, while P has Lie algebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. An important result is that P is actually the semidirect product of G_0 and a vector group. More precisely, one proves (see e.g. [**4**, proposition 2.10]) that for any element $g \in P$ there are unique elements $g_0 \in G_0$ and $Z \in \mathfrak{g}_1$ such that $g = g_0 \exp(Z)$. Hence if we define $P_+ \subset P$ as the image of $\mathfrak{g}_1$ under the exponential map, then $\exp : \mathfrak{g}_1 \to P_+$ is a diffeomorphism and P is the semidirect product of G_0 and P_+.

If neither $\mathfrak{g}$ nor its complexification is isomorphic to $\mathfrak{sl}(n, \mathbb{C})$ with the $|1|$–grading given in block form by $\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix}$, where the blocks are of size 1 and $n-1$, then a parabolic geometry of type (G, P) on a smooth manifold M (of the same dimension as $\mathfrak{g}_{-1}$) is defined to be a first order G_0–structure on the manifold M, where G_0 is viewed as a subgroup of $GL(\mathfrak{g}_{-1})$ via the adjoint action. We will henceforth refer to these structures as the structures which are not of projective type.

On the other hand, if either $\mathfrak{g}$ or its complexification is isomorphic to $\mathfrak{sl}(n,\mathbb{C})$ with the above grading, then this is some type of a projective structure, which is given by a choice of a class of affine connections on M (details below). See [**5**, 3.3] for a discussion of various examples of irreducible parabolic geometries.

Given a $|1|$–graded Lie algebra $\mathfrak{g}$, the simplest choice of group is $G = \mathrm{Aut}(\mathfrak{g})$, the group of all automorphisms of the Lie algebra $\mathfrak{g}$. Note that, for this choice of the group G, P is exactly the group $\mathrm{Aut}_f(\mathfrak{g})$ of all automorphism of the *filtered* Lie algebra $\mathfrak{g} \supset \mathfrak{p} \supset \mathfrak{g}_1$, while G_0 is exactly the group $\mathrm{Aut}_{gr}(\mathfrak{g})$ of all automorphisms of the *graded* Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. For a general choice of G, the adjoint action shows that P (respectively G_0) is a covering of a subgroup of $\mathrm{Aut}_f(\mathfrak{g})$ (respectively $\mathrm{Aut}_{gr}(\mathfrak{g})$) which contains the connected component of the identity. Note however, that in any case the group P_+ is exactly the group of those automorphisms φ of $\mathfrak{g}$ such that for each $i = -1, 0, 1$ and each $A \in \mathfrak{g}_i$ the image $\varphi(A)$ is congruent to A modulo $\mathfrak{g}_{i+1} \oplus \mathfrak{g}_{i+2}$.

In any case, as shown in [**3**, 4.2, 4.4], on any manifold M equipped with a parabolic geometry of type (G,P) one has the following basic data:

(1) A principal G_0–bundle $p : \mathcal{G}_0 \to M$ which defines a first order G_0–structure on M. (In the non–projective cases, this defines the structure, while in the projective cases it is a full first order frame bundle.) The tangent bundle TM and the cotangent bundle T^*M are the associated bundles to $\mathcal{G}_0$ corresponding to the adjoint action of G_0 on $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$, respectively. There is an induced bundle $\mathrm{End}_0 TM$ which is associated to $\mathcal{G}_0$ via the adjoint action of G_0 on $\mathfrak{g}_0$. This is canonically a subbundle of $T^*M \otimes TM$ and so we can view sections of this bundle either as endomorphisms of TM or of T^*M.

(2) An algebraic bracket $\{\ ,\ \} : TM \otimes T^*M \to \mathrm{End}_0 TM$, which together with the trivial brackets on $TM \otimes TM$ and on $T^*M \otimes T^*M$, the brackets $\mathrm{End}_0 TM \otimes TM \to TM$ given by $\{\Phi, \xi\} = \Phi(\xi)$ and $\mathrm{End}_0 TM \otimes T^*M \to T^*M$ given by $\{\Phi, \omega\} = -\Phi(\omega)$, and the bracket on $\mathrm{End}_0 TM \otimes \mathrm{End}_0 TM \to \mathrm{End}_0 TM$ given by the commutator of endomorphisms of TM, makes $T_xM \oplus \mathrm{End}_0 T_xM \oplus T^*_xM$, for each point $x \in M$, into a graded Lie algebra isomorphic to $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. (This algebraic bracket is induced from the Lie algebra bracket of $\mathfrak{g}$.)

(3) A preferred class of affine connections on M induced from principal connections on $\mathcal{G}_0$, such that for two preferred connections ∇ and $\hat{\nabla}$ there is a unique smooth one–form $\Upsilon \in \Omega^1(M)$ such that $\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \{\{\Upsilon, \xi\}, \eta\}$ for all vector fields ξ, η on M. (In the projective cases, the structure is defined by the choice of this class of connections, while in the non–projective cases their existence is a nontrivial but elementary result.) Moreover, there is a restriction on the torsion of preferred connections, see below.

There is a nice reinterpretation of (1) and (2): Define the bundle $\overrightarrow{\mathcal{A}} = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1 \to M$ by $\mathcal{A}_{-1} = TM$, $\mathcal{A}_0 = \mathrm{End}_0 TM$ and $\mathcal{A}_1 = T^*M$. Then the algebraic bracket from (2) makes $\overrightarrow{\mathcal{A}}$ into a bundle of graded Lie algebras. Moreover, since $\mathcal{A}_i$ is the associated bundle $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_i$ the definition of the algebraic bracket implies that each point $u_0 \in \mathcal{G}_0$ lying over $x \in M$ leads to an isomorphism $\underline{u_0} : \mathfrak{g} \to \mathcal{A}_x$ of graded Lie algebras. In this picture, the principal right action of G_0 on $\mathcal{G}_0$ leads to $\underline{u_0 \cdot g} = \underline{u_0} \circ \mathrm{Ad}(g)$.

There are a few important facts on preferred connections that have to be noted. First, since they are induced from principal connections on $\mathcal{G}_0$, the algebraic brackets from (2) are covariantly constant with respect to any of the preferred connections. Second, the Jacobi identity immediately implies that $\{\{\Upsilon, \xi\}, \eta\}$ is symmetric in ξ and η, so all preferred connections have the same torsion $T \in \Gamma(\Lambda^2 T^*M \otimes TM)$. Hence, this torsion is an invariant of the parabolic geometry. The normalisation condition on the torsion mentioned above is that the trace over the last two entries of the map $\Lambda^2 TM \otimes T^*M \to \mathrm{End}_0 TM$ defined by $(\xi, \eta, \omega) \mapsto \{T(\xi, \eta), \omega\}$ vanishes. That is, in the language of [**3**], the torsion is is ∂^*–closed.

There are also a few facts on the curvature of preferred connections that we will need in the sequel: Namely, if ∇ is a preferred connection, and $R \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}_0 TM)$ is its curvature, then by [**3**, 4.6] one may split R canonically as $R(\xi, \eta) = W(\xi, \eta) - \{\mathsf{P}(\xi), \eta\} + \{\mathsf{P}(\eta), \xi\}$, where $\mathsf{P} \in \Gamma(T^*M \otimes T^*M)$ is the *rho–tensor* and $W \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}_0 TM)$ is called the *Weyl–curvature* of the preferred connection. What makes this splitting canonical is the requirement that the trace over the last two entries of the map $\Lambda^2 T^*M \otimes T^*M \to T^*M$ defined by $(\xi, \eta, \omega) \mapsto W(\xi, \eta)(\omega) = -\{W(\xi, \eta), \omega\}$ vanishes. Referring, once again, to the language of [**3**], this is the condition that W is ∂^*–closed. The change of both P and W under a change of preferred connection is relatively simple. Namely, from [**3**, 4.6] we get for $\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \{\{\Upsilon, \xi\}, \eta\}$ the expressions

$$\hat{\mathsf{P}}(\xi) = \mathsf{P}(\xi) - \nabla_\xi \Upsilon + \tfrac{1}{2}\{\Upsilon, \{\Upsilon, \xi\}\}$$
$$\hat{W}(\xi, \eta) = W(\xi, \eta) + \{\Upsilon, T(\xi, \eta)\}$$

In particular, if the torsion of M vanishes, then the Weyl–curvature is independent of the choice of the preferred connection and thus an invariant of the parabolic geometry on M.

The *Cotton–York tensor* $CY \in \Gamma(\Lambda^2 T^*M \otimes T^*M)$ of a preferred connection ∇ is defined as the covariant exterior derivative of the rho–tensor, i.e.

$$CY(\xi, \eta) = (\nabla \mathsf{P})(\xi, \eta) - (\nabla \mathsf{P})(\eta, \xi) + \mathsf{P}(T(\xi, \eta)).$$

It turns out that if both the torsion and the Weyl–curvature vanish, then CY is independent of the choice of the preferred connection and thus an invariant of the parabolic geometry. Finally, it can be shown that if for one (equivalently any) preferred connection the torsion, the Weyl–curvature and the Cotton–York tensor vanish, then the manifold is locally isomorphic (as a parabolic geometry) to the flat model G/P.

1.2. $(\mathfrak{g}, P)$**–modules.** — The basic ingredient for a tractor bundle on a manifold M equipped with a parabolic geometry of type (G, P) is a (finite–dimensional) nontrivial $(\mathfrak{g}, P)$–module $\mathbb{V}$. This means that on $\mathbb{V}$ one has given actions ρ of P and ρ' of $\mathfrak{g}$ such that the restriction of ρ' to the subalgebra $\mathfrak{p}$ coincides with the derivative of ρ and such that $\rho'(\mathrm{Ad}(g) \cdot A) = \rho(g) \circ \rho'(A) \circ \rho(g^{-1})$ for all $g \in P$ and $A \in \mathfrak{g}$. The basic examples of $(\mathfrak{g}, P)$–modules are provided by representations of the group G, by simply restricting the representation to P but keeping its derivative defined on $\mathfrak{g}$. Since G is simple, any finite dimensional $\mathfrak{g}$–module splits as a direct sum of irreducible modules, so we will henceforth assume that $\mathbb{V}$ is irreducible as a $\mathfrak{g}$–module.

Clearly we can restrict the action of P on $\mathbb{V}$ to G_0 and hence view $\mathbb{V}$ as a G_0–module (and thus also as a $\mathfrak{g}_0$–module). The grading element E is contained in the centre of $\mathfrak{g}_0$, and thus Schur's lemma implies that it acts by a scalar on any irreducible $\mathfrak{g}_0$–module. In particular, we may split $\mathbb{V}$ as $\oplus_j \mathbb{V}_j$ according to eigenvalues of E. For $A \in \mathfrak{g}_i$ and $v \in \mathbb{V}_j$ note the computation $E{\cdot}A{\cdot}v = [E, A]{\cdot}v + A{\cdot}E{\cdot}v = (i + j)A{\cdot}v$. So the action of $\mathfrak{g}_i$ maps each $\mathbb{V}_j$ to $\mathbb{V}_{j+i}$ (where we define $\mathbb{V}_k = 0$ if an integer k is not an eigenvalue of E acting on $\mathbb{V}$). Since any nontrivial representation of a simple Lie algebra is faithful, it follows that there are at least two nonzero components in the sum $\oplus_j \mathbb{V}_j$, and in particular, $\mathbb{V}$ is never an irreducible $\mathfrak{g}_0$–module. Finally, note that since $\mathbb{V}$ is an irreducible $\mathfrak{g}$–module, it is generated by a single element. This implies that if j_0 is the lowest eigenvalue of E occurring in $\mathbb{V}$ all other eigenvalues are obtained by adding positive integers to j_0, so the splitting actually has the form $\mathbb{V} = \oplus_{j=0}^N \mathbb{V}_{j_0+j}$. The upshot of this is that we can encode the $\mathfrak{g}$–module structure as the sequence $(\mathbb{V}_j)$ of $\mathfrak{g}_0$–modules, together with the actions $\mathfrak{g}_{\pm 1} \times \mathbb{V}_j \to \mathbb{V}_{j\pm 1}$.

1.3. Let us henceforth fix a simple Lie group G with $|1|$–graded Lie algebra $\mathfrak{g}$, an irreducible $(\mathfrak{g}, P)$–module $\mathbb{V}$ with decomposition $\mathbb{V} = \oplus \mathbb{V}_j$ according to eigenvalues of the grading element E, and a smooth manifold M endowed with a parabolic geometry of type (G, P). Then since each $\mathbb{V}_j$ is a G_0–submodule of $\mathbb{V}$, we can form the associated bundle $\mathcal{V}_j = \mathcal{G}_0 \times_{G_0} \mathbb{V}_j \to M$ and put $\overrightarrow{\mathcal{V}} = \oplus_j \mathcal{V}_j$. Moreover, the action $\mathfrak{g} \to L(\mathbb{V}, \mathbb{V})$ induces a bundle map $\rho : \overrightarrow{\mathcal{A}} \to L(\overrightarrow{\mathcal{V}}, \overrightarrow{\mathcal{V}})$, which has the property that $\rho(\mathcal{A}_i)(\mathcal{V}_j) \subset \mathcal{V}_{i+j}$ for all $i = -1, 0, 1$ and all j. By construction, we have $\rho(\{s, t\}) = \rho(s) \circ \rho(t) - \rho(t) \circ \rho(s)$ for all sections s, t of $\overrightarrow{\mathcal{A}}$. Note that in particular, we can take $\mathbb{V} = \mathbb{A} := \mathfrak{g}$, in which case we recover the bundle $\overrightarrow{\mathcal{A}}$. Since in this case the action is given by the algebraic bracket, we denote it by ad (instead of ρ). If we want to deal with both actions simultaneously, or if there is no risk of confusion, we will also simply write $\bullet$ for the action, i.e. $s \bullet t$ equals $\rho(s)(t)$ or $\mathrm{ad}(s)(t) = \{s, t\}$.

