Electromagnetism, metric deformations, ellipticity and gauge operators on conformal 4-manifolds *

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Abstract

On Riemannian signature conformal 4-manifolds we give a conformally invariant extension of the Maxwell operator on 1-forms. We show the extension is in an appropriate sense injectively elliptic, and recovers the invariant gauge operator of Eastwood and Singer. The extension has a natural compatibility with the de Rham complex and we prove that, given a certain restriction, its conformally invariant null space is isomorphic to the first de Rham cohomology. General machinery for extending this construction is developed and as a second application we describe an elliptic extension of a natural operator on perturbations of conformal structure. This operator is closely linked to a natural sequence of invariant operators that we construct explictly. In the conformally flat setting this yields a complex known as the conformal deformation complex and for this we describe a conformally invariant Hodge theory which parallels the de Rham result.

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1 Introduction

Many differential operators that are important in physics and differential geometry are deficient from the point of view of ellipticity or hyperbolicity. An example in the setting of Riemannian 4-manifolds is given by a certain natural 4th order conformally invariant operator on metric perturbations. There are analogues in higher even dimensions. It seems to us these operators should have an important role in the relevant deformation theory, and with a view to applications in this area we were led to consider whether there are some form of gauge-fixing operators which would extend these to elliptically coercive operators. The path to solving this has exposed a rather rich theory which blends classical elliptic theory with new tools emerging from representation theory. This enables a systematic approach to a whole class of problems of this nature. Details of this are developed in [4, 5]. Here as a means of introducing and surveying the key ideas we discuss two examples in 4 dimensions. By confining ourselves (for the most part) to this dimension and these very concrete cases we are able to present a self-contained treatment using rather elementary tools. In particular representation theoretic aspects are entirely suppressed. One of the examples is the above-mentioned operator in deformation theory and the other is the Maxwell operator of electromagnetism. Included in the results are conformally invariant, elliptically coercive extensions of these operators that lead to a notion of conformally invariant gauge-fixing. The latter extends and develops the result in [13]. As an application we show that (given an appropriate restriction) the conformally invariant null space of the Maxwell extension is precisely the first de Rham cohomology. Similarly, in the conformally flat case we give a conformally invariant Hodge theory for the deformation complex.

In the classical theory of electromagnetism, the *Maxwell equations* on a 2-form F over a pseudo-Riemannian 4-manifold are $dF = 0$, $\delta F = 0$. Here δ is the formal adjoint of the exterior derivative d; in the original physical problem, the metric has Lorentz signature. If our manifold is a simply connected region in $\mathbb{R}^4$, then by the Poincare Lemma, the equation $dF = 0$ implies that $F = dA$ for a 1-form A; this is traditionally called the *vector potential* for the Maxwell field F. The Maxwell system then reduces to the single equation $\delta dA = 0$. Of course the potential is only determined up to the "gauge" freedom of replacing A with $A + df$ for some function f. This ambiguity can be restricted by imposing further so-called "gauge-fixing" equations. A traditional choice is the (first order) *Lorentz gauge* equation $\delta A = 0$. With this added to the Maxwell equation $\delta dA = 0$, the vector potential A is determined by initial data on a Cauchy hypersurface.

An important feature of the Maxwell equations is that they are conformally invariant. This means, among other things, that the equations are well defined on a *conformal space-time*; that is, a 4-manifold equipped with a conformal equivalence class of Lorentzian metrics, rather than a single distinguished Lorentzian metric. The equivalence relation involved is given by $g \sim \widehat{g}$ iff $\widehat{g} = \Omega^2 g$ for some smooth positive function Ω. However, the Lorentz gauge equation is not conformally invariant, and so is not well defined in the conformal setting. (The equation $\delta A = 0$ is invariant on form-densities of a certain weight, but not on the form-0-densities where the Maxwell operator δd acts. See below for specifics on densities.)

In [13], Eastwood and Singer propose a third-order gauge fixing operator, which we shall denote S, with principal part $\delta d \delta$. They show that their operator is not conformally invariant on general 1-forms, but *is* invariant on the conformally invariant subspace of 1-forms in the kernel of the Maxwell operator δd.

We shall show that in fact the Eastwood-Singer operator S and the Maxwell operator δd can be naturally viewed as parts of a naturally arising single conformally invariant operator, which we shall denote by E. In fact, in the space-time setting (i.e. the setting of a fixed metric, or *scale*), this operator precisely recovers the system $A \mapsto (-\delta dA, \mathsf{S}A)$. But the important feature of E is that it has a clear and well-defined interpretation in the weaker setting of a conformal structure. Furthermore, our treatment is directly linked to the representation theory that gives rise to the natural vector bundles of conformal geometry. As a result, it generalises to similar situations; in particular, that of the so-called *metric deformation complex* as mentioned above (see also below). Our treatment, however, will employ the conformally invariant *tractor calculus*, which plays roughly the same role in conformal geometry that tensor-spinor calculus plays in pseudo-Riemannian geometry. This calculus, although defined here geometrically, encodes the required representation theoretic structures and so enables a self-contained treatment which, apart from some notational conventions, does not directly appeal to results from representation theory.

We shall work mainly in the case of Riemannian conformal structures initially, but all formulas and results on invariant operators continue in signature to conformal structures of other signatures. Note that in the discussion above of the Maxwell equations in the Lorentzian regime, the issue of determination on Cauchy surfaces came into play. This is a hyperbolicity property, and to some extent, such properties tend to correspond to ellipticity properties in the Riemannian regime. One concrete way to make the link, and one that applies in many situations of interest, is to verify that in each scale there is an operator $T = (T_1, T_2)$ and a positive integer m with the property that

$$(T_1, T_2) \begin{pmatrix} \text{operator} \\ \text{gauge} \end{pmatrix} = (\nabla^a \nabla_a)^m + (\text{lower order}).$$

For example, in the Maxwell-Eastwood-Singer discussion above, we have

$$(\delta d, d) \begin{pmatrix} \delta d \\ \delta d \delta + (\text{lower order}) \end{pmatrix} = (\nabla^a \nabla_a)^2 + (\text{lower order}).$$

This property of being a factor (up to lower-order terms) of a power the d'Alembertian $\nabla^a \nabla_a$ in the Lorentzian regime immediately gives some hyperbolicity properties, while the corresponding property with the Laplacian $\nabla^a \nabla_a$ in the Riemannian regime guarantees elliptic coercivity.

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2 Elliptically coercive extensions and gauge operators

In accordance with the remarks directly above, we shall work in the Riemannian conformal case. Let M be a Riemannian 4-manifold with Levi-Civita connection ∇ for the metric g. Let $\mathcal{E}^1$ be the space of smooth sections of the cotangent bundle T^*M. In fact, we shall often also informally refer to $\mathcal{E}^1$ as the cotangent bundle. Consider the Maxwell equations $d\omega = 0$, $\delta\omega = 0$ on a 2-form ω. Unless cohomology intervenes, the relation $d\omega = 0$ implies that $\omega = d\Phi$ for some 1-form Φ; as in the Lorentzian case, we shall call Φ the *vector potential.* The *Maxwell operator* δd on vector potentials fails to be elliptic – its leading symbol at a covector ξ is $\iota(\xi)\varepsilon(\xi)$, where ι and ε are, respectively, interior and exterior multiplication – and this symbol annihilates the range of $\varepsilon(\xi)$.