Now we are ready to formulate the main result of this paper:

Theorem. — *Suppose that $\mathcal{V} \to M$ is a vector bundle, and suppose that for each preferred connection ∇ on M we can construct an isomorphism $\mathcal{V} \to \overrightarrow{\mathcal{V}} = \oplus_j \mathcal{V}_j$, which we write as $t \mapsto \overrightarrow{t} = (\dots, t_j, t_{j+1}, \dots)$ both on the level of elements and of sections. Suppose, further, that changing from ∇ to $\hat{\nabla}$ with corresponding one–form Υ, this isomorphism changes to $t \mapsto \widehat{\overrightarrow{t}} = (\dots, \hat{t}_j, \hat{t}_{j+1}, \dots)$, where*

$$\hat{t}_k = \sum_{i \geq 0} \tfrac{1}{i!} \rho(\Upsilon)^i (t_{k-i}).$$

Then for a point $x \in M$ the set $\mathcal{A}_x$ of all linear maps $\varphi : \mathcal{V}_x \to \mathcal{V}_x$ for which there exists an element $\overrightarrow{\varphi} \in \overrightarrow{\mathcal{A}}_x$ such that $\overrightarrow{\varphi(t)} = \rho(\overrightarrow{\varphi})(\overrightarrow{t})$ for all $t \in \mathcal{V}_x$ is independent of the choice of the preferred connection ∇. The spaces $\mathcal{A}_x$ form a smooth subbundle

$\mathcal{A}$ of $L(\mathcal{V},\mathcal{V}) = \mathcal{V}^* \otimes \mathcal{V}$, which is an adjoint tractor bundle on M in the sense of [**3**, 2.2]. *Moreover the isomorphism $\mathcal{A} \to \overrightarrow{\mathcal{A}}$ defined by $\varphi \mapsto \overrightarrow{\varphi}$ (given above) has the same transformation property as the isomorphism above, i.e.*

$$\hat{\varphi}_k = \sum_{i\geq 0} \tfrac{1}{i!}\,\mathrm{ad}(\Upsilon)^i(\varphi_{k-i}).$$

Then $\mathcal{V}$ is the $\mathbb{V}$–tractor bundle for an appropriate adapted frame bundle for $\mathcal{A}$.

The expression (in the isomorphism corresponding to ∇)

$$\overrightarrow{\nabla^{\mathcal{V}}_\xi t} = \nabla_\xi \overrightarrow{t} + \big(\rho(\xi) + \rho(\mathsf{P}(\xi))\big)(\overrightarrow{t})$$

for $\xi \in \mathfrak{X}(M)$ and $t \in \Gamma(\mathcal{V})$ defines a normal tractor connection on $\mathcal{V}$, and the same formula with $\mathcal{V}$ replaced by $\mathcal{A}$ and ρ replaced by ad *defines a normal tractor connection on $\mathcal{A}$. Thus, $\mathcal{V}$ and $\mathcal{A}$ are the (up to isomorphism unique) normal tractor bundles on M corresponding to $\mathbb{V}$ and $\mathfrak{g}$, respectively.*

Finally, the curvature R of both these connections is (in the isomorphism corresponding to ∇) given by

$$\overrightarrow{R(\xi,\eta)(s)} = (T(\xi,\eta) + W(\xi,\eta) + CY(\xi,\eta)) \bullet \overrightarrow{s},$$

where T, W and CY are the torsion, the Weyl–curvature and the Cotton–York tensor of ∇.

The remainder of this section is dedicated to the proof of this theorem.

1.4. The adjoint tractor bundle determined by $\mathcal{V}$. — To follow the approach to tractor bundles developed in [**3**], we need first an adjoint tractor bundle $\mathcal{A} \to M$ before we can deal with (or even define) general tractor bundles. So we first discuss the bundle $\mathcal{A}$ from theorem 1.3.

First note, that we can nicely rewrite the change of isomorphisms from theorem 1.3 as $\hat{\overrightarrow{t}} = e^{\rho(\Upsilon)}(\overrightarrow{t})$, where the exponential is defined as a power series as usual. Since $\rho(\Upsilon)$ is by construction nilpotent, this sum is actually finite. Moreover, since ρ corresponds to the infinitesimal action of the Lie algebra $\mathfrak{g}$, $e^{\rho(\Upsilon)}$ in that picture corresponds to the (group) action of $\exp(Z)$, where $Z \in \mathfrak{g}_1$ corresponds to Υ. From the definition of a $(\mathfrak{g}, P)$–module in 1.2 it follows that for each $A \in \mathfrak{g}$ and $v \in \mathbb{V}$ we have

$$\exp(-Z)\cdot A\cdot \exp(Z)\cdot v = (\mathrm{Ad}(\exp(-Z))(A))\cdot v = (e^{-\,\mathrm{ad}(Z)}(A))\cdot v,$$

and thus $A\cdot \exp(Z)\cdot v = \exp(Z)\cdot(e^{-\,\mathrm{ad}(Z)}(A))\cdot v$. Transferring this to the manifold, we obtain

$$\rho(s) \circ e^{\rho(\Upsilon)} = e^{\rho(\Upsilon)} \circ \rho(e^{-\,\mathrm{ad}(\Upsilon)}(s)), \tag{1}$$

for each $\Upsilon \in \Omega^1(M)$ and each $s \in \Gamma(\overrightarrow{\mathcal{A}})$. Note further, that $e^{-\,\mathrm{ad}(\Upsilon)}$ is just the identity on $\mathcal{A}_1 = T^*M$, while for $\Phi \in \mathcal{A}_0 = \mathrm{End}_0\, TM$ we have $e^{-\,\mathrm{ad}(\Upsilon)}(\Phi) = \Phi - \{\Upsilon, \Phi\} \in \mathcal{A}_0 \oplus \mathcal{A}_1$ and for $\xi \in \mathcal{A}_{-1} = TM$, we have $e^{-\,\mathrm{ad}(\Upsilon)}(\xi) = \xi - \{\Upsilon, \xi\} + \frac{1}{2}\{\Upsilon, \{\Upsilon, \xi, \}\}$.

The defining equation for $\varphi \in L(\mathcal{V}_x, \mathcal{V}_x)$ to lie in $\mathcal{A}_x$ from theorem 1.3 is just $\overrightarrow{\varphi(t)} = \rho(\overrightarrow{\varphi})(\overrightarrow{t})$ for some element $\overrightarrow{\varphi}$ of $\overrightarrow{\mathcal{A}}_x$ (and all $t \in \mathcal{V}_x$). If $\hat{\nabla}$ is another preferred connection and $\Upsilon \in \Omega^1(M)$ is the corresponding one–form, then using formula (1) from above, we compute

$$\widehat{\overrightarrow{\varphi(t)}} = e^{\rho(\Upsilon)} \circ \rho(\overrightarrow{\varphi})(\overrightarrow{t}) = \rho(e^{\mathrm{ad}(\Upsilon)}(\overrightarrow{\varphi})) \circ e^{\rho(\Upsilon)}(\overrightarrow{t}) = \rho(e^{\mathrm{ad}(\Upsilon)}(\overrightarrow{\varphi}))(\widehat{\overrightarrow{t}}),$$

which shows both that $\mathcal{A}_x$ is independent of the choice of preferred connection, and that $\widehat{\overrightarrow{\varphi}} = e^{\mathrm{ad}(\Upsilon)}(\overrightarrow{\varphi})$, so the change of isomorphisms $\mathcal{A} \to \overrightarrow{\mathcal{A}}$ induced by preferred connections is proved. A preferred connection thus induces a global isomorphism $\mathcal{A} \to \overrightarrow{\mathcal{A}}$, so $\mathcal{A} \subset \mathcal{V}^* \otimes \mathcal{V}$ is a smooth subbundle. Next, the (pointwise) commutator of endomorphisms defines an algebraic bracket $\{\ ,\ \}$ on $\mathcal{A}$, making it into a bundle of Lie algebras. From the fact that ρ comes from a representation of $\mathfrak{g}$ we conclude that $\overrightarrow{\{\varphi_1, \varphi_2\}} = \{\overrightarrow{\varphi_1}, \overrightarrow{\varphi_2}\}$, so for each preferred connection the isomorphism $\mathcal{A} \to \overrightarrow{\mathcal{A}}$ is an isomorphism of bundles of Lie algebras.

From the formula $\widehat{\overrightarrow{\varphi}} = e^{\mathrm{ad}(\Upsilon)}(\overrightarrow{\varphi})$ it follows that if $\overrightarrow{\varphi}$ lies in $\mathcal{A}_0 \oplus \mathcal{A}_1$ then the same is true for $\widehat{\overrightarrow{\varphi}}$, and moreover their components in $\mathcal{A}_0$ are equal. Similarly, if $\overrightarrow{\varphi} \in \mathcal{A}_1$ then $\widehat{\overrightarrow{\varphi}} = \overrightarrow{\varphi}$. Thus, we get an invariantly defined filtration $\mathcal{A} = \mathcal{A}^{-1} \supset \mathcal{A}^0 \supset \mathcal{A}^1$ of $\mathcal{A}$. Furthermore, writing $\mathrm{gr}(\mathcal{A})$ to denote the associated graded vector bundle of $\mathcal{A}$ (i.e. $\mathrm{gr}(\mathcal{A}) = (\mathcal{A}^{-1}/\mathcal{A}^0) \oplus (\mathcal{A}^0/\mathcal{A}^1) \oplus \mathcal{A}^1$) then we also get a canonical isomorphism from $\mathrm{gr}(\mathcal{A}) \to \overrightarrow{\mathcal{A}}$. In particular, since $\overrightarrow{\mathcal{A}}$ is a locally trivial bundle of graded Lie algebras modelled on $\mathfrak{g}$ and the isomorphism $\mathcal{A} \to \overrightarrow{\mathcal{A}}$ provided by any preferred connection is filtration preserving, we see that $\mathcal{A}$ is a locally trivial bundle of filtered Lie algebras over M modelled on $\mathfrak{g}$, and thus an adjoint tractor bundle in the sense of [**3**, 2.2].

Next, we can can use $\mathcal{A}$ to construct a corresponding adapted frame bundle (see [**3**, 2.2]), that is a principal P–bundle $\mathcal{G} \to M$ such that $\mathcal{A} = \mathcal{G} \times_P \mathfrak{g}$, the associated bundle with respect to the adjoint action. First note that if $\mathcal{A}$ is given as an associated bundle in this way then, by definition, any point $u \in \mathcal{G}$ lying over $x \in M$ induces an isomorphism $\underline{u} : \mathfrak{g} \to \mathcal{A}_x$ of filtered Lie algebras. Now if $\psi : \mathfrak{g} \to \mathcal{A}_x$ is any such isomorphism, then we can pass to the associated graded Lie algebras on both sides and, in view of the canonical isomorphism from $\mathrm{gr}(\mathcal{A})$ to $\overrightarrow{\mathcal{A}}$ constructed above, the result is an isomorphism $\mathfrak{g} \to \overrightarrow{\mathcal{A}}_x$. With this observations at hand, we now define $\mathcal{G}_x$ to be the set of all pairs (u_0, ψ), where $u_0 \in (\mathcal{G}_0)_x$ and $\psi : \mathfrak{g} \to \mathcal{A}_x$ is an isomorphism of filtered Lie algebras such that the induced isomorphism $\mathfrak{g} \to \overrightarrow{\mathcal{A}}_x$ of graded Lie algebras equals $\underline{u_0}$, see 1.1. Putting $\mathcal{G} = \cup_{x \in M} \mathcal{G}_x$ we automatically get a smooth structure on $\mathcal{G}$, since we can view $\mathcal{G}$ as a submanifold the fibred product of $\mathcal{G}_0$ with the linear frame bundle of $\mathcal{A}$. The first projection is a surjective submersion from this fibred product onto $\mathcal{G}_0$ and we can compose with this the usual projection from $\mathcal{G}_0$ to M. Moreover, for each $u_0 \in \mathcal{G}_0$, composing with $\underline{u_0}$ the inverse of the isomorphism $\mathcal{A}_x \to \overrightarrow{\mathcal{A}}_x$ provided by any preferred connection, gives by construction an isomorphism ψ such that $(u_0, \psi) \in \mathcal{G}$. Hence, the restriction of this surjective submersion to $\mathcal{G}$ is still surjective.

Next, we define a right action of P on $\mathcal{G}$ by $(u_0, \psi) \cdot g := (u_0 \cdot g_0, \psi \circ \mathrm{Ad}(g))$, where $g = g_0 \exp(Z)$ and in the first component we use the principal right action on $\mathcal{G}_0$. Clearly, this is well defined (i.e. $(u_0, \psi) \cdot g$ lies again in $\mathcal{G}$) and a right action. We claim that this action is free and transitive on each fibre of the projection $\mathcal{G} \to M$. If $(u_0, \psi)\cdot g = (u_0, \psi)$ for one point, then we must have $g_0 = e$ since the principal action of $\mathcal{G}_0$ is free, so we must have $g = \exp(Z)$. But for $Z \in \mathfrak{g}_1$ the adjoint action of $\exp(Z)$ equals the identity if and only if $Z = 0$, see [**12**, lemma 3.2], so freeness follows. On the other hand, the principal action on $\mathcal{G}_0$ is transitive on each fibre, so it suffices to deal with the case of two points of the form (u_0, ψ_1) and (u_0, ψ_2). But in this case, by construction $\psi_1^{-1} \circ \psi_2 : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of the filtered Lie algebra $\mathfrak{g}$ which induces the identity on the associated graded Lie algebra, and we have observed in 1.1 that any such isomorphism is of the form $\mathrm{Ad}(\exp(Z))$ for some $Z \in \mathfrak{g}_1$. Thus, from $\mathcal{G}$ we have on the one hand a principal P_+ bundle (with a G_0–equivariant projection) $\mathcal{G} \to \mathcal{G}_0$ and on the other hand a principal P bundle $\mathcal{G} \to M$.

Next, consider the map $\mathcal{G} \times \mathfrak{g} \to \mathcal{A}$ defined by $((u_0, \psi), X) \mapsto \psi(X)$. This clearly maps both $((u_0, \psi)\cdot g, X)$ and $((u_0, \psi), \mathrm{Ad}(g)(X))$ to $\psi(\mathrm{Ad}(g)(X))$, so it induces a homomorphism $\mathcal{G} \times_P \mathfrak{g} \to \mathcal{A}$ of vector bundles. The restriction of this to each fibre by construction is a linear isomorphism and, in fact, an isomorphism of filtered Lie algebras, so the whole map is an isomorphism of bundles of filtered Lie algebras.

Finally, we have to show that $\mathcal{V} = \mathcal{G} \times_P \mathbb{V}$. To do this, choose a preferred connection ∇. This defines a smooth map $\tau : \mathcal{G} \to \mathcal{G}_0 \times P_+$ as follows: For $(u_0, \psi) \in \mathcal{G}$ consider the composition consisting of $\psi : \mathfrak{g} \to \mathcal{A}_x$ followed by the isomorphism $\mathcal{A}_x \to \overrightarrow{\mathcal{A}}_x$ provided by ∇ and then the isomorphism $\underline{u_0}^{-1} : \overrightarrow{\mathcal{A}}_x \to \mathfrak{g}$. By construction, this is an isomorphism of filtered Lie algebras which induces the identity on the associated graded Lie algebra, so it is given as $\mathrm{Ad}(\tau(u_0, \psi))$ for a unique element $\tau(u_0, \psi) \in P_+$. Clearly $\mathrm{Ad} \circ \tau$ is smooth and so τ is smooth. From the defining equation one immediately verifies that for $g_0 \in G_0$ and $g' \in P_+$ we get $\tau((u_0, \psi)\cdot g_0) = g_0^{-1} \tau(u_0, \psi) g_0$ and $\tau((u_0, \psi)\cdot g') = \tau(u_0, \psi) g'$, respectively.

Now we define a map $f : \mathcal{G} \times \mathbb{V} \to \mathcal{V}$ by requiring that $\overrightarrow{f((u_0, \psi), v)} = \underline{u_0}(\tau(u_0, \psi)\cdot v)$, where the action on the right hand side is in the $\mathfrak{g}$–module $\mathbb{V}$, and the isomorphism $\underline{u_0} : \mathbb{V} \to \overrightarrow{\mathcal{V}}_x$ comes from the fact that $\overrightarrow{\mathcal{V}}$ is an associated bundle to $\mathcal{G}_0$. Using the fact that $\underline{u_0\cdot g_0}(v) = \underline{u_0}(g_0\cdot v)$ and the equivariancy properties of τ we see that $\overrightarrow{f((u_0, \psi)\cdot g, v)} = \overrightarrow{f((u_0, \psi), g\cdot v)}$ for all g which are either in G_0 or in P_+ and thus for all $g \in P$. consequently, f factors to a homomorphism $\mathcal{G} \times_P \mathbb{V} \to \mathcal{V}$ of vector bundles, which by construction induces a linear isomorphism in each fibre and thus is an isomorphism of vector bundles. Hence, $\mathcal{V}$ is the $\mathbb{V}$–tractor bundle corresponding to the adapted frame bundle $\mathcal{G}$ for the adjoint tractor bundle $\mathcal{A}$.

It should be noted, at this point, that the isomorphism $\mathcal{G} \times_P \mathbb{V} \to \mathcal{V}$ constructed above is actually independent of the choice of the preferred connection ∇. Indeed, if $\hat{\nabla}$ is another preferred connection corresponding to $\Upsilon \in \Omega^1(M)$, then the definition of τ easily implies that $\hat{\tau}(u_0, \psi) = \exp(\underline{u_0}^{-1}(\Upsilon))\tau(u_0, \psi)$. Using this, and the formula for $\overrightarrow{f(\widehat{(u_0, \psi)}, v)}$, one easily verifies directly, that even the map f itself is independent of the choice of ∇. Finally a point of notation. Since $\mathcal{V}$ may be viewed as an associated bundle as established here it is clear that any point $u \in \mathcal{G}$ lying over $x \in M$ induces a $(\mathfrak{g}, P)$–isomorphism $\underline{u} : \mathbb{V} \to \mathcal{V}_x$.

1.5. The tractor connections. — The next step is to prove that the definition of the connection $\nabla^{\mathcal{V}}$ in theorem 1.3 is independent of the choice of the preferred connection ∇ and that $\nabla^{\mathcal{V}}$ is a tractor connection on $\mathcal{V}$. Since this uses only the formula for the transformation of isomorphisms induced by a change of preferred connection, we recover at the same time the result for $\mathcal{A}$, since this is just the special case $\mathbb{V} = \mathfrak{g}$.

The definition of $\nabla^{\mathcal{V}}$ in theorem 1.3 reads as

$$\overrightarrow{\nabla^{\mathcal{V}}_{\xi} t} = \nabla_{\xi} \overrightarrow{t} + (\rho(\xi) + \rho(\mathsf{P}(\xi)))(\overrightarrow{t}).$$

Since any preferred connection ∇ is induced by a principal connection on $\mathcal{G}_0$, and $\rho : \overrightarrow{\mathcal{A}} \to L(\overrightarrow{\mathcal{V}}, \overrightarrow{\mathcal{V}})$ is induced by a G_0–homomorphism $\mathfrak{g} \to L(\mathbb{V}, \mathbb{V})$ we conclude that

$$\nabla_{\xi}(\rho(\Upsilon)(\overrightarrow{t})) = \rho(\nabla_{\xi}\Upsilon)(\overrightarrow{t}) + \rho(\Upsilon)(\nabla_{\xi}\overrightarrow{t}),$$

for any vector field $\xi \in \mathfrak{X}(M)$, any one–form Υ and section $\overrightarrow{t}$ of $\overrightarrow{\mathcal{V}}$. Taking into account that the bracket $\{\ ,\ \}$ is trivial on $\Omega^1(M)$ and hence the actions of one–forms via ρ always commute, we get this implies that

$$\nabla_{\xi}(\rho(\Upsilon)^i(\overrightarrow{t})) = i\rho(\nabla_{\xi}\Upsilon)\rho(\Upsilon)^{i-1}(\overrightarrow{t}) + \rho(\Upsilon)^i(\nabla_{\xi}\overrightarrow{t}),$$

which in turn leads to

$$\nabla_{\xi}(e^{\rho(\Upsilon)}(\overrightarrow{t})) = \rho(\nabla_{\xi}\Upsilon)(e^{\rho(\Upsilon)}(\overrightarrow{t})) + e^{\rho(\Upsilon)}(\nabla_{\xi}\overrightarrow{t}). \tag{2}$$

If $\hat{\nabla}$ is another preferred connection and Υ is the corresponding one–form, then $\hat{\nabla}_{\xi}\overrightarrow{t} = \nabla_{\xi}\overrightarrow{t} + \rho(\{\Upsilon, \xi\})(\overrightarrow{t})$. Replacing in this formula $\overrightarrow{t}$ by $\widehat{\overrightarrow{t}} = e^{\rho(\Upsilon)}(\overrightarrow{t})$ and using formula (2) to compute $\nabla_{\xi}\widehat{\overrightarrow{t}}$, we get

$$\hat{\nabla}_{\xi}\widehat{\overrightarrow{t}} = \widehat{(\nabla_{\xi}\overrightarrow{t})} + \rho(\nabla_{\xi}\Upsilon)(\widehat{\overrightarrow{t}}) + \rho(\{\Upsilon, \xi\})(\widehat{\overrightarrow{t}}).$$

From formula (1) of 1.4 we have $\rho(\hat{\overrightarrow{s}})\widehat{\overrightarrow{t}} = \widehat{\rho(\overrightarrow{s})\overrightarrow{t}}$ for any sections $\overrightarrow{s} \in \Gamma(\overrightarrow{\mathcal{A}})$ and $\overrightarrow{t} \in \Gamma(\overrightarrow{\mathcal{V}})$. For example in the case that $\mathcal{V} = \mathcal{A}$ we have on one hand that for $\omega \in \Omega^1(M)$, we have $\rho(\omega)(\widehat{\overrightarrow{t}}) = \widehat{\rho(\omega)(\overrightarrow{t})}$. On the other hand for $\xi \in \mathfrak{X}(M)$, we get

$$\rho(\xi)(\widehat{\overrightarrow{t}}) = \widehat{\rho(\xi)(\overrightarrow{t})} - \rho(\{\Upsilon, \xi\})(\widehat{\overrightarrow{t}}) - \tfrac{1}{2}\rho(\{\Upsilon, \{\Upsilon, \xi\}\})(\widehat{\overrightarrow{t}}).$$

From 1.1 we know that $\hat{\mathsf{P}}(\xi) = \mathsf{P}(\xi) - \nabla_{\xi}\Upsilon + \frac{1}{2}\{\Upsilon, \{\Upsilon, \xi\}\}$. Thus, together with the above we arrive at

$$(\rho(\xi) + \rho(\hat{\mathsf{P}}(\xi)))(\widehat{\overrightarrow{t}}) = \widehat{\rho(\xi)(\overrightarrow{t})} + \widehat{\rho(\mathsf{P}(\xi))(\overrightarrow{t})} - \rho(\nabla_{\xi}\Upsilon)(\widehat{\overrightarrow{t}}) - \rho(\{\Upsilon, \xi\})(\widehat{\overrightarrow{t}}),$$

which exactly cancels with the contribution $\hat{\nabla}_{\xi}\widehat{\overrightarrow{t}} - \widehat{(\nabla_{\xi}\overrightarrow{t})}$ calculated above, so $\nabla^{\mathcal{V}}$ is independent of the choice of the preferred connection ∇.

To verify that $\nabla^{\mathcal{V}}$ is a tractor connection, we first verify the non–degeneracy condition from [**3**, definition 2.5(2)], which is very simple. In fact, the canonical filtration $\dots \supset \mathcal{V}^j \supset \mathcal{V}^{j+1} \supset \dots$ on $\mathcal{V}$ is simply given by $t \in \mathcal{V}^j$ if and only if $\overrightarrow{t} \in \mathcal{V}_j \oplus \mathcal{V}_{j+1} \oplus \dots$, which is clearly independent of the choice of the preferred connection. In particular, as we observed for $\mathcal{A}$ in 1.4, we get a canonical isomorphism between $\operatorname{gr}(\overrightarrow{\mathcal{V}})$, the graded vector bundle associated to $\mathcal{V}$, and $\overrightarrow{\mathcal{V}}$. But by construction, for each vector field ξ, ∇_ξ preserves the decomposition $\overrightarrow{\mathcal{V}} = \oplus \mathcal{V}_j$. Hence for a smooth section t of $\mathcal{V}^j$, we see that $\nabla^{\mathcal{V}}_\xi t$ is a section of $\mathcal{V}^{j-1}$ and its class in $\mathcal{V}^{j-1}/\mathcal{V}^j$ is mapped under the above isomorphism to $\rho(\xi)(t_j)$. Thus, the fact that any nontrivial representation of $\mathfrak{g}$ is faithful implies the non–degeneracy condition, since it implies that for nonzero $\xi \in T_xM$ we find a j and $t_j \in (\mathcal{V}_j)_x$ such that $\rho(\xi)(t_j)$ is nonzero.

The second condition is to verify that $\nabla^{\mathcal{V}}$ is a $\mathfrak{g}$–connection in the sense of [**3**, definition 2.5(1)]. So what we have to do is the following: For a smooth section $t \in \Gamma(\mathcal{V})$ consider the corresponding P–equivariant map $\tilde{t} : \mathcal{G} \to \mathbb{V}$. Then take a point $u \in \mathcal{G}$ lying over $x \in M$, a tangent vector $\bar{\xi} \in T_u\mathcal{G}$ and its image $\xi \in T_xM$, and consider the difference $\bar{\xi}\cdot\tilde{t} - \underline{u}^{-1}(\nabla^{\mathcal{V}}_\xi t(x)) \in \mathbb{V}$. The condition to verify is that this is given by the action of an element of $\mathfrak{g}$ on $\tilde{t}(u)$. Note first, that if $\bar{\xi}$ is vertical, the second term vanishes so the condition is automatically satisfied by (the infinitesimal version of) equivariancy of $\tilde{t}$.