Let Δ denote the Bochner Laplacian $\nabla^a \nabla_a$. For a natural differential operators P on tensors, the condition that there exists another natural operator (a *quasi-inverse*) Q with $QP = \Delta^m +$ (lower order) implies that P is elliptically coercive – for example, its distributional null space is finite-dimensional and consists of smooth sections. If P takes an irreducible bundle to itself, [1] shows that the existence of such a Q is actually equivalent to ellipticity. One natural notion of ellipticity for operators like the *Euclidean Coulomb gauge* system $(\delta d, \delta)$ or the Eastwood-Singer system $(\delta d, \delta d \delta +$ (lower order)) is the *graded ellipticity* of Douglis and Nirenberg [10], which handles block arrays in which the entries have different orders. In order not to become unnecessarily enmeshed in the technicalities of this, we shall keep as our goal the existence of a quasi-inverse Q in the sense above. For purposes of this paper, we may take the term "elliptically coercive", applied to a natural operator, to mean the existence of a natural quasi-inverse.

Suppose we have a natural differential operator $\mathcal{B} : \mathcal{V}_2 \to \mathcal{V}_3$ where $\mathcal{V}_i$ is a space of smooth sections of some bundle. As for $\mathcal{E}^1$, we shall often refer to such spaces as bundles to simplify the discussion. It may be that the realization of $\mathcal{B}$ in some (and thus any) conformal scale is elliptic. If not, we seek a bundle $\mathcal{W}$ with $\mathcal{V}_3$ as a quotient and an operator $\mathcal{X} : \mathcal{V}_2 \to \mathcal{W}$ such that $\pi \circ \mathcal{X} = \mathcal{B}$, where π is the bundle map $\pi : \mathcal{W} \to \mathcal{V}_3$. Any (elliptically coercive) operator $\mathcal{X}$ with this property will be termed an (*elliptically coercive*) *extension* of $\mathcal{B}$.

Now suppose there is an operator $\mathcal{A} : \mathcal{V}_1 \to \mathcal{V}_2$ such that image($\mathcal{A}$) is a subspace of ker($\mathcal{B}$). Then we have a sequence $\mathcal{V}_1 \to \ker(\mathcal{B}) \subset \mathcal{V}_2 \to \mathcal{V}_3$. We are interested in whether ker($\mathcal{B}$)/image($\mathcal{A}$) can be naturally and invariantly identified with a subspace of ker($\mathcal{B}$). Thus we will view image($\mathcal{A}$) as the "gauge freedom" of the solutions to the $\mathcal{B}$ equation. Let $\mathcal{U}$ be the kernel in $\mathcal{W}$ of the bundle map π. Then we can write an exact sequence, or equivalently a composition series,

$$0 \to \mathcal{U} \to \mathcal{W} \to \mathcal{V}_3 \to 0 \quad \Leftrightarrow \quad \mathcal{W} = \mathcal{V}_3 \,+\!\!\!\!\!\!\bigcirc\, \mathcal{U}$$

to summarise this information about about the filtration of $\mathcal{W}$. Note that when restricted to ker($\mathcal{B}$), $\mathcal{X}$ takes values in the subspace $\mathcal{U}$. Let us denote the resulting operator $\mathcal{G} : \ker(\mathcal{B}) \to \mathcal{U}$ and call this the *gauge operator* given by $\mathcal{X}$. The situation so far is summarised in the commutative operator diagram in Figure 2. Here $\mathcal{P}$ is the composition of the gauge operator $\mathcal{G}$ with $\mathcal{A}$.

In the best of worlds $\mathcal{X}$ could turn out to be what might be called a "gauge-fixing extension" of $\mathcal{B}$ (relative to $\mathcal{A}$). This would mean for any $v_2 \in$

$$
\begin{tikzcd}
 & & & 0 \arrow[d] \\
 & & & \mathcal{U} \arrow[d] \\
 & & & \mathcal{W} \arrow[d] \\
\mathcal{V}_1 \arrow[r, "\mathcal{A}"'] \arrow[uurrr, "\mathcal{P}"] & \ker(\mathcal{B}) \arrow[r] \arrow[uurr, "\mathcal{G}"] & \mathcal{V}_2 \arrow[r, "\mathcal{B}"'] \arrow[ur, "\mathcal{X}"] & \mathcal{V}_3 \arrow[d] \\
 & & & 0
\end{tikzcd}
$$

Figure 1: The extension $\mathcal{X}$ of $\mathcal{B}$ is a gauge extension if $\mathcal{P}$ is elliptic.

$\ker(\mathcal{B})$ there is $v_1 \in \mathcal{V}_1$ such that $\mathcal{X}(v_2 + \mathcal{A}(v_1)) = 0$. That is, there should exist v_1 solving $\mathcal{P}(v_1) = -\mathcal{G}(v_2)$. The remaining freedom in $v_2 + \mathcal{A}(v_1)$ is then reduced to adding some u_1 from $\ker(\mathcal{P})$. It is ideal if this remaining freedom has no impact on the quotient $\ker(\mathcal{B})/\mathrm{image}(\mathcal{A})$. That is, a gauge-fixing extension should have $\ker(\mathcal{P}) \subset \ker(\mathcal{A})$. Whether a particular extension is gauge-fixing in this way is in general a non-local problem. One class of extensions in which there is at least a good chance arises when $\mathcal{P}$ is elliptic. Let us term such an extension $\mathcal{X}$ a *gauge extension* of $\mathcal{B}$ if $\mathcal{P} : \mathcal{V}_1 \to \mathcal{U}$ is elliptic. We will say the kernel of $\mathcal{P}$ is *harmless* if $\ker(\mathcal{P}) \subset \ker(\mathcal{A})$.

It is useful to note what all of these objects are in the Maxwell setting. Here $\mathcal{B}$ is the Maxwell operator $\delta d : \mathcal{E}^1 \to \mathcal{E}^1$ and $\mathcal{A}$ the exterior derivative $d : \mathcal{E} \to \mathcal{E}^1$, where we write $\mathcal{E}$ for (the smooth sections of) the trivial bundle. As in the space-time case, the traditional choice for $\mathcal{G}$ has been the divergence δ; in Riemannian signature this is called the (Euclidean) *Coulomb gauge.* So then $\mathcal{X}$ is just the operator $(\delta d, \delta) : \mathcal{E}^1 \to \mathcal{E}^1 \oplus \mathcal{E}$. Observe that this is a gauge extension of the Maxwell operator, as the $\mathcal{P}$ of Figure 2 can be taken to be $-\Delta$. (In this connection, note that $-\Delta = \delta d$ on functions.) To see that this is gauge fixing in the sense we have described, we need to know the gauge can be attained. In the compact setting, for example, this is always possible because to reach the Coulomb gauge involves solving $\delta(\Phi + df) = 0$

for f; that is, solve $\Delta f = \delta\Phi$. This achievable by standard elliptic theory. In addition, the operator $(\delta d, \delta)$ is an elliptically coercive extension, since composing with $(1, d) : \mathcal{E}^1 \oplus \mathcal{E} \to \mathcal{E}^1$ yields $\delta d + d\delta$, the form Laplacian (which agrees with $-\Delta$ up to lower order terms). The problem here is just that the operator $(\delta d, \delta)$ does not correspond (in the sense of the discussion of Figure 2) to any conformally invariant operator.

On a Riemannian 4-manifold the Eastwood-Singer gauge operator may also be viewed as giving an extension of the form $(-\delta d, \mathsf{S}) : \mathcal{E}^1 \to \mathcal{E}^1 \oplus \mathcal{E}$. In the conformal setting, however, the replacement for $\mathcal{E}^1 \oplus \mathcal{E}$ does not split as a direct sum. It turns out that there is a conformally invariant elliptically coercive gauge extension of Maxwell operator, $E : \mathcal{E}^1 \to \mathcal{E}_{\bar{A}}[-3]$ where $\mathcal{E}_{\bar{A}}[-3]$ (cf. $\mathcal{W}$ in Figure 2) is a bundle with the composition series $\mathcal{F} = \mathcal{E}^1[-2] \mathbin{+\!\!\!\!\supset} \mathcal{E}[-4]$ (cf. $\mathcal{W} = \mathcal{V}_3 \mathbin{+\!\!\!\!\supset} \mathcal{U}$), $\mathcal{E}^1[-2]$ is a bundle isomorphic to the cotangent bundle, and $\mathcal{E}[-4]$ is isomorphic to the trivial bundle. In a choice of metric from the conformal class, E recovers the operator $(-\delta d, \mathsf{S})$.

3 Conformal geometry and tractor calculus

Tractor calculus is a conformally invariant calculus based on natural bundles in conformal geometry. It includes the Cartan connection manifested as an induced vector bundle connection. However, one of the main misunderstandings in the area is that this is the end of the story. There are several other fundamental and equally important invariant operators involved. Perhaps even more importantly, the calculus provides the right forum for using results and ideas from representation theory in the "curved" (for example, conformally curved) differential geometric setting. Although this aspect has been suppressed in the current article, we should point out that it has been very influential in this work and will be described in [4, 5]. We summarise here some key tools of tractor calculus. This is mainly drawn from the development presented in [8], but many of the ideas and tools had their origins in [16], [2], and [15]. The notation and conventions in general follow the last two sources.

For the remainder of this section there is no real advantage in restricting to dimension 4, so we work on a real conformal n-manifold M, where $n \geq 3$. That is, we have a pair $(M, [g])$, where M is a smooth n-manifold and $[g]$ is a conformal equivalence class of Riemannian metrics. (In fact *most* results in this section are signature independent.) Two metrics g and $\widehat{g}$ are said to be *conformally equivalent* if $\widehat{g} = \Omega^2 g$ for some positive smooth function Ω. (The replacement of the metric g by the metric $\widehat{g}$ is called a *conformal transformation.*) For a given conformal manifold $(M, [g])$, we shall denote by $\mathcal{Q}$ the bundle of metrics. $\mathcal{Q}$ is a subbundle of S^2T^*M with fibre $\mathbb{R}_+$. ¿From this principal bundle there are natural line bundles $\mathcal{E}[w]$, $w \in \mathbb{R}$, on $(M, [g])$ induced from the irreducible representations of $\mathbb{R}_+$. A section of $\mathcal{E}[w]$ corresponds to a real-valued function f on $\mathcal{Q}$ with the homogeneity property $f(\Omega^2 g, x) = \Omega^w f(g, x)$.

For many discussions it will be convenient to use Penrose's abstract index notation. Thus for example we will sometimes use $\mathcal{E}_a$ as an alternative notation for the cotangent bundle $\mathcal{E}^1$ or its smooth sections. We then write $\mathcal{E}_{ab}$ for $\otimes^2\mathcal{E}_a$, $\mathcal{E}_{(ab)}$ for the symmetrisation of this and so forth. Similarly $\mathcal{E}^a$ indicates the tangent bundle or its smooth sections. An index which appears twice, once raised and once lowered, indicates a contraction. These conventions will be extended in an obvious way to the tractor bundles described below. In all settings, indices may also be omitted if the meaning is clear. We use the notation $\mathcal{E}_a[w]$ for $\mathcal{E}_a \otimes \mathcal{E}[w]$ and so on.

With $\mathcal{E}_+[-2]$ denoting the fibre subbundle of $\mathcal{E}[-2]$ corresponding to $\mathbb{R}_+ \subset \mathbb{R}$, it is easily verifed that $\mathcal{E}_+[-2]$ is canonically isomorphic to $\mathcal{Q}$. The *conformal metric* $\boldsymbol{g}_{ab}$ is the tautological section of $\mathcal{E}_{ab}[2]$ that represents the map $\mathcal{E}_+[-2] \cong \mathcal{Q} \to \mathcal{E}_{(ab)}$. Then $\boldsymbol{g}^{ab}$ is the section of $\mathcal{E}^{ab}[-2]$ such that $\boldsymbol{g}_{ab}\boldsymbol{g}^{bc} = \delta_a{}^c$, the identity endomorphism on $\mathcal{E}_c$. The conformal metric and its inverse will be used to raise and lower indices without further mention. Given a choice of metric g from the conformal class, we write ∇_a for the corresponding Levi-Civita connection. With these conventions the Laplacian Δ is given $\Delta = \boldsymbol{g}^{ab}\nabla_a\nabla_b = \nabla^b\nabla_b$. In view of the isomorphism $\mathcal{E}_+[-2] \cong \mathcal{Q}$, a choice of metric also trivialises the bundles $\mathcal{E}[w]$. This determines a connection on $\mathcal{E}[w]$ via the exterior derivative on functions. We shall also denote such a connection by ∇_a and refer to it the Levi-Civita connection. Defined in this way the Levi-Civita connection preserves the conformal metric.

The Riemannian curvature is defined by $(\nabla_a\nabla_b - \nabla_b\nabla_a)v^c = R_{ab}{}^c{}_d v^d$, on tangent vector fields v. This can be decomposed into the totally trace-free Weyl curvature C_{abcd} and a remaining part described by the symmetric *Rho-tensor* P_{ab}, according to

$$R_{abcd} = C_{abcd} + 2\boldsymbol{g}_{c[a}\mathsf{P}_{b]d} + 2\boldsymbol{g}_{d[b}\mathsf{P}_{a]c},$$

where $[\cdots]$ indicates the antisymmetrisation over the enclosed indices. The

Rho-tensor is a trace modification of the Ricci tensor R_{ab}. We write J for the trace $\mathsf{P}^a{}_a$ of P.