Effectively, we have already observed in 1.4 above that any preferred connection ∇ induces a global section σ of $\mathcal{G} \to \mathcal{G}_0$ by mapping $u_0 \in (\mathcal{G}_0)_x$ to $(u_0, \psi) \in \mathcal{G}_x$, where ψ is the composition of the inverse of the isomorphism $\mathcal{A}_x \to \overrightarrow{\mathcal{A}}_x$ defined by ∇ with $\underline{u_0} : \mathfrak{g} \to \overrightarrow{\mathcal{A}}_x$. Moreover, by construction this section is G_0–equivariant. Now if $(u_0, \psi) \in \mathcal{G}_x$ is any point, then there is an element $g' \in P_+$ such that $(u_0, \psi) = \sigma(u_0)\cdot g'$. This means that ψ is the composition of ψ' with $\mathrm{Ad}(g')$, where $\sigma(u_0) = (u_0, \psi')$ and $g' = \exp(Z)$ for a unique $Z \in \mathfrak{g}_1$. Extend $\underline{u_0}(Z) \in T_xM$ to a one–form $\Upsilon \in \Omega^1(M)$ and consider the connection $\hat{\nabla}$ corresponding to Υ. Then using $\underline{u_0} \circ \mathrm{Ad}(\exp(Z)) = e^{\mathrm{ad}(\Upsilon(x))} \circ \underline{u_0}$, we see that the section $\hat{\sigma}$ corresponding to $\hat{\nabla}$ has the property that $\hat{\sigma}(u_0) = (u_0, \psi)$.

Returning to our original problem, we may thus assume without loss of generality that $(u_0, \psi) = \sigma(u_0)$ for the section σ corresponding to a preferred connection ∇. Moreover, adding an appropriate vertical vector, we may assume that $\bar{\xi} = T_{u_0}\sigma\cdot\xi'$ for some $\xi' \in T_{u_0}\mathcal{G}_0$, which still projects to $\xi \in T_xM$. But then $\bar{\xi}\cdot\tilde{t}(u) = \xi'\cdot(\tilde{t} \circ \sigma)(u_0)$. Now we just have to make a final observation. The correspondence between sections and equivariant functions is given by $\tilde{t}(u_0, \psi) = \psi^{-1}(t(x))$. Moreover, since $(u_0, \psi) = \sigma(u_0)$, we see from 1.4 that $\psi^{-1}(t(x)) = \underline{u_0}^{-1}(\overrightarrow{t}(x))$. Consequently, $(\tilde{t} \circ \sigma) : \mathcal{G}_0 \to \mathbb{V}$ is exactly the G_0–equivariant function corresponding to $\overrightarrow{t}$. Since the preferred connection ∇ is induced from a principal connection on $\mathcal{G}_0$, the difference $\xi'\cdot(\tilde{t} \circ \sigma) - \underline{u_0}(\nabla_\xi \overrightarrow{t})$ is given by the action of an element of $\mathfrak{g}_0$ (namely the value of the connection form on ξ') on $\tilde{t}(\sigma(u_0))$. Thus, also $\xi'\cdot(\tilde{t} \circ \sigma) - \underline{u_0}(\overrightarrow{\nabla^{\mathcal{V}}_\xi t})$ is given by the action of an element of $\mathfrak{g}$ on this, namely the one just described plus the ones corresponding to ξ and $\mathsf{P}(\xi)$. But since $\psi = \underline{(u_0, \psi)}$ and thus $\psi = \underline{\sigma(u_0)}$, we see from above that $\underline{u_0}^{-1}(\overrightarrow{\nabla^{\mathcal{V}}_\xi t}) = \underline{\sigma(u_0)}^{-1}(\nabla^{\mathcal{V}}_\xi t)$, so $\nabla^{\mathcal{V}}$ is indeed a $\mathfrak{g}$–connection and thus a tractor connection.

1.6. Curvature. — The final thing is to compute the curvature and, as above, it suffices to do this for $\nabla^{\mathcal{V}}$ since $\mathcal{A}$ is the special case $\mathbb{V}=\mathfrak{g}$. By definition

$$\overrightarrow{\nabla^{\mathcal{V}}_\eta t}=\nabla_\eta \vec{t}+(\rho(\eta)+\rho(\mathsf{P}(\eta)))(\vec{t}).$$

Since ρ is covariantly constant for any preferred connection, we get

$$\begin{aligned}\nabla_\xi\overrightarrow{\nabla^{\mathcal{V}}_\eta t}=\nabla_\xi\nabla_\eta \vec{t}+\rho(\nabla_\xi\eta)(\vec{t})+\rho(\eta)(\nabla_\xi\vec{t})+\\ +\rho(\nabla_\xi(\mathsf{P}(\eta)))(\vec{t})+\rho(\mathsf{P}(\eta))(\nabla_\xi\vec{t}).\end{aligned}\tag{3}$$

Thus, $\overrightarrow{\nabla^{\mathcal{V}}_\xi\nabla^{\mathcal{V}}_\eta t}$ is given by adding to the above sum the terms

$$\begin{aligned}&\rho(\xi)(\nabla_\eta\vec{t})+\rho(\xi)\circ\rho(\eta)(\vec{t})+\rho(\xi)\circ\rho(\mathsf{P}(\eta))(\vec{t}))+\\ &\rho(\mathsf{P}(\xi))(\nabla_\eta\vec{t})+\rho(\mathsf{P}(\xi))\circ\rho(\eta)(\vec{t})+\rho(\mathsf{P}(\xi))\circ\rho(\mathsf{P}(\eta))(\vec{t}).\end{aligned}\tag{4}$$

Finally, directly from the definition of $\nabla^{\mathcal{V}}$, we get

$$\overrightarrow{\nabla^{\mathcal{V}}_{[\xi,\eta]}t}=\nabla_{[\xi,\eta]}\vec{t}+(\rho([\xi,\eta])+\rho(\mathsf{P}([\xi,\eta])))(\vec{t}).\tag{5}$$

To obtain the formula for $\overrightarrow{R^{\mathcal{V}}(\xi,\eta)(t)}$, by definition of the curvature, we have to take all terms from (3) and (4), then subtract the same terms with ξ and η exchanged and finally subtract the terms from (5). Since $\{\xi,\eta\}=\{\mathsf{P}(\xi),\mathsf{P}(\eta)\}=0$, the second and last term in (4) are symmetric in ξ and η (see 1.3), so we may forget those. Moreover the first term in (4) together with the third term in the right hand side of (3), as well as the fourth term in (4) together with the last term in the right hand side of (3) are again symmetric, so we may forget all those. Now the first term in the right hand side of (3) together with its alternation and the negative of the first term in the right hand side of (5) add up to $\rho(R(\xi,\eta))(\vec{t})$, where $R\in\Gamma(\Lambda^2T^*M\otimes\mathrm{End}_0TM)$ is the curvature of ∇ (viewed as a connection on TM). On the other hand, the two remaining terms in (4) together with their alternations add up to $\rho(\{\mathsf{P}(\xi),\eta\}-\{\mathsf{P}(\eta),\xi\})(\vec{t})$. Together with the curvature term from above, this exactly leads to $\rho(W(\xi,\eta))(\vec{t})$. Then the second term in the right hand side of (3) together with its alternation and minus the second term in the right hand side of (5) give $\rho(T(\xi,\eta))(\vec{t})$ by the definition of the torsion. The remaining part is simply

$$\rho(\nabla_\xi(\mathsf{P}(\eta))-\nabla_\eta(\mathsf{P}(\xi))-\mathsf{P}([\xi,\eta]))(\vec{t}).$$

Inserting $[\xi,\eta]=\nabla_\xi\eta-\nabla_\eta\xi-T(\xi,\eta)$ we see that this simply equals $\rho(CY(\xi,\eta))(\vec{t})$ by definition of the Cotton–York tensor.

Note that this immediately implies that $\nabla^{\mathcal{V}}$ is a normal tractor connection on $\mathcal{V}$, since by construction T and W are ∂^*–closed, while for CY this is trivially true because of homogeneity ($\partial^*(CY):\vec{\mathcal{A}}\to\vec{\mathcal{A}}$ would be homogeneous of degree three).

1.7. The fundamental D–operators and a summary. — Starting from a bundle $\mathcal{V}\to M$ with an appropriate class of isomorphisms $\mathcal{V}\to\vec{\mathcal{V}}$ provided by preferred connections, we have constructed the normal adjoint tractor bundle $\mathcal{A}\to M$ and proved that $\mathcal{V}$ is the $\mathbb{V}$–tractor bundle corresponding to $\mathcal{A}$. Moreover, for any preferred connection ∇ we get an isomorphism $\mathcal{A} \to \overrightarrow{\mathcal{A}}$ which is compatible with the isomorphisms for $\mathcal{V}$ in the sense that denoting the canonical action $\mathcal{A} \otimes \mathcal{V} \to \mathcal{V}$ by $(s \otimes t) \mapsto s \bullet t$, then $\overrightarrow{s \bullet t} = \rho(\overrightarrow{s})(\overrightarrow{t})$. So we are able to work consistently both with $\mathcal{A}$ and $\mathcal{V}$ by working with the bundles $\overrightarrow{\mathcal{A}}$ and $\overrightarrow{\mathcal{V}}$ which are simply direct sums of familiar, easily understood bundles. Moreover, we have constructed explicitly the normal tractor connections on $\mathcal{V}$ and $\mathcal{A}$.

The fundamental D-operators are first order invariant differential operators which for parabolic geometries generalise the notion of covariant derivatives in a rather natural way. For weighted tensor bundles, tractor bundles and tensor products of these the fundamental D-operators are described explicitly in [**3**] in terms of the tractor connection. In particular via proposition 3.2 of that work and the results above for the tractor connection we can compute, in our current setting, the fundamental D–operators both on $\mathcal{V}$ and on $\mathcal{A}$. Explicitly, on $\mathcal{V}$, the fundamental D–operator is given by

$$\overrightarrow{D_s t} = \nabla_\xi \overrightarrow{t} - \rho(\Phi)(\overrightarrow{t}) - \rho(\omega - \mathsf{P}(\xi))(\overrightarrow{t}),$$

where $t \in \Gamma(\mathcal{V})$ and $s \in \Gamma(\mathcal{A})$ is such that $\overrightarrow{s} = (\xi, \Phi, \omega)$. In a similar notation, we get on $\mathcal{A}$ the formula

$$\overrightarrow{D_{s_1} s_2} = \nabla_\xi \overrightarrow{s_2} - \{\Phi, \overrightarrow{s_2}\} - \{\omega - \mathsf{P}(\xi), \overrightarrow{s_2}\},$$

which expanded into components exactly gives the formula in [**3**, 4.14]. By naturality of the fundamental D–operators (see [**3**, proposition 3.1]) this implies that on any of the bundles $\mathcal{V}_j$ (or of any of the subbundles of any such bundle corresponding to a G_0-invariant component of $\mathbb{V}_j$), the fundamental D–operator is given by $D_s\sigma = \nabla_\xi\sigma - \Phi \bullet \sigma$, where again $\overrightarrow{s} = (\xi, \Phi, \omega)$. Since the fundamental D–operator is $\mathcal{A}^*$-valued and we know the fundamental D–operator on $\mathcal{A} \cong \mathcal{A}^*$, we may iterate this operator. For example, the formula for the square of D from [**3**, 4.14] continues to hold in this case.

2. Conformal Standard tractors

In this section we show that our results are very easy to apply in concrete situations. Moreover, we show how to relate the bracket notation we have used here to a standard abstract index notation. Among particular results we construct a normal tractor bundle with connection, which we term the standard tractor bundle, and observe that this is isomorphic to the tractor bundle in [**1**]. This establishes that the latter is consistent with the canonical Cartan connection.

2.1. Conformal manifolds. — We shall work on a real conformal n-manifold M where $n \geq 3$. That is, we have a pair $(M, [g])$ where M is a smooth n-manifold and $[g]$ is a conformal equivalence class of metrics. Two metrics g and $\widehat{g}$ are said to be *conformally equivalent*, or just *conformal*, if $\widehat{g}$ is a positive scalar function multiple of g. In this case it is convenient to write $\widehat{g} = \Omega^2 g$ for some positive smooth function Ω. (The transformation $g \mapsto \widehat{g}$, which changes the choice of metric from the conformal class, is termed a *conformal rescaling*.) We shall allow the metrics in the equivalence class to have any fixed signature. For a given conformal manifold $(M, [g])$ we will denote by $\mathcal{L}$ the bundle of metrics. That is $\mathcal{L}$ is a subbundle of S^2T^*M with fibre $\mathbb{R}^+$ whose points correspond to the values of the metrics in the conformal class.