Under a conformal transformation the Levi-Civita connection then transforms as follows:

$$\widehat{\nabla_a u_b} = \nabla_a u_b - \Upsilon_a u_b - \Upsilon_b u_a + \boldsymbol{g}_{ab}\Upsilon^c u_c, \qquad \widehat{\nabla_a \sigma} = \nabla_a \sigma + w\Upsilon_a \sigma. \tag{1}$$

Here $u_b \in \mathcal{E}_b$, $\sigma \in \mathcal{E}[w]$, and $\Upsilon_a := \Omega^{-1}\nabla_a\Omega$.

Specialising for the moment to dimension 4, on a 2-form ω, Maxwell's equations $d\omega = 0$, $\delta\omega = 0$ may alternatively be written as $3\nabla_{[a}\omega_{bc]} = 0$ and $-\nabla^c\omega_{ca} = 0$ respectively. Similarly the *Maxwell operator* on vector potentials is the operator $\delta d : \Phi_a \mapsto -2\nabla^b(\nabla_b\Phi_a - \nabla_a\Phi_b)$. It is straightforward to verify directly, using the transformation formulae, that these equations are conformally invariant. On the other hand for $\varphi \in \mathcal{E}_a[w]$ we have

$$\widehat{\nabla}^a\varphi_a = \nabla^a\varphi_a + (w+2)\Upsilon^a\varphi_a.$$

This shows that the Coulomb gauge operator δ is invariant $\mathcal{E}_a[-2] \to \mathcal{E}[-4]$ but *is not* conformally invariant on $\mathcal{E}_a$. In fact it is clearly not even invariant on the subspace of exact 1-forms. Thus this is incompatible with the invariance of the Maxwell operator δd, which acts invariantly on $\mathcal{E}_a = \mathcal{E}_a[0]$.

A natural generalisation of the Maxwell equations to even dimensions $n = 2\ell$ is the system $d\omega = 0$, $\delta\omega = 0$ on $\mathcal{E}_{a_1\cdots a_\ell}$. Again, unless cohomology intervenes, there is a vector potential $\Phi \in \mathcal{E}_{a_1\cdots a_{\ell-1}}$, and the equations reduce to the invariant equation $\delta d\Phi = 0$. Appending the Coulomb gauge equation $\delta A = 0$ in a scale, we have an elliptically coercive system, but again this system is not conformally invariant: δ carries $\mathcal{E}_{a_1\cdots a_{\ell-1}}[-2]$ to $\mathcal{E}_{a_1\cdots a_{\ell-2}}[-4]$ invariantly, but does not act invariantly on $\mathcal{E}_{a_1\cdots a_{\ell-1}}[0]$. There is, however, an analogue of the invariant Eastwood-Singer gauge [4].

The Weyl curvature is conformally invariant, that is $\widehat{C}_{ab}{}^c{}_d = C_{ab}{}^c{}_d$, and the Rho-tensor transforms by

$$\widehat{\mathsf{P}}_{ab} = \mathsf{P}_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \tfrac{1}{2}\Upsilon^c\Upsilon_c\boldsymbol{g}_{ab}. \tag{2}$$

Let us write $\mathcal{E}_{(ab)_0}[1]$ for the symmetric trace-free part of $\mathcal{E}_{ab}[1]$. Then $\mathcal{E}_{(ab)_0}[1]$ is naturally a smooth subbundle of the bundle of 2-jets $J^2(\mathcal{E}[1])$ of the density bundle $\mathcal{E}[1]$. The *standard tractor bundle* $\mathcal{E}^A$ is defined by the exact sequence

$$0 \to \mathcal{E}_{(ab)_0}[1] \to J^2(\mathcal{E}[1]) \to \mathcal{E}^A \to 0. \tag{3}$$

The jet exact sequence at 2-jets and the corresponding sequence at 1-jets, viz. $0 \to \mathcal{E}_a[1] \to J^1(\mathcal{E}[1]) \to \mathcal{E}[1] \to 0$, determine a composition series for $\mathcal{E}^A$ which we can summarise via the semi-direct sum notation by $\mathcal{E}^A = \mathcal{E}[1] \oplus\!\!\!\!+ \mathcal{E}_a[1] \oplus\!\!\!\!+ \mathcal{E}[-1]$. We denote by X^A the canonical section of $\mathcal{E}^A[1] := \mathcal{E}^A \otimes \mathcal{E}[1]$ corresponding to the mapping $\mathcal{E}[-1] \to \mathcal{E}^A$. A choice of metric from the equivalence class determines an isomorphism $\mathcal{E}^A \to \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1] =: [\mathcal{E}^A]_g$ of vector bundles. If the image of $V^A \in \mathcal{E}^A$ is $[V^A]_g = (\sigma, \mu_a, \tau)$, then for $\hat{g} = \Omega^2 g$ we have

$$[V^A]_{\hat{g}} = (\widehat{\sigma, \mu_a, \tau}) = (\sigma, \mu_a + \sigma \Upsilon_a, \tau - \Upsilon_b \mu^b - \tfrac{1}{2} \sigma \Upsilon_b \Upsilon^b).$$

This transformation formula characterises sections of $\mathcal{E}^A$ in terms of triples in $\mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$ at all possible scales. In this notation $[X^A]_g = (0, 0, 1)$. It is convenient to introduce scale-dependent sections $Z^{Ab} \in \mathcal{E}^{Ab}[-1]$ and $Y^A \in \mathcal{E}^A[-1]$ mapping into the other slots of these triples so that $[V^A]_g = (\sigma, \mu_a, \tau)$ is equivalent to

$$V^A = Y^A \sigma + Z^{Ab} \mu_b + X^A \tau. \tag{4}$$

The standard tractor bundle has an invariant metric h_{AB} of signature $(p+1, q+1)$ and an invariant connection, which we shall also denote by ∇_a, preserving h_{AB}. If V^A is as above and $\underline{V}^B \in \mathcal{E}^B$ is given by $[\underline{V}^B]_g = (\underline{\sigma}, \underline{\mu}_b, \underline{\tau})$, then

$$h_{AB} V^A \underline{V}^B = \mu^b \underline{\mu}_b + \sigma \underline{\tau} + \tau \underline{\sigma}.$$

Using h_{AB} and its inverse to raise and lower indices, we immediately see that

$$Y_A X^A = 1, \quad Z_{Ab} Z^A{}_c = \boldsymbol{g}_{bc},$$

and that all other quadratic combinations that contract the tractor index vanish. In fact the metric may be decomposed into a sum of projections, $h_{AB} = Z_A{}^c Z_{Bc} + X_A Y_B + Y_A X_B$. The tractor metric will be used to raise and lower indices without further comment. We shall use either "horizontal" (as in $[V^A]_g = (\sigma, \mu_a, \tau)$ or (4)) or "vertical" (as in (5) below) notation, depending on which is clearer in each given situation.

If, for a metric g from the conformal class, $V^A \in \mathcal{E}^A$ is given by $[V^A]_g = (\sigma, \mu_a, \tau)$, then the invariant tractor connection is given by

$$[\nabla_a V^A]_g = \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + \boldsymbol{g}_{ab} \tau + \mathsf{P}_{ab} \sigma \\ \nabla_a \tau - \mathsf{P}_{ab} \mu^b \end{pmatrix}. \tag{5}$$

Tensor products of the standard tractor bundle, skew or symmetric parts of these, and so forth are all termed tractor bundles. The bundle tensor product of such a bundle with $\mathcal{E}[w]$, for some real number weight w, is termed a weighted tractor bundle. Given a choice of conformal scale we have the corresponding Levi-Civita connection on tensor and density bundles. In this setting we can use the coupled Levi-Civita tractor connection to act on sections of the tensor product of a tensor bundle with a tractor bundle. This is defined by the Leibniz rule in the usual way. In particular we have

$$\nabla_a X_A = Z_{Aa}\,, \quad \nabla_a Z_{Ab} = -\mathsf{P}_{ab} X_A - Y_A \boldsymbol{g}_{ab}\,, \quad \nabla_a Y_A = \mathsf{P}_{ab} Z_A{}^b, \tag{6}$$

which are useful for calculations.

The adjoint tractor bundle $\mathcal{E}^\alpha$ is simply the second exterior power of the tractor bundle, i.e. $\mathcal{E}^\alpha := \mathcal{E}^{[AB]}$. It follows that it has a composition series

$$\mathcal{E}^a \mathbin{\oplus\!\!\!\!+} (\mathcal{E} \oplus \mathcal{E}_{[ab]}[2]) \mathbin{\oplus\!\!\!\!+} \mathcal{E}_a\,.$$

Given a choice of metric, this decomposes so that the semi-direct sum becomes a direct sum (i.e. $\mathbin{\oplus\!\!\!\!+}$ gets replaced by $\oplus$), and it is convenient to write sections $\mathbb{V}^\beta$ of $\mathcal{E}^\beta$ as corresponding 4-tuples

$$[\mathbb{V}^\alpha]_g = (\xi^a, \Phi_b{}^a, \varphi, \omega_a).$$

Under a conformal transformation $g \mapsto \hat{g}$, we have

$$[\mathbb{V}^\alpha]_{\hat{g}} = (\xi^a, \widehat{\Phi_b{}^a, \varphi}, \omega_a) =$$
$$(\xi^a, \Phi_b{}^a + \xi^a \Upsilon_b - \xi_b \Upsilon^a, \varphi + \xi^a \Upsilon_a, \omega_a - \Phi_a{}^b \Upsilon_b - \varphi \Upsilon_a - \xi^b \Upsilon_b \Upsilon_a + \tfrac{1}{2} \xi_a \Upsilon_k \Upsilon^k).$$

We can view the adjoint tractor bundle as the bundle of filtration and metric-preserving endomorphisms of the standard tractor bundle, and we take one-half of the trace form as the inner product $B_{\alpha\beta}$ on $\mathcal{E}^\beta$. (The typical fibre of $\mathcal{E}^\alpha$ is the Lie algebra $\mathfrak{so}(n+1,1)$.) That is if $[\mathbb{V}]_g = (\xi^a, \Phi_b{}^a, \varphi, \omega_a)$ and $[\underline{\mathbb{V}}]_g = (\underline{\xi}^a, \underline{\Phi}_b{}^a, \underline{\varphi}, \underline{\omega}_a)$, then

$$B_{\alpha\beta} \mathbb{V}^\alpha \underline{\mathbb{V}}^\beta = \frac{1}{2} \Phi_a{}^b \underline{\Phi}_b{}^a + \varphi \underline{\varphi} + \xi^a \underline{\rho}_a + \rho_a \underline{\xi}^a.$$

The connection on the standard tractor bundle gives a connection on its tensor powers by the Leibniz rule, and in particular on $\mathcal{E}^\beta$. For a section $\mathbb{V}^\alpha$ of $\mathcal{E}^\alpha$ with $[\mathbb{V}^\beta]_g = (\xi^b, \Phi_c{}^b, \varphi, \omega_b)$, this is given by

$$[\nabla_a \mathbb{V}^\beta]_g = \begin{pmatrix} \nabla_a \xi^b - \Phi_a{}^b - \delta_a{}^b \varphi \\ \nabla_a \Phi_c{}^b + \delta_a{}^b \omega_c - \boldsymbol{g}_{ac}\omega^b + \xi^b \mathsf{P}_{ac} - \xi_c \mathsf{P}_a{}^b \\ \nabla_a \varphi + \omega_a + \xi^k \mathsf{P}_{ka} \\ \nabla_a \omega_b - \mathsf{P}_{ka}\Phi_b{}^k - \mathsf{P}_{ba}\varphi \end{pmatrix} \tag{7}$$

Alternatively, in analogy with the standard tractor calculations above, we can write

$$\mathbb{V}^\beta = \mathbb{Y}^\beta{}_a \xi^a + \mathbb{Z}^\beta{}_a{}^b \Phi_b{}^a + \mathbb{W}^\beta \varphi + \mathbb{X}^{\beta a}\omega_a\,,$$

where $\mathbb{X}^{\beta a}$ is an invariant section, and $\mathbb{Y}^\beta{}_a$, $\mathbb{Z}^\beta{}_a{}^b$, and $\mathbb{W}^\beta$ are scale-dependent sections. It is straightforward to write formulae for ∇ on these (cf. (6)).

We conclude with some observations we will need later. One is that the Yamabe operator extends to a conformally operator on tractor bundles of the appropriate weight. That is there is a conformally invariant differential operator $\Box : \mathcal{E}^\Psi[1-n/2] \to \mathcal{E}^\Psi[-1-n/2]$, where $\mathcal{E}^\Psi[w]$ indicates any tractor bundle of weight w. This is given by the usual formula,

$$\Box V := \nabla_p \nabla^p V + w \mathsf{J} V, \tag{8}$$

except now ∇ indicates the coupled tractor-Levi-Civita connection.

Now consider $\mathbb{R}^{n+2}$ equipped with an inner product $\boldsymbol{h}$ of signature $(n+1,1)$. The space of null lines is a quadric in the projectivisation $\mathbb{P}_{n+1} = \mathbb{P}\mathbb{R}^{n+2}$ with a (conformally flat) conformal structure. This n-sphere is usually regarded as the standard flat model for a conformal structure and we will refer to this as the conformal sphere. The orthogonal group $G := \mathrm{O}(\boldsymbol{h})$ acts conformally on this space which may be identified with G/P, where P is a certain parabolic subgroup of G. Now G is a principal P-bundle over G/P and in this setting the standard tractor bundle is induced from the defining representation of G regarded as a P-module. Since this space carries a representation of G, the bundle is trivialisable. It follows easily from the normality of the tractor connection (see [8] and [7]) that under this trivialisation the operator $\Box$ agrees with the trivially extended Yamabe operator. Thus $\Box$ is elliptic in this flat model but therefore also in general.