Following the usual conventions in abstract index notation, we will write $\mathcal{E}$ for the trivial bundle over M, $\mathcal{E}^i$ for TM and $\mathcal{E}_i$ for T^*M. Tensor products of these bundles will be indicated by adorning the symbol $\mathcal{E}$ with appropriate indices. For example, in this notation $\otimes^2 T^*M$ is written $\mathcal{E}_{ij}$ and we write $\mathcal{E}_{(ij)}$ to indicate the symmetric part of this bundle, so in this notation $\mathcal{L} \subset \mathcal{E}_{(ij)}$. Unless otherwise indicated, our indices will be *abstract indices* in the sense of Penrose [**10**]. An index which appears twice, once raised and once lowered, indicates a contraction. In case a frame is chosen and the indices are concrete, use of the Einstein summation convention (to implement the contraction) is understood. Given a choice of metric, indices will be raised and lowered using the metric without explicit mention. Finally we point out that these conventions will be extended in an obvious way to the tractor bundles described below.

We may view $\mathcal{L}$ as a principal bundle with group $\mathbb{R}_+$, so there are natural line bundles on $(M, [g])$ induced from the irreducible representations of $\mathbb{R}_+$. We write $\mathcal{E}[w]$ for the line bundle induced from the representation of weight $-w/2$ on $\mathbb{R}$ (that is $\mathbb{R}_+ \ni x \mapsto x^{-w/2} \in \mathrm{End}(\mathbb{R})$). Thus a section of $\mathcal{E}[w]$ is a real valued function f on $\mathcal{L}$ with the homogeneity property $f(\Omega^2 g, x) = \Omega^w f(g, x)$ where Ω is a positive function on M, $x \in M$ and g is a metric form the conformal class [g]. We will use the notation $\mathcal{E}_i[w]$ for $\mathcal{E}_i \otimes \mathcal{E}[w]$ and so on. Note that, as we shall see below, this convention differs in sign from the one of [**3**, 4.15]. We have kept with this convention in order to be consistent with [**1**].

Let $\mathcal{E}_+[w]$ be the fibre subbundle of $\mathcal{E}[w]$ corresponding to $\mathbb{R}_+ \subset \mathbb{R}$. Choosing a metric g from the conformal class defines a function $f : \mathcal{L} \to \mathbb{R}$ by $f(\hat{g}, x) = \Omega^{-2}$, where $\hat{g} = \Omega^2 g$, and this clearly defines a smooth section of $\mathcal{E}[-2]_+$. Conversely, if f is such a section, then $f(g, x)g$ is constant up the fibres of $\mathcal{L}$ and so defines a metric in the conformal class. So $\mathcal{E}_+[-2]$ is canonically isomorphic to $\mathcal{L}$, and the *conformal metric* $\boldsymbol{g}_{ij}$ is the tautological section of $\mathcal{E}_{ij}[2]$ that represents the map $\mathcal{E}_+[-2] \cong \mathcal{L} \to \mathcal{E}_{(ij)}$. On the other hand, for a section g_{ij} of $\mathcal{L}$ consider the map $\varphi_{ij} \mapsto g^{k\ell}\varphi_{k\ell} g_{ij}$, which is visibly independent of the choice of g. Thus, we get a canonical section $\boldsymbol{g}^{ij}$ of $\mathcal{E}^{ij}[-2]$ such that $\boldsymbol{g}_{ij}\boldsymbol{g}^{jk} = \delta_i^k$.

2.2. To identify conformal structures as a parabolic geometry we first need a $|1|$–graded Lie algebra $\mathfrak{g}$. To do this, for signature (p, q) $(p + q = n)$ consider $\mathbb{R}^{n+2}$ with coordinates $x_0, \dots, x_{n+1}$ and the inner product associated to the quadratic form $2x_0x_{n+1} + \sum_{i=1}^p x_i^2 - \sum_{i=p+1}^n x_i^2$, and let $\mathfrak{g}$ be the orthogonal Lie algebra with respect to this inner product, so $\mathfrak{g} = \mathfrak{so}(p+1, q+1)$. Let $\mathbb{I}$ be the $n \times n$ diagonal matrix with p 1's and q (-1)'s in the diagonal and put

$$\mathbb{J} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbb{I} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $\mathfrak{g}$ is the set of all $(n+2)\times(n+2)$ matrices $\tilde{A}$ such that $\tilde{A}^t\mathbb{J} = -\mathbb{J}\tilde{A}$, so in $(1,n,1)\times(1,n,1)$ block form, these are exactly the matrices of the form

$$\begin{pmatrix} a & Z & 0 \\ X & A & -\mathbb{I}Z^t \\ 0 & -X^t\mathbb{I} & -a \end{pmatrix}$$

with $X\in\mathbb{R}^n$, $Z\in\mathbb{R}^{n*}$, $a\in\mathbb{R}$ and $A\in\mathfrak{so}(p,q)$ (that is $A^t\mathbb{I} = -\mathbb{I}A$). The grading is given by assigning degree -1 to the entry corresponding to X, degree zero to the ones corresponding to a and A and degree one to the one corresponding to Z. Will use the notation $X\in\mathfrak{g}_{-1}$, $(a,A)\in\mathfrak{g}_0$ and $Z\in\mathfrak{g}_1$. Then the actions of $\mathfrak{g}_0$ on $\mathfrak{g}_{\mp1}$ induced by the bracket are given by $[(a,A),X] = AX - aX$ and $[(a,A),Z] = aZ - ZA$, which immediately implies that the grading element E is given by $E = (1,0)\in\mathfrak{g}_0$. As an appropriate $\mathfrak{g}$–invariant bilinear form on $\mathfrak{g}$ we choose $\frac{1}{2}$ times the trace form on $\mathfrak{g}$ and denote this by B. The advantage of this choice is that then the induced $\mathfrak{g}_0$–invariant pairing between $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ is exactly given by the standard dual pairing between $\mathbb{R}^n$ and $\mathbb{R}^{n*}$. For later use, we also note that the bracket $\mathfrak{g}_{-1}\times\mathfrak{g}_1\to\mathfrak{g}_0$ is given by $[X,Z] = (-ZX, XZ - \mathbb{I}Z^tX^t\mathbb{I})$.

2.3. The group level. — Consider the group $SO(p+1,q+1)$ which has Lie algebra $\mathfrak{g}$. By definition, this consists of all matrices M such that $M^t\mathbb{J}M = \mathbb{J}$ and such that M has determinant one. Since the grading element E lies in the centre of $\mathfrak{g}_0$, any element g of the corresponding subgroup G_0 must satisfy $\mathrm{Ad}(g)(E) = E$. Using these two facts, a straightforward computation shows that any such element must be block diagonal and of the form

$$\begin{pmatrix} c & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & c^{-1} \end{pmatrix}$$

with $c\in\mathbb{R}$ and $C\in SO(p,q)$ with respect to the standard inner product (that is the inner product given by $\mathbb{I}$). Moreover, the adjoint action of such an element on $\mathfrak{g}_{-1}$ is given by $(c,C)\cdot X = c^{-1}CX$. Hence we see that choosing $G = SO(p+1,q+1)$ in the case $n = p+q$ odd (where $-\mathrm{id}$ is orientation reversing) and $G = SO(p+1,q+1)/\pm\mathrm{id}$ in the case n even, we get a group G such that the adjoint action of G_0 on $\mathfrak{g}_{-1}$ induces an isomorphism of G_0 with the group of all conformal isometries of $\mathfrak{g}_{-1}$ (with the standard inner product of signature (p,q)), so this will be our choice of groups.

Now we can immediately interpret explicitly all the objects described in 1.1 on a conformal manifold M. The fibre of the principal G_0–bundle $\mathcal{G}_0\to M$ over $x\in M$ is exactly the set of all conformal isometries $u:\mathfrak{g}_{-1}\to T_xM$, and the principal right action of $g\in G_0$ is given by $u\cdot g = u\circ\mathrm{Ad}(g)$. This is by construction free and transitive on each fibre, so we really get a principal bundle. By construction $TM = \mathcal{G}_0\times_{G_0}\mathfrak{g}_{-1}$. The bilinear form B from 2.2 identifies $\mathfrak{g}_1$ with the dual G_0–module of $\mathfrak{g}_{-1}$, so $T^*M = \mathcal{G}_0\times_{G_0}\mathfrak{g}_1$. In the picture of equivariant functions, the pairing between vector fields and one–forms induced by this identification is induced by the pointwise pairing between $\mathfrak{g}_{-1} = \mathbb{R}^n$ and $\mathfrak{g}_1 = \mathbb{R}^{n*}$ by our choice of B.

Next, we want to identify the associated bundle $\mathrm{End}_0 TM = \mathcal{G}_0\times_{G_0}\mathfrak{g}_0$. As stated in 1.1 we identify $\mathfrak{g}_0$ with a set of linear maps $\mathfrak{g}_{-1}\to\mathfrak{g}_{-1}$ by mapping $(a,A)\in$

$\mathfrak{g}_0$ to $X \mapsto [(a,A),X] = AX - aX$. As such endomorphisms, elements of $\mathfrak{g}_0$ are characterised by the fact that

$$\langle[(a,A),X],Y\rangle + \langle X,[(a,A),Y]\rangle = -2a\langle X,Y\rangle,$$

where $\langle\ ,\ \rangle$ is the standard inner product of signature (p,q). Thus we see that $\mathrm{End}_0\, TM$ consists of all bundle maps $\Phi : TM \to TM$ such that $g(\Phi(\xi),\eta)+g(\xi,\Phi(\eta)) = 2\varphi g(\xi,\eta)$ for some real number φ for one (or, equivalently, any) metric g from the conformal class. In abstract index notation, this reads as $g_{kj}\Phi_i^k + g_{ik}\Phi_j^k = 2\varphi g_{ij}$, and so $n\varphi$ is exactly the trace of Φ, that is $\varphi = \frac{1}{n}\Phi_i^i$. Note that, in the picture of $\mathfrak{g}_0$–valued functions, $\Phi - \varphi\,\mathrm{id}$ corresponds to A and $-\varphi$ corresponds to the a of (a,A) (cf. section 2.2).

Next, we have to identify the density bundles as associated bundles to $\mathcal{G}_0$. By definition, any point $u \in \mathcal{G}_0$ lying over $x \in M$ is a conformal isometry $u : \mathfrak{g}_{-1} \to T_xM$. Consequently, $(\xi,\eta) \mapsto \langle u^{-1}(\xi), u^{-1}(\eta)\rangle$ defines an element of $\mathcal{L}_x$. For $(c,C) \in G_0$, we see from the definition of the principal right action that $(u{\cdot}(c,C))^{-1}(\xi) = cC^{-1}u^{-1}(\xi)$, so acting with this changes the corresponding element of $\mathcal{L}_x$ by multiplication with c^2. Consequently, we see that considering the representation $\lambda : G_0 \to \mathbb{R}^+$ defined by $\lambda(c,C) = c^2$, the mapping which assigns to (u,α) the inner product $(\xi,\eta) \mapsto \alpha\langle u^{-1}(\xi), u^{-1}(\eta)\rangle$ induces an isomorphism $\mathcal{G}_0 \times_{G_0} \mathbb{R}^+ \cong \mathcal{L}$. Since $\mathcal{L} \cong \mathcal{E}[-2]$, we see that $\mathcal{E}[w]$ is the associated bundle to $\mathcal{G}_0$ with respect to the representation $(c,C) \mapsto |c|^{-w}$ or infinitesimally $E \mapsto -w$, so our convention differs in sign from the one of [**3**, 4.15].

As we have noted in 1.1, the brackets $\mathrm{End}_0\, TM \otimes TM \to TM$ and $T^*M \otimes \mathrm{End}_0\, TM \to T^*M$ are given by the evaluation of endomorphisms, so in abstract index notation we have $\{\Phi,\xi\}^i = \Phi_j^i\xi^j$ and $\{\Phi,\omega\}_i = -\Phi_i^j\omega_j$. To describe the bracket $TM \otimes T^* \to \mathrm{End}_0\, TM$, recall from 2.2 that for $X, Y \in \mathfrak{g}_{-1}$ and $Z \in \mathfrak{g}_1$ we have $[[X,Z],Y] = ZXY + XZY - \mathbb{I}Z^tX^t\mathbb{I}Y$. The first term is obtained by multiplying Y by the pairing of Z and X, while the second one is obtained by multiplying X by the pairing of Z and Y. For the last term, note that $X^t\mathbb{I}Y$ is the standard inner product of X and Y, while $\mathbb{I}Z^t$ is just the element of $\mathfrak{g}_{-1}$ corresponding to $Z \in \mathfrak{g}_1$ under the isomorphism provided by the inner product. This easily implies that the bracket $TM \otimes T^*M \to \mathrm{End}_0\, TM$ is given by

$$\{\xi,\omega\}_j^i = \xi^i\omega_j - g_{jk}\xi^k g^{i\ell}\omega_\ell + \xi^k\omega_k\delta_j^i.$$

An affine connection ∇ on M is induced by a principal connection on $\mathcal{G}_0$ if and only if it preserves the conformal class $[g]$ given on M. Moreover, there are torsion free connections preserving this conformal class (e.g. the Levi–Civita connection of any given metric in the class), so the (unique possible) ∂^*–closed value of the torsion must be zero. Hence, the preferred connections on M are exactly those torsion free connections on M which preserve the conformal class, i.e. the Weyl–structures on the conformal manifold M.

If ∇ and $\hat\nabla$ are two such Weyl–structures, then we know from 1.1 that there is a unique one–form $\Upsilon \in \Omega^1(M)$ such that $\hat\nabla_\xi\eta = \nabla_\xi\eta + \{\{\Upsilon,\xi\},\eta\}$. In abstract index notation, this reads as

$$\hat\nabla_i\eta^j = \nabla_i\eta^j - \Upsilon_i\eta^j + g_{ik}\eta^k g^{j\ell}\Upsilon_\ell - \Upsilon_k\eta^k\delta_i^j.$$

Suppose that ∇ and $\hat{\nabla}$ are the Levi–Civita connections for g and $\hat{g} = \Omega^2 g$, respectively. In terms of g, the condition $\hat{\nabla}\hat{g} = 0$ implies

$$2\Omega(\xi\cdot\Omega)g(\eta,\zeta) = \Omega^2 g(\{\{\Upsilon,\xi\},\eta\},\zeta) + \Omega^2(\eta,\{\{\Upsilon,\xi\},\zeta\}).$$

Since $\{\Upsilon,\xi\} \in \Gamma(\mathrm{End}_0 TM)$ and the above formula for the bracket implies $\{\Upsilon,\xi\}^i_i = -n\xi^i\Upsilon_i$, this leads to $\Upsilon = -\Omega^{-1}d\Omega$. Comparing with the formula in [**1**, 2.1] we see that our Υ in the case of Levi–Civita connections is the negative of the Υ there.

For s a section of any associated bundle to $\mathcal{G}_0$, we have the formula $\hat{\nabla}_\xi s = \nabla_\xi s + \{\Upsilon,\xi\}\bullet s$ by definition of the action of induced connections. In particular, for $\sigma \in \Gamma(\mathcal{E}[w])$ we get (cf. [**3**, 4.15]) $\hat{\nabla}_\xi\sigma = \nabla_\xi\sigma + \frac{w}{n}\{\Upsilon,\xi\}^i_i\sigma$, or in abstract index notation $\hat{\nabla}_i\sigma = \nabla_i\sigma - w\Upsilon_i\sigma$. For later use, we note the formula for $\omega \in \Gamma(\mathcal{E}_i[w])$, which is given by $\hat{\nabla}_\xi\omega = \nabla_\xi\omega + \{\{\Upsilon,\xi\},\omega\} + \frac{w}{n}\{\Upsilon,\xi\}^i_i\omega$, i.e.

$$\hat{\nabla}_i\omega_j = \nabla_i\omega_j + \Upsilon_j\omega_i - g_{ij}g^{k\ell}\Upsilon_k\omega_\ell + (1-w)\Upsilon_i\omega_j.$$

2.4. The final things we have to describe are the rho–tensor P_{ij}, the Weyl–curvature $W_{ij}{}^k{}_\ell$ and the Cotton–York tensor CY_{ijk} for a preferred connection ∇. Let $R_{ij}{}^k{}_\ell$ be the curvature of ∇. Put $\mathrm{Ric}_{ij} = R_{ki}{}^k{}_j$, the Ricci curvature of ∇, which is a section of $\mathcal{E}_{ij}$. Note however, that for general preferred connections Ric is not symmetric (in contrast to the special case of Levi–Civita connections). Finally, consider the scalar curvature $R \in \Gamma(\mathcal{E}[-2])$ defined by $R = \boldsymbol{g}^{ij}\mathrm{Ric}_{ij}$. By definition, $R(\xi,\eta) = W(\xi,\eta) - \{\mathsf{P}(\xi),\eta\} + \{\mathsf{P}(\eta),\xi\}$, and $W_{ki}{}^k{}_j = 0$. The defining equation can be written as

$$R_{ij}{}^k{}_\ell = W_{ij}{}^k{}_\ell + 2\mathsf{P}_{\ell[i}\delta^k_{j]} - 2\boldsymbol{g}^{km}\mathsf{P}_{m[i}\boldsymbol{g}_{j]\ell} - 2\mathsf{P}_{[ij]}\delta^k_\ell.$$

From this formula it is visible, that W is exactly the trace-free part of R (with respect to the indices i and k). Contracting over the indices i and k in this equation and renaming some indices, we obtain

$$\mathrm{Ric}_{ij} = -(n-1)\mathsf{P}_{ji} + \mathsf{P}_{ij} - \mathsf{P}\boldsymbol{g}_{ij},$$

where we define $\mathsf{P} \in \Gamma(\mathcal{E}[-2])$ by $\mathsf{P}_{ij}\boldsymbol{g}^{ij}$. Contracting the above equation with $\boldsymbol{g}^{ij}$, we obtain $\mathsf{P} = -\frac{1}{2n-2}R$. Reinserting this, we easily get

$$\mathsf{P}_{ij} = -\tfrac{1}{n-2}\left(\tfrac{1}{n}\mathrm{Ric}_{ij} + \tfrac{n-1}{n}\mathrm{Ric}_{ji} - \tfrac{1}{2n-2}R\boldsymbol{g}_{ij}\right).$$

In particular, if ∇ is a Levi–Civita connection, then by the Bianchi identity Ric_{ij} is symmetric, so we obtain the usual simpler formula $\mathsf{P}_{ij} = -\frac{1}{n-2}(\mathrm{Ric}_{ij} - \frac{1}{2n-2}R\boldsymbol{g}_{ij})$, which shows that our Rho–tensor is the negative of the one used in [**1**]. Moreover, this shows that for a Levi–Civita connection, the Rho–tensor is symmetric. In that case, we further know that $R_{ij}{}^k{}_k = 0$, and together with symmetry of the rho–tensor we may conclude from the decomposition of R above that also $W_{ij}{}^k{}_k = 0$.

Finally, since the torsion is trivial in this case, the formula for the Cotton–York tensor of ∇ is simply given by $CY_{ijk} = \nabla_i\mathsf{P}_{kj} - \nabla_j\mathsf{P}_{ki}$.

2.5. Here we use the results of section 1 to construct the tractor bundle $\mathcal{E}^I$ (where I is an abstract index) corresponding to the standard representation $\mathbb{V}=\mathbb{R}^{n+2}$ of G. If we split an element of $\mathbb{V}$ as a triple, with components of sizes 1, n and 1, then the action of the Lie algebra is given by

$$(X,(a,A),Z)\cdot\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}au+Z(v)\\uX+Av-w\mathbb{I}Z^t\\-X^t\mathbb{I}v-aw\end{pmatrix}.$$

This immediately shows that we have found the splitting $\mathbb{V}=\mathbb{V}_{-1}\oplus\mathbb{V}_0\oplus\mathbb{V}_1$ into eigenspaces for the action of the grading element $E=(1,0)\in\mathfrak{g}_0$. Moreover, this immediately allows us to read off all the data we need: The bundles $\mathcal{V}_{\pm1}$ corresponding to $\mathbb{V}_{\pm1}$ visibly are simply $\mathcal{E}[\mp1]$. Comparing with the action of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ we further see, that the bundle corresponding to $\mathbb{V}_0$ is just $\mathcal{E}^i[-1]$. For further use, it will be more useful to view this as $\mathcal{E}_i[1]$ (via contracting with $\boldsymbol{g}_{ij}$). Finally, denoting an element of $\overrightarrow{\mathcal{V}}=\mathcal{E}[1]\oplus\mathcal{E}_i[1]\oplus\mathcal{E}[-1]$ by (σ,μ_i,τ), the action ρ of $\overrightarrow{\mathcal{A}}=TM\oplus\operatorname{End}_0TM\oplus T^*M$ on $\overrightarrow{\mathcal{V}}$ is given by

$$\rho((\xi,\Phi,\omega))(\sigma,\mu,\tau)=(-\xi^i\mu_i+\tfrac{1}{n}\Phi^i_i\sigma,\tau\xi^j\boldsymbol{g}_{ij}-\Phi^j_i\mu_j+\tfrac{1}{n}\Phi^j_j\mu_i-\sigma\omega_i,-\tfrac{1}{n}\Phi^i_i\tau+\boldsymbol{g}^{ij}\mu_i\omega_j).$$

2.6. The standard tractor bundle. — Consider the two–jet prolongation $J^2(\mathcal{E}[1])$ of the density bundle $\mathcal{E}[1]$. By definition, we have the jet exact sequences

$$0\to\mathcal{E}_{(ij)}[1]\to J^2(\mathcal{E}[1])\to J^1(\mathcal{E}[1])\to0 \tag{6}$$

$$0\to\mathcal{E}_i[1]\to J^1(\mathcal{E}[1])\to\mathcal{E}[1]\to0 \tag{7}$$

As we have observed in 2.1 the conformal structure splits $\mathcal{E}_{(ij)}$ as $\mathcal{E}_{(ij)_0}\oplus\mathcal{E}[-2]$, where the first space is the kernel of the contraction with $\boldsymbol{g}^{ij}$. Tensoring this with $\mathcal{E}[1]$, we see that $\mathcal{E}_{(ij)_0}[1]$ sits as a smooth subbundle in $J^2(\mathcal{E}[1])$, and we define $\mathcal{E}^I$ to be the quotient bundle. So by definition, we have an exact sequence

$$0\to\mathcal{E}_{(ij)_0}[1]\to J^2(\mathcal{E}[1])\to\mathcal{E}^I\to0, \tag{8}$$

while the 2-jet sequence gives us an exact sequence

$$0\to\mathcal{E}[-1]\to\mathcal{E}^I\to J^1(\mathcal{E}[1])\to0. \tag{9}$$

In particular, we see that the kernel of $\mathcal{E}^I\to J^1(\mathcal{E}[1])\to\mathcal{E}[1]$ sits as subbundle within the kernel of $\mathcal{E}^I\to J^1(\mathcal{E}[1])$ and so there is a canonical filtration of $\mathcal{E}^I$ such that the associated graded bundle is isomorphic to $\mathcal{E}[1]\oplus\mathcal{E}_i[1]\oplus\mathcal{E}[-1]=\overrightarrow{\mathcal{V}}$. Consequently, this is a good candidate for the standard tractor bundle.

Note that our definition of $\mathcal{E}^I$ has the advantage that it immediately implies the existence of a second order invariant differential operator $D^I:\Gamma(\mathcal{E}[1])\to\Gamma(\mathcal{E}^I)$, which is given by composing the canonical projection $J^2(\mathcal{E}[1])\to\mathcal{E}^I$ with the two–jet operator j^2.

Proposition. — *For a preferred connection ∇, the map*

$$j^2_x\sigma\mapsto\overrightarrow{D^I}\sigma(x)=(\sigma(x),\nabla_i\sigma(x),\tfrac{1}{n}\boldsymbol{g}^{ij}(-\nabla_i\nabla_j\sigma(x)+\mathsf{P}_{ij}\sigma(x))),$$

induces an isomorphism $\mathcal{E}^I \to \mathcal{E}[1] \oplus \mathcal{E}_i[1] \oplus \mathcal{E}[-1]$ of vector bundles. Moreover, changing ∇ to $\hat{\nabla}$ with the corresponding one–form Υ, we obtain a normal tractor bundle transformation as required in theorem 1.3, i.e.

$$\widehat{(\sigma,\mu,\tau)} = (\sigma, \mu_i - \sigma\Upsilon_i, \tau + \boldsymbol{g}^{ij}\Upsilon_i\mu_j - \tfrac{1}{2}\sigma\boldsymbol{g}^{ij}\Upsilon_i\Upsilon_j).$$

Proof. — Clearly, the formula in the proposition defines a bundle map $J^2(\mathcal{E}[1]) \to \mathcal{E}[1] \oplus \mathcal{E}_i[1] \oplus \mathcal{E}[-1]$. Moreover, if $j^2_x\sigma$ lies in the kernel of this map, then we must have $j^1_x\sigma = 0$ in order to have the first two components vanish. But then the last component becomes $\boldsymbol{g}^{ij}\nabla_i\nabla_j\sigma(x)$ which by definition vanishes if and only if $\nabla_i\nabla_j\sigma(x)$ lies in $\mathcal{E}_{(ij)_0}[1] \subset J^2(\mathcal{E}[1])$. (Note that the double covariant derivative is symmetric since $j^1_x\sigma = 0$.) Consequently, the map factors to a bundle map $\mathcal{E}^I \to \overrightarrow{\mathcal{V}}$ which is injective on each fibre. Since both bundles have the same rank, it is an isomorphism of vector bundles.