4 Gauge extension of the Maxwell operator

The exterior derivative operators are well defined diffeomorphism-invariant operators on any smooth manifold, and so in particular are well defined on a conformal manifold. However there are other conformally invariant operators between forms. On the dimension 4 conformal sphere (section 3) the following diagram gives all G-invariant operators [12] between the forms (via the isomorphisms $\mathcal{E}^1[-2] \cong \mathcal{E}^3$ and $\mathcal{E}[-4] \cong \mathcal{E}^4$; see immediately below). In fact these are the only G-invariant operators on forms which take values in irreducible tensor bundles.

$$\begin{tikzcd}
 & & \mathcal{E}^2_+ \arrow[dr] & & \\
\mathcal{E} \arrow[r] \arrow[rrrr, bend right=50] & \mathcal{E}^1 \arrow[ur] \arrow[dr] \arrow[rr, bend left=50] & & \mathcal{E}^1[-2] \arrow[r] & \mathcal{E}[-4] \\
 & & \mathcal{E}^2_- \arrow[ur] & &
\end{tikzcd} \tag{9}$$

Here $\mathcal{E}^2_\pm$ are the self-dual and anti-self-dual 2-forms. Proceeding from the left, the first short horizontal operator is the exterior derivative on functions. The first diagonal operators are given by the exterior derivative followed with projections into $\mathcal{E}^2_\pm$ and the remaining short arrow operators are formal adjoints of these. (Formal adjoints are with respect to the conformally invariant inner product of section 4.1 below.) The operator $\mathcal{E}^1 \to \mathcal{E}^1[-2]$ is of course the Maxwell operator δd (which is up to scale is the composition around either edge of the diamond) and the longest operator has principal part Δ^2. The generalisation of these to invariant operators on general conformal 4-manifolds is straightforward except for the last of these, which in that generality is known as the Paneitz operator. This operator, which we shall denote P_4, is given by the formula

$$P_4 := \nabla_b(\nabla^b\nabla^c + 4\mathsf{P}^{bc} - 2\mathsf{J}\boldsymbol{g}^{bc})\nabla_c : \mathcal{E} \to \mathcal{E}[-4].$$

The Paneitz operator is formally self-adjoint and annihilates constant functions. Among operators with these properties it is known to be the unique (up to constant multiples) conformally invariant natural operator between these bundles. Most of the operators from diagram (9) play a role in the gauge extension of the Maxwell operator.

Let us temporarily work in a general dimension $n \geq 3$. Recall that the standard tractor bundle has a composition series

$$\mathcal{E}_A = \mathcal{E}[1] \oplus\!\!\!\!+ \mathcal{E}_a[1] \oplus\!\!\!\!+ \mathcal{E}[-1]$$

and the bundle injection $\mathcal{E}[-1] \to \mathcal{E}_A$ is given by $\rho \mapsto \rho X_A$. Let us denote by $\mathcal{E}_{\bar{A}}$ the quotient of $\mathcal{E}_A$ by the image of this map, and let $\mathcal{E}^{\bar{A}}$ be the dual bundle. Extending the conventions from above, we write $\mathcal{E}_{\bar{A}}[w]$ to mean $\mathcal{E}_{\bar{A}} \otimes \mathcal{E}[w]$ and so forth. Clearly $\mathcal{E}_{\bar{A}}[w-1]$ has the composition series $\mathcal{E}_{\bar{A}}[w-1] = \mathcal{E}[w] \oplus\!\!\!\!+ \mathcal{E}_a[w]$. We define

$$\begin{aligned} I_{\bar{A}}{}^a : \mathcal{E}_a[w] &\to \mathcal{E}_{\bar{A}}[w-1], \\ \mu_a &\mapsto I_{\bar{A}}{}^a \mu_a \end{aligned}$$

to be the canonical inclusion. Given choice of metric g, we have $[\mathcal{E}_{\bar{A}}[w-1]]_g = \mathcal{E}[w] \oplus \mathcal{E}_a[w]$ and the inclusion is given by $[I_{\bar{A}}{}^a \mu_a]_g = (0, \mu_a)$.

Now provided $w \notin \{1 - n/2, 2 - n\}$, the algebraic bundle surjection $P_{\bar{A}}{}^A : \mathcal{E}_A[w-1] \to \mathcal{E}_{\bar{A}}[w-1]$ has an invariant differential splitting. That is, there is an operator

$$S_A{}^{\bar{A}} : \mathcal{E}_{\bar{A}}[w-1] \to \mathcal{E}_A[w-1] \tag{10}$$

such that the composition $P_{\bar{B}}{}^A S_A{}^{\bar{A}}$ is the identity $\delta_{\bar{B}}{}^{\bar{A}}$ on $\mathcal{E}_{\bar{A}}[w-1]$. In terms of the decomposition $[\mathcal{E}_A[w-1]]_g = \mathcal{E}[w] \oplus \mathcal{E}_a[w] \oplus \mathcal{E}[w-2]$ this is given by

$$\begin{pmatrix} \sigma \\ \mu_a \end{pmatrix} \mapsto \begin{pmatrix} \sigma \\ \mu_a \\ -\dfrac{1}{n+2w-2}\left(\dfrac{1}{n+w-2}\Delta + \mathsf{J}\right)\sigma - \dfrac{1}{n+w-2}\nabla^b \mu_b \end{pmatrix}.$$

In the alternative notation,

$$\zeta_{\bar{A}} = Y_{\bar{A}}\sigma + Z_{\bar{A}}{}^a \mu_a$$

is carried to

$$\begin{aligned} &S_A{}^{\bar{A}} \zeta_{\bar{A}} = Y_A X^{\bar{A}} \zeta_{\bar{A}} + Z_A{}^a Z^{\bar{A}}{}_a \zeta_{\bar{A}} \\ &\quad - \frac{1}{n+2w-2} X_A \left(\frac{1}{n+w-2}\Delta + \mathsf{J}\right)(X^{\bar{A}}\zeta_{\bar{A}}) - \frac{1}{n+w-2} X_A \nabla^b (Z^{\bar{A}}{}_b \zeta_{\bar{A}}). \end{aligned}$$

Here $Y_{\bar{A}}$ is the image of Y_A under $\mathcal{E}_A[-1] \to \mathcal{E}_{\bar{A}}[-1]$, $X^{\bar{A}}$ the section of $\mathcal{E}^{\bar{A}}[1]$ with image X^A under $\mathcal{E}^{\bar{A}}[1] \to \mathcal{E}^A[1]$ and so forth.