If $\hat{\nabla}$ is another preferred connection corresponding to Υ, then clearly the first component stays the same. For the second component, we get $\hat{\nabla}_i\sigma = \nabla_i\sigma - \sigma\Upsilon_i$ from 2.3, so we get the transformation law for the second component. Differentiating this once more, we obtain

$$\nabla_i(\hat{\nabla}_j\sigma) = \nabla_i\nabla_j\sigma - \Upsilon_j\nabla_i\sigma - \sigma\nabla_i\Upsilon_j.$$

According to the last formula in 2.3, to get $\hat{\nabla}_i(\hat{\nabla}_j\sigma)$ we have to add to this $\Upsilon_j\hat{\nabla}_i\sigma - g_{ij}g^{k\ell}\Upsilon_k\hat{\nabla}_\ell\sigma$. Then expanding the result yields

$$\hat{\nabla}_i\hat{\nabla}_j\sigma = \nabla_i\nabla_j\sigma - \sigma\nabla_i\Upsilon_j - \Upsilon_i\Upsilon_j\sigma - g_{ij}g^{k\ell}\Upsilon_k\nabla_\ell\sigma + g_{ij}g^{k\ell}\Upsilon_k\Upsilon_\ell\sigma.$$

On the other hand, we have to compute the change of the rho–tensor in abstract index notation. From 1.1 we know that $\hat{\mathsf{P}}(\xi) = \mathsf{P}(\xi) - \nabla_\xi\Upsilon + \frac{1}{2}\{\Upsilon,\{\Upsilon,\xi\}\}$. From the formulae in 2.3, one immediately reads off that the last term is given by $-\xi^j\Upsilon_j\Upsilon_i + \frac{1}{2}g_{ij}\xi^j g^{k\ell}\Upsilon_k\Upsilon_\ell$. This immediately implies that

$$\hat{\mathsf{P}}_{ij} = \mathsf{P}_{ij} - \nabla_j\Upsilon_i - \Upsilon_i\Upsilon_j + \tfrac{1}{2}g_{ij}g^{kl}\Upsilon_k\Upsilon_\ell.$$

Subtracting the above expression for $\hat{\nabla}_i\hat{\nabla}_j\sigma$ from $\sigma\hat{\mathsf{P}}_{ij}$ and contracting with $\frac{1}{n}\boldsymbol{g}^{ij}$, we immediately get the transformation law claimed for the last component. That the formula for the transformation coincides with $e^{\rho(\Upsilon)}$ can be immediately read off the formula for ρ in 2.5. □

2.7. Now we can apply all the machinery developed in section 1 directly to this case. Using the formulae for $\rho(\xi)$ and $\rho(\mathsf{P}(\xi))$ from 2.5, we immediately see that by theorem 1.3 the normal tractor connection $\nabla^{\mathcal{V}}$ on $\mathcal{E}^I$ is given by

$$\overrightarrow{\nabla^{\mathcal{V}}_\xi t} = (\nabla_\xi\sigma - \xi^i\mu_i, \nabla_\xi\mu_i + \tau\boldsymbol{g}_{ij}\xi^j - \sigma\mathsf{P}_{ij}\xi^j, \nabla_\xi\tau + \boldsymbol{g}^{ij}\mu_i\mathsf{P}_{jk}\xi^k),$$

for $\overrightarrow{t} = (\sigma,\mu_i,\tau)$, which just means

$$\overrightarrow{\nabla^{\mathcal{V}}_i t} = (\nabla_i\sigma - \mu_i, \nabla_i\mu_j + \tau\boldsymbol{g}_{ij} - \sigma\mathsf{P}_{ji}, \nabla_i\tau + \boldsymbol{g}^{jk}\mu_j\mathsf{P}_{ki}).$$

The differences between this formula and the one in [**1**, 2.3] are due to the different sign of the Rho–tensor. Thus it follows immediately from theorem 1.3 that the tractor bundle and connection constructed in [**1**] is the normal tractor bundle with normal connection corresponding to the defining representation of $\mathfrak{so}(p+1,p+1)$.

Using the fact that the torsion vanishes and that the Weyl–curvature satisfies $W_{ij}{}^{k}{}_{k}=0$, we conclude from theorem 1.3 and the formulae for ρ from 2.5 that the curvature of the normal tractor connection is given by

$$\overrightarrow{R^{\mathcal{V}}(\xi,\eta)(t)}=(0,W(\xi,\eta)^j_i\mu_j-\sigma CY(\xi,\eta)_i,\boldsymbol{g}^{ij}\mu_i CY(\xi,\eta)_j),$$

where again $\overrightarrow{t}=(\sigma,\mu_i,\tau)$.

Next, we want to interpret the results of theorem 1.3 concerning the adjoint tractor bundle. By definition, the adjoint tractor bundle $\mathcal{A}$ is a subbundle of $\mathcal{E}^I_J$, the bundle of endomorphisms of $\mathcal{E}^I$ which consists of those endomorphisms which are of the form $\overrightarrow{\varphi(t)}=\rho(\overrightarrow{\varphi})(\overrightarrow{t})$ for some $\overrightarrow{\varphi}\in A$. But these endomorphisms can be easily characterised: On $\overrightarrow{\mathcal{V}}$ we have the obvious analog of the inner product on $\mathbb{R}^{n+2}$ from 2.2, i.e. $((\sigma,\mu_i,\tau),(\sigma',\mu'_i\tau'))\mapsto\sigma\tau'+\tau\sigma'+\boldsymbol{g}^{ij}\mu_i\mu'_j$. From its construction it is clear, that $\overrightarrow{\mathcal{A}}$ consists of all endomorphisms of $\overrightarrow{\mathcal{V}}$ preserving this inner product. But this inner product induces an inner product $h_{IJ}\in\Gamma(\mathcal{E}_{(IJ)})$ on $\mathcal{E}^I$, since the change between two isomorphisms is given by $e^{\rho(\Upsilon)}$, which in the group picture corresponds to the action of an element of P_+, which leaves the metric invariant. (Alternatively, this may also be verified by a simple direct computation.) So by construction $\mathcal{A}\subset\mathcal{E}^I_J$ consists of those endomorphisms s^I_J which satisfy $h_{KI}s^K_J+h_{KJ}s^K_I=0$, so we may as well identify $\mathcal{A}$ with the bundle $\mathcal{E}_{[IJ]}$. Moreover, we have the inverse isomorphism $h^{IJ}\in\Gamma(\mathcal{E}^{(IJ)})$, so we can always raise and lower tractor indices.

Any preferred connection leads to an isomorphism $\mathcal{A}\cong\overrightarrow{\mathcal{A}}$, and using the formulae for the algebraic brackets from 2.3 and the formula for $\{\Upsilon,\{\Upsilon,\xi\}\}$ from the proof of proposition 2.6, we see that the change of these isomorphisms is given by

$$\widehat{(\xi^i,\Phi^i_j,\omega_i)}=(\xi^i,\Phi^i_j-\xi^i\Upsilon_j+g_{jk}\xi^k g^{i\ell}\Upsilon_\ell-\xi^k\Upsilon_k\delta^i_j,\omega_i+\Phi^j_i\Upsilon_j-\xi^j\Upsilon_j\Upsilon_i+\tfrac{1}{2}g_{ij}\xi^j g^{k\ell}\Upsilon_k\Upsilon_\ell).$$

The normal tractor connection $\nabla^{\mathcal{A}}$ on $\overrightarrow{s}=(\xi,\Phi,\omega)$ is given by

$$\overrightarrow{\nabla^{\mathcal{A}}_\eta s}=(\nabla_\eta\xi+\{\eta,\Phi\},\nabla_\eta\Phi+\{\eta,\omega\}+\{\mathsf{P}(\eta),\xi\},\nabla_\eta\omega+\{\mathsf{P}(\eta),\Phi\}),$$

and using the formulae for the brackets we immediately see that the first component of $\overrightarrow{\nabla^{\mathcal{A}}_i s}$ equals $\nabla_i\xi^j-\Phi^j_i$, the last component is given by $\nabla_i\omega_j+\Phi^k_j\mathsf{P}_{ki}$, while for the middle component we get

$$\nabla_i\Phi^j_k+\delta^j_i\omega_k-g_{ik}g^{j\ell}\omega_\ell+\omega_i\delta^j_k-\xi^j\mathsf{P}_{ki}+g_{k\ell}\xi^\ell g^{jm}\mathsf{P}_{mi}-\xi^\ell\mathsf{P}_{\ell i}\delta^j_k.$$

3. Tractor Calculus

In this final part we describe and use some of the basic machinery of the standard tractor calculus. In our current setting there are two main reasons for this. Firstly it enables us to relate, in a simple and explicit manner, adjoint tractor expressions to the corresponding standard tractor expressions. The second use, which draws from the first, is that it enables us to extract, from our constructions here (which have been developed in the general setting of irreducible parabolic geometries and then specialised to the conformal case), the key objects of conformal tractor calculus as developed for example in [**1, 7, 8**]. Along the way the treatment should give the reader some insight into the techniques used to calculate explicitly via tractor calculus.

3.1. The inclusion $\mathcal{E}[-1] \hookrightarrow \mathcal{E}^I$ in (9) is equivalent to a canonical section $\boldsymbol{X}^I$ of $\mathcal{E}^I[1]$. On the other hand, we have the projection $\mathcal{E}^I \to \mathcal{E}[1]$, which can similarly be viewed as a canonical section of $\mathcal{E}_I[1]$. From the definition of the inner product h it follows immediately, that this section is given by $h_{IJ}\boldsymbol{X}^J =: \boldsymbol{X}_I$. From these definitions, we have immediately that $\boldsymbol{X}_I\boldsymbol{X}^I = h_{IJ}\boldsymbol{X}^I\boldsymbol{X}^J = 0$. For any choice of preferred connection ∇ and any section $\tau \in \Gamma(\mathcal{E}[-1])$, the tractor section $\tau\boldsymbol{X}^I$ is mapped to $(0,0,\tau) \in \Gamma(\overrightarrow{\mathcal{E}^I})$ under the isomorphism described in proposition 2.6. Thus $\boldsymbol{X}^I$ is equivalent to the section $\overrightarrow{\boldsymbol{X}}^I = (0,0,1)$ of $\overrightarrow{\mathcal{E}^I}[1] = \overrightarrow{\mathcal{E}^I} \otimes \mathcal{E}[1]$.

Next, choosing a metric g from the conformal class is equivalent to choosing a global nonzero section σ_0 of $\mathcal{E}[1]$. Up to constant multiples, σ_0 is characterised by the fact that $\nabla\sigma_0 = 0$, where ∇ denotes the Levi–Civita connection of g, which is one of the preferred connections. According to proposition 2.6 via the Levi–Civita connection, the choice of g induces an isomorphism $\mathcal{E}^I \cong \mathcal{E}[1] \oplus \mathcal{E}_i[1] \oplus \mathcal{E}[-1]$. In particular, this gives us a splitting $\mathcal{E}^I \to \mathcal{E}[-1]$ of the canonical inclusion, which can be viewed as a section Y_I of $\mathcal{E}_I[-1]$ such that $Y_I\boldsymbol{X}^I = 1$, and a splitting $\mathcal{E}[1] \to \mathcal{E}^I$ of the canonical projection, which we may view as a section Y^I of $\mathcal{E}^I[-1]$. By definition, $X_IY^I = 1$, which immediately leads to $Y_I = h_{IJ}Y^J$. Further, proposition 2.6 immediately implies the explicit formula $Y^I = (\sigma_0)^{-1}D^I\sigma_0 - \frac{1}{n}\mathsf{P}\boldsymbol{X}^I$. Finally we denote by Z^{Ii} the section of $\mathcal{E}^{Ii}[-1]$ which gives the bundle injection $\mathcal{E}_i[1] \to \mathcal{E}^I$ induced by the above isomorphism.

In this notation, if $\overrightarrow{t} = (\sigma, \mu, \tau) \in \overrightarrow{\mathcal{V}}$ (in the isomorphism corresponding to g), then $t^I = \sigma Y^I + Z^{Ii}\mu_i + \tau\boldsymbol{X}^I$. We will raise and lower indices using the conformal metric $\boldsymbol{g}_{ij}$, the tractor metric h_{IJ} and their inverses without further mention. For example $Z_{Ii} := \boldsymbol{g}_{ij}h_{IJ}Z^{Jj}$.

These tractor bundle sections can be used effectively in the explicit description of relationship between the bundles $\mathcal{A}$ and $\mathcal{V} = \mathcal{E}^I$. In the previous section we observed that $\mathcal{A}$ may be identified with $\mathcal{E}_{[IJ]}$. Choosing a preferred connection this may be understood by describing $\overrightarrow{\mathcal{A}}$ as a subbundle of $\overrightarrow{\mathcal{E}}^I \otimes \overrightarrow{\mathcal{E}}_J$. In 2.5 we have already explicitly described the action ρ of $\overrightarrow{\mathcal{A}}$ on $\overrightarrow{\mathcal{V}}$. For $\overrightarrow{s} = (\xi, \Phi, \omega) \in \Gamma(\overrightarrow{\mathcal{A}})$ we can view $\rho((\xi, \Phi, \omega))$ as a section of $(\overrightarrow{\mathcal{V}})^* \otimes \overrightarrow{\mathcal{V}}$ and using the result from section 2.5 we see that in $(1, n, 1) \times (1, n, 1)$ block form this is given by,

$$\begin{pmatrix} -\varphi & \omega & 0 \\ \xi & \Phi^{(0)} & -g^{-1}(\omega, \cdot) \\ 0 & -g(\xi, \cdot) & \varphi \end{pmatrix}$$

where $\Phi^{(0)}$ is the trace-free part of Φ and $\varphi := \Phi^i_i/n$. Or, in terms of the notation introduced above, we can describe s as the section of $\mathcal{E}^I{}_J$ as follows:

$$s^I{}_J = \xi^I Y_J - Y^I\xi_J + {\Phi^{(0)}}^I{}_J + (Y^I\boldsymbol{X}_J - \boldsymbol{X}^I Y_J)\varphi + \boldsymbol{X}^I\omega_J - \omega^I\boldsymbol{X}_J,$$

where we have used the shorthand notation $\xi^I = Z^I{}_i\xi^i$ etcetera. Thus using h_{IJ} to lower indices, we have $s_{IJ} = 2\xi_{[I}Y_{J]} + \Phi^{(0)}{}_{IJ} + 2Y_{[I}\boldsymbol{X}_{J]}\varphi + 2\boldsymbol{X}_{[I}\omega_{J]} \in \Gamma(\mathcal{E}_{[IJ]})$. Note that $E_{IJ} := 2Y_{[I}\boldsymbol{X}_{J]}$ is the *grading tractor* corresponding to the choice of g. That is, identifying $\mathcal{E}_{[IJ]}$ with the bundle of endomorphisms of $\mathcal{V}$ preserving h, then the splitting $\mathcal{V} = \mathcal{V}_{-1} \oplus \mathcal{V}_0 \oplus \mathcal{V}_1$ corresponding to the eigenvalues $i = -1, 0, 1$ of E_{IJ}, just recovers the isomorphism $\mathcal{V} \to \overrightarrow{\mathcal{V}}$ induced by g (via the Levi–Civita connection ∇). Since E_{IJ} is the unique section with this property we are justified in referring to it as the grading tractor corresponding to ∇.

To conclude we note that the adjoint tractor metric B is easily described in terms of the standard tractor notation. Let $\overrightarrow{s}_1 = (\xi_1, \varphi_1, \omega_1)$ and $\overrightarrow{s}_2 = (\xi_2, \varphi_2, \omega_2)$. Recall that the inner product B on $\mathcal{A}$ is induced by $\frac{1}{2}$ of the trace form on $\mathfrak{g}$. Thus, it is given by $B(s_1, s_2) = \omega_1(\xi_2) + \omega_2(\xi_1) + \operatorname{tr}(\Phi_1\Phi_2)$ and we can rewrite this as $B(s_1, s_2) = \frac{1}{2}s_1{}^I{}_J s_2{}^J{}_I = \frac{1}{2}h_{IL}h_{JK}s_1^{IJ}s_2^{KL}$.

3.2. The fundamental D–operator and the tractor $\mathbb{D}$. — The fundamental D–operator can be described in terms of this notation. Recall that, given a choice of preferred connection ∇, for t a section of a weighted tensor bundle we have $D_s t = \nabla_\xi t - \Phi \bullet t$, where $\overrightarrow{s} = (\xi, \Phi, \omega)$. In particular, if σ is a section of the line bundle $\mathcal{E}[w]$ then we have $D_s\sigma = \nabla_\xi\sigma - w\varphi\sigma$. Now in terms of the standard tractors we have observed that s is given by $2\xi^{[I}Y^{J]} + \Phi^{(0)}{}^{IJ} + 2Y^{[I}X^{J]}\varphi + 2X^{[I}\omega^{J]}$ and so it follows immediately that on $\sigma \in \Gamma(\mathcal{E}[w])$ the ($\mathcal{A}$-valued) operator D is given by

$$D_{IJ}\sigma = X_{[J}\tilde{D}_{I]}\sigma$$

where, $\tilde{D}_I\sigma = (Z_I^i\nabla_i + wY_I)\sigma$ or, equivalently, $\overrightarrow{\tilde{D}\sigma} = (w\sigma, \nabla\sigma, 0)$. (Of course $\tilde{D}\sigma$ depends on the choice of ∇ but the operator $\sigma \mapsto 2X_{[J}\tilde{D}_{I]}\sigma$ is independent of this choice.)

One can use the fundamental D-operator to generate other invariant operators. For example we can construct the second order "tractor D–operator" as given in [**1**] (but first discovered by Thomas [**11**]). For any tractor bundle $\mathcal{T}$, this operator maps sections of $\mathcal{T}[w]$ to sections of $\mathcal{T}[w-1] \otimes \mathcal{E}^I$. Here we will denote this operator by $\mathbb{D}$ to distinguish it from the fundamental D–operator.

We first deal with the bundle $\mathcal{E}[w]$. For $s_1, s_2 \in \mathcal{A}$ and t any weighted tensor field it is straightforward to show that

$$\begin{aligned} DDt(s_1, s_2) = \nabla^2 t(\xi_1, \xi_2) - \Phi_1 \bullet \nabla_{\xi_2} t - \Phi_2 \bullet \nabla_{\xi_1} t + \nabla_{\{\Phi_1, \xi_2\}} t + \\ + \Phi_2 \bullet \Phi_1 \bullet t - \{\omega_1, \xi_2\} \bullet t + \{\mathsf{P}(\xi_1), \xi_2\} \bullet t, \end{aligned}$$

where $s_i = (\xi_i, \Phi_i, w_i)$, $i = 1, 2$. (This expression is derived explicitly in [**3**].) Thus for $\sigma \in \mathcal{E}[w]$ this simplifies to

$$\begin{aligned} DD\sigma(s_1, s_2) = \xi_1^i\xi_2^j\nabla_i\nabla_j\sigma + (1-w)\varphi_1\xi_2^j\nabla_j\sigma - w\varphi_2\xi_1^i\nabla_i\sigma + \Phi^1_{(0)}{}^i{}_j\xi_2^j\nabla_i\sigma + \\ + w^2\varphi_1\varphi_2\sigma + w\xi^i w_i\sigma - w\mathsf{P}_{ij}\xi_1^i\xi_2^j\sigma. \end{aligned}$$

Since this is $s_1^{IJ}s_2^{KL}D_{IJ}D_{KL}\sigma$ it is easy to write down the (lengthy) expression for $D_{IJ}D_{KL}\sigma$ in terms of $\boldsymbol{X}_I, Y_J$ and $\nabla_K := Z_K^i\nabla_i$. Contracting with $4h^{IK}$ we obtain

$$\begin{aligned}4h^{IK}D_{IJ}D_{KL}\sigma =&\boldsymbol{X}_J\boldsymbol{X}_L(\Delta\sigma - w\mathsf{P}\sigma) - (w-1)\boldsymbol{X}_J\nabla_L\sigma - (n+w-1)\boldsymbol{X}_L\nabla_J\sigma\\ &- w(w-1)\boldsymbol{X}_JY_L\sigma - w(n+w-1)\boldsymbol{X}_LY_J\sigma - wh_{JL}\sigma,\end{aligned}$$

where $\Delta = \boldsymbol{g}^{ij}\nabla_i\nabla_j$. Thus

$$\begin{aligned}4h^{IK}D_{I(J}D_{L)_0K}\sigma =& -\boldsymbol{X}_J\boldsymbol{X}_L(\Delta\sigma - w\mathsf{P}\sigma) + (n+2w-2)\boldsymbol{X}_{(J}\nabla_{L)_0}\sigma\\ &+ (n+2w-2)w\boldsymbol{X}_{(J}Y_{L)_0}\sigma\end{aligned}$$

where $(\cdots)_0$ indicates the symmetric trace-free (with respect to h^{IJ}) part of the enclosed indices. It is easily verified explicitly that the map $s_K \mapsto \boldsymbol{X}_{(I}S_{K)_0}$ determines a bundle monomorphism $\mathcal{E}_K[-1] \hookrightarrow \mathcal{E}_{(IK)_0}$. Thus we may deduce immediately that $\sigma \mapsto -\boldsymbol{X}_C(\Delta\sigma - w\mathsf{P}\sigma) + (n+2w-2)\nabla_C\sigma + (n+2w-2)wY_C\sigma$ is an invariant differential operator. In fact this is precisely the tractor D–operator $\sigma \mapsto \mathbb{D}_C\sigma$ for $\mathcal{E}[w]$.

In fact, as stated in [**1**], this tractor D–operator generalises to weighted tractor bundles. The easiest route to this result is via another simple observation. If $\mathcal{T}$ is a tractor bundle then, since both D_{IJ} and $\nabla^{\mathcal{T}}$ satisfy a Leibniz rule, the map

$$t\otimes\sigma \mapsto (\boldsymbol{X}_{[J}Z_{I]}^i\nabla_i^{\mathcal{T}}t)\otimes\sigma + t\otimes\boldsymbol{X}_{[J}\tilde{D}_{I]}\sigma,$$

where $t \in \Gamma(\mathcal{T})$, determines a well defined linear operator on the weighted tractor bundle $\mathcal{T}\otimes\mathcal{E}[w]$. This is (apart from a factor of 2) precisely operator D_{IJ} described in [**7, 8**]; several applications of this operator are also described in those sources. Here we will denote this coupled operator by $D_{IJ}^{\mathcal{T}}$ to distinguish it from the fundamental D operator. In this notation the $\mathcal{T}$ simply indicates any tractor bundle rather than any given fixed such bundle.

To simplify the computation let us write $\tilde{D}^{\mathcal{T}}$ to mean the tractor connection coupled generalisation of $\tilde{D}$. That is if $\sigma \in \Gamma(\mathcal{E}[w])$ and t is a section of some tractor bundle then $\tilde{D}^{\mathcal{T}}$ is defined by the rule

$$\tilde{D}_I^{\mathcal{T}}t\otimes\sigma = (Z_I^i\nabla_i^{\mathcal{T}}t)\otimes\sigma + t\otimes\tilde{D}_I\sigma$$

and that it satisfy the Leibniz rule $\tilde{D}_I^{\mathcal{T}}fs = (Z_I^i\nabla_if)s + f\tilde{D}_I^{\mathcal{T}}s$ if s is a weighted tractor field and f a function. As with $\tilde{D}$, $\tilde{D}^{\mathcal{T}}$ depends on the choice of a preferred connection. However we have the identity $D_{IJ}^{\mathcal{T}}s = \boldsymbol{X}_{[J}\tilde{D}_{I]}^{\mathcal{T}}s$, for any weighted tractor field s. Next note that it follows easily from the definition of $\tilde{D}^{\mathcal{T}}$ and the explicit formula for $\overrightarrow{\nabla^{\mathcal{V}}}$ in section 2.7 that for any weighted tractor field s we have $\tilde{D}_I^{\mathcal{T}}\boldsymbol{X}_Js - \boldsymbol{X}_J\tilde{D}_I^{\mathcal{T}}s = (h_{IJ} - \boldsymbol{X}_IY_J)s$. Combining these two observations it is a very short calculation to verify that

$$4h^{JK}D_{J(I}^{\mathcal{T}}D_{L)_0K}^{\mathcal{T}}s = -\boldsymbol{X}_I\boldsymbol{X}_L(\Delta^{\mathcal{T}} - w\mathsf{P})s + (n+2w-2)\boldsymbol{X}_{(I}\tilde{D}_{L)_0}^{\mathcal{T}}s,$$

with $\Delta^{\mathcal{T}}s := \boldsymbol{g}^{ij}\nabla_i^{\mathcal{T}}\nabla_j^{\mathcal{T}}s$ and where, at this point, we mean by $\nabla^{\mathcal{T}}$ the coupled Levi-Civita–tractor connection. This constructs an invariant 2nd order operator

$$\mathbb{D}_Is = \boldsymbol{X}_I(\Delta^{\mathcal{T}} - w\mathsf{P})s + (n+2w-2)\tilde{D}^{\mathcal{T}}s$$

for s a section of any tractor bundle tensored with $\mathcal{E}[w]$. This derivation follows [**7**] and has recovered the usual 'tractor-D' operator on weighted tractor bundles as in [**1**].

Note that if $w = 1 - n/2$ then the second term on the right hand side of the display above vanishes. This immediately implies that $\Delta^{\mathcal{T}} - (1-n/2)\mathsf{P}$ is invariant on tractor sections of any type and with weight $1 - n/2$. This is the tractor generalisation of the Yamabe operator.

It is clear from the last display that acting on $\mathcal{E}[w]$ this recovers the the operator constructed above. This may at first seem rather surprising since the ingredients seem different. So as a final point we explain why $4h^{JK}D^{\mathcal{T}}_{J(I}D^{\mathcal{T}}_{L)_0K}$ and $4h^{JK}D_{J(I}D_{L)_0K}$ agree on $\mathcal{E}[w]$. Note that for $\sigma \in \Gamma(\mathcal{E}[w])$ we have, from the definition of $D^{\mathcal{T}}_{IJ}$, that $D^{\mathcal{T}}D^{\mathcal{T}}\sigma = D^{\mathcal{T}}D\sigma$. Using this observation and [**3**] proposition 3.2, $D^{\mathcal{T}}D\sigma(s_1,s_2) - DD\sigma(s_1,s_2) = D_{\{s_2,s_1\}}\sigma$. It follows that if we let $\sigma_{IL} := h^{JK}D^{\mathcal{T}}_{JI}D_{LK}\sigma - h^{JK}D_{JI}D_{LK}\sigma$ then, since $h^{JK} \in \mathcal{E}^{(JK)}$, we have that $\sigma_{IL} = -\sigma_{LI}$ and so $h^{JK}D^{\mathcal{T}}_{JI}D_{LK}\sigma - h^{JK}D_{JI}D_{LK}\sigma$ vanishes upon symmetrisation over the indices IL.

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September 19, 2001

ANDREAS ČAP, Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A–1090 Wien, Austria and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, A–1090 Wien, Austria • *E-mail :* `Andreas.Cap@esi.ac.at`

A. ROD GOVER, Department of Mathematics, The University of Auckland, Private Bag 92019, Auckland 1, New Zealand • *E-mail :* `gover@math.auckland.ac.nz`