We would now like to introduce the formal adjoints $\bar{S}^{\bar{A}A} : \mathcal{E}_A[-3] \to \mathcal{E}^{\bar{A}}[-3]$ and $\bar{I}^a{}_{\bar{A}} : \mathcal{E}^{\bar{A}}[-3] \to \mathcal{E}^a[-4]$ of the operators above. Recall that the *formal adjoint* of a differential operator between vector bundles, $\mathcal{D} : E \to F$, is a differential operator $\mathcal{D}^* : F^* \to E^*$, provided a smooth measure is fixed. Given a metric g from the conformal class we have the Riemannian measure. This depends on the choice g. However there is a canonical conformal volume form $\boldsymbol{\epsilon}$, that is the canonical section of $\mathcal{E}_{[a_1 a_2 \cdots a_n]}[n]$ compatible with the conformal metric. Thus we can invariantly integrate densities of weight $-n$. As a result, the formal adjoint, computed with respect to conformal structure, of a conformally invariant differential operator $\mathcal{D} : \mathcal{E}^s_t[w] \to \mathcal{E}^u_v[w']$, where s, t, u, v are index arrays rather than single indices, will be a *conformally invariant* differential operator

$$\bar{\mathcal{D}} : \mathcal{E}^v_u[-n-w'] \to \mathcal{E}^t_s[-n-w]. \tag{11}$$

Setting $\underline{w} = 1 - w - n$, the formal adjoint of $S_A{}^{\bar{A}}$ is easily found (integrating by parts) to be the operator $\bar{S}^{\bar{A}}{}_A : \mathcal{E}^A[\underline{w}] \to \mathcal{E}^{\bar{A}}[\underline{w}]$ given by

$$[\bar{S}^{\bar{A}}{}_A V^A]_g = \begin{pmatrix} \mu^a - \dfrac{1}{\underline{w}+1} \nabla^a \sigma \\ \dfrac{1}{n+2\underline{w}} \left(-\dfrac{1}{\underline{w}+1} \Delta + \mathsf{J} \right) \sigma + \tau \end{pmatrix}$$

if $[V^A]_g = (\sigma, \mu^a, \tau)$. In the alternative notation,

$$\begin{aligned} \bar{S}^{\bar{A}}{}_A V^A = {} & Z^{\bar{A}}{}_a Z_A{}^a V^A - \frac{1}{\underline{w}+1} Z^{\bar{A}}{}_a \nabla^a (X_A V^A) \\ & + \frac{1}{n+2\underline{w}} X^{\bar{A}} \left(-\frac{1}{\underline{w}+1} \Delta + \mathsf{J} \right) (X_A V^A) + X^{\bar{A}} Y_A V^A. \end{aligned}$$

It follows from the splitting property of S that $\bar{S}$ splits the canonical bundle injection $\mathcal{E}^{\bar{A}}[-3] \to \mathcal{E}_A[-3]$. That is, upon restriction to $\mathcal{E}^{\bar{A}}[-3]$, regarded as a subbundle of $\mathcal{E}_A[-3]$, the operator $\bar{S}$ is the identity.

The formal adjoint $\bar{I}^a{}_{\bar{A}}$ of $I_{\bar{A}}{}^a$ is the map which simply takes $\mathcal{E}^{\bar{A}}[\underline{w}] = \mathcal{E}_a[\underline{w}+1] \oplus \mathcal{E}[\underline{w}-1]$ to its quotient by the subbundle $\mathcal{E}[\underline{w}-1]$, so if $V^{\bar{A}} \in \mathcal{E}^{\bar{A}}[\underline{w}]$ is given by $[(\mu^a, \tau)]_g$, then $[\bar{I}^a{}_{\bar{A}} V^{\bar{A}}]_g = \mu^a$.

To construct a gauge extension of the Maxwell operator we merely have to specialise to $n = 4, w = 0$ and compose with $\Box$ appropriately. We obtain the following.

Theorem 4.1 *The operator*

$$(E^{\bar{A}a} := \bar{S}^{\bar{A}A}\Box S_A{}^{\bar{C}} I_{\bar{C}}{}^a) : \mathcal{E}_a \to \mathcal{E}^{\bar{A}}[-3]$$

is a conformally invariant operator such that
(i) $\bar{I}_{b\bar{A}} E^{\bar{A}a} : \mathcal{E}_a \to \mathcal{E}_b[-2]$ *is a non-zero multiple of the Maxwell operator.*
(ii) $-2E^{\bar{A}a}\nabla_a f = X^{\bar{A}} P_4 f$
(iii) E *is elliptically coercive.*

To adapt figure 2 to this setting, we have that $\mathcal{A}$ is d on functions, and $\mathcal{B}$ is the Maxwell operator δd. Then (i) is stating that E is an extension (cf. $\mathcal{L}$) of the Maxwell operator; (ii) means it is a gauge extension (since the Paneitz operator is elliptic), and (iii) says finally that it is an elliptically coercive gauge extension.

Proof of the theorem: The formulae are conformally invariant by construction. To obtain (i) the key point is to establish that E is non-trivial. Since $\Box$ is elliptic (see section 3) it has finite dimensional null space in any compact setting. Thus it follows that the composition $\Box SI : \mathcal{E}_a \to \mathcal{E}_A[-3]$ is not trivial. Now $\mathcal{E}_A[-3] = \mathcal{E}[-2] \oplus \mathcal{E}_a[-2] \oplus \mathcal{E}[-4]$. Composing with $\Box SI$ the map $\mathcal{E}_A[-3] \to \mathcal{E}[-2]$ yields an invariant differential operator $\mathcal{E}_a \to \mathcal{E}[-2]$. Consulting the diagram (9) we see that there is no such operator in the flat model. Thus in that homogeneous setting this last operator must be trivial, meaning that the image of $\Box SI$ lies in the sub-bundle $\mathcal{E}^{\bar{A}}[-3] = \mathcal{E}_a[-2] \oplus \mathcal{E}[-4]$ of $\mathcal{E}^A[-3]$. It follows easily that on the conformal sphere $\bar{S}$ acts as the identity on the image of $\Box SI$. Thus $E = \bar{S}\Box SI$ is also non-trivial in the flat model and so non-trivial in general.

Now suppose $\bar{I}E$ were trivial on the conformal sphere. Then, since $\mathcal{E}^{\bar{A}}[-3]$ has the composition series $\mathcal{E}^{\bar{A}}[-3] = \mathcal{E}_a[-2] \oplus \mathcal{E}[-4]$, E would give a non-trivial invariant operator $\mathcal{E}_a \to \mathcal{E}[-4]$. But according to the diagram above there is no such operator. Thus $\bar{I}E$ is non-trivial in this flat model and hence non-trivial in general. Once again from the diagram it follows that on the conformal sphere $\bar{I}E$ is the Maxwell operator (up to a non-zero scale). In fact it is easily verified that even in the general case the Maxwell operator is the unique conformally invariant differential operator between the bundles $\mathcal{E}_a$ and $\mathcal{E}_a[-2]$. This concludes the proof of (i).

Now since exact 1-forms are annihilated by the Maxwell operator it follows from (i) that Edf takes values in the subspace $X\mathcal{E}[-4]$ in $\mathcal{E}^{\bar{A}}[-3] =$

$\mathcal{E}_a[-2] \oplus \mathcal{E}[-4]$. That is

$$E^{\bar{A}a}\nabla_a f = X^{\bar{A}} P f$$

for some invariant operator $P : \mathcal{E} \to \mathcal{E}[-4]$. By construction P factors through d and so annihilates constant functions. Using once again the ellipticity of $\Box$ we can also deduce that P is non-trivial and so by uniqueness P is the Paneitz operator as claimed in (ii).

Finally observe that it is straightforward to directly calculate the operators in the proposition. Choosing some metric g from the conformal class for the purpose of calculations, observe that $(n+w-2)S_A{}^{\bar{A}} I_{\bar{A}}{}^a : \mathcal{E}_a[w] \to \mathcal{E}_A[w-1]$ is simply $\varphi_a \to (0,\ (n+w-2)\varphi_a,\ \ -\nabla^b\varphi_b)$ (this is the operator E^{bC} of [11]). Setting $n=4, w=0$ and composing with $\Box$ yields

$$\Box S_A{}^{\bar{C}} I_{\bar{C}}{}^a \varphi_a = \begin{pmatrix} 0 \\ 2\nabla^b\nabla_{[b}\varphi_{a]} \\ -\frac{1}{2}\nabla_b(\nabla^b\nabla^c + 4\mathsf{P}^{bc} - 2\mathsf{J}\boldsymbol{g}^{bc})\varphi_c \end{pmatrix}$$

Thus $\bar{S}^{\bar{A}B}\Box S_B{}^{\bar{C}} I_{\bar{C}}{}^a\varphi_a$ is just

$$\left(2\nabla^b\nabla_{[b}\varphi_{a]}\,, -\tfrac{1}{2}\nabla_b(\nabla^b\nabla^c + 4\mathsf{P}^{bc} - 2\mathsf{J}\boldsymbol{g}^{bc})\varphi_c\right).$$

So we have that $[E\varphi]_g = -(\delta d\varphi, \delta d\delta\varphi + (\text{lower order}))$. As mentioned already, the elliptic coercivity of this is verified by composing with $(\delta d, d)$ which yields $\Delta^2 + (\text{lower order})$. $\Box$

Some points are worth making here. Firstly note that the second component, $\mathsf{S}\varphi := -\frac{1}{2}\nabla_b(\nabla^b\nabla^c + 4\mathsf{P}^{bc} - 2\mathsf{J}\boldsymbol{g}^{bc})\varphi_c$, of $[E]_g$ is Eastwood and Singer's gauge operator, at least modulo a factor of $-1/2$. From this explicit formula we see that we have $\mathsf{S} = \delta T$ for a second order operator T, a fact that we will use below. Next from its construction here, we see that the conditional conformal invariance of the gauge operator S (i.e. the fact that it is conformally invariant on solutions of the Maxwell operator) is an immediate consequence of the invariance of the operator $E^{\bar{A}a}$ and the conformal transformation law $[V^{\bar{A}}]_g = (\mu_b, \tau) \mapsto \widehat{(\mu_a, \tau - \Upsilon^c\mu_c)} = [V^{\bar{A}}]_{\hat{g}}$ for sections $V^{\bar{A}}$ of $\mathcal{E}^{\bar{A}}[-3]$. Finally we should say that, although the operator $\bar{S}^{\bar{A}B}$ effectively plays no role here, we can only know this by actually performing the calculation in some detail.

4.1 Application: A conformal Hodge theory

Here we suppose that M is an oriented compact 4-manifold. If M is equipped with a Riemannian metric, then Hodge-de Rham theory identifies the i^{th} de Rham cohomology $H^i(M)$ with the space of harmonics $\mathcal{H}^i(M)$. This is the kernel of the form Laplacian $\delta d + d\delta$ on i-forms or, alternatively, it is recovered by $\mathcal{H}^i(M) = \ker(d : \mathcal{E}^i \to \mathcal{E}^{i+1}) \cap \ker(\delta : \mathcal{E}^i \to \mathcal{E}^{i-1})$ (with obvious qualifications at either extreme of the de Rham complex). As before, $\mathcal{E}^i := \mathcal{E}_{[a_1 \cdots a_i]}$.

In general then we would expect the subspace of harmonics to move around as we change to different metrics in the conformal class. In fact in dimension 4, $\mathcal{H}^2(M)$ is a conformally invariant subspace of $\mathcal{E}^2$. This is obvious as both d and δ are conformally invariant on $\mathcal{E}^2$. Also $\mathcal{H}^0(M)$ is just the invariant subspace of locally constant functions. On the other hand $\mathcal{H}^1(M)$ is not stable in this way in $\mathcal{E}^1$. Verifying this is the same calculation as verifying the failure of the Coulomb gauge to be conformally invariant (see Section 3). It is interesting to ask whether there is a conformally invariant replacement for $\mathcal{H}^1(M)$. In fact $\ker(E)$ is, at the very least, a good candidate.

Theorem 4.2 *Suppose M is an oriented compact manifold such that the null space of P_4 is the space of locally constant functions. Then* $\ker(E : \mathcal{E}_a \to \mathcal{E}^{\bar{A}}[-3])$ *is a conformally invariant subspace of $\mathcal{E}^1$ isomorphic to $H^1(M)$.*

Note that since, in a choice of scale, $[E\varphi]_g$ has the form $(-\delta d\varphi, \mathsf{S}\varphi)$ it follows that $\ker(E)$ is just $\ker(\delta d : \mathcal{E}^1 \to \mathcal{E}^1[-2]) \cap \ker(\mathsf{S} : \mathcal{E}^1 \to \mathcal{E}[-4])$. This intersection is conformally invariant because S is invariant on $\ker(\mathsf{S})$.

Note that for $i = 0, 1, 2$ there is an invariant pairing between $\mathcal{E}^i$ and $\mathcal{E}^{4-i}$ given simply by

$$\varphi, \psi \mapsto \int_M \varphi \wedge \psi \tag{12}$$

Note this does not require a Riemannian or even a conformal structure; it is well defined on any oriented compact 4-manifold M. So of course, in particular, it is conformally invariant.

At a Riemannian scale, the Hodge $\star$ operator $\varphi_{a_1 \cdots a_k} \mapsto \epsilon^{c_1 \cdots c_k}{}_{b_1 \cdots b_{n-k}} \varphi_{c_1 \cdots c_k}$ is a natural bundle isometry $\mathcal{E}^k \to \mathcal{E}^{n-k}$ in the form inner products $f_k(\varphi, \psi) := (k!)^{-1} \varphi_{a_1 \cdots a_k} \psi^{a_1 \cdots a_k}$, and we have the identity $f_k(\varphi, \psi)\epsilon = \varphi \wedge \star\psi$. Given just a conformal structure, $\star$ carries $\mathcal{E}^k[w]$ to $\mathcal{E}^{n-k}[w + n - 2k]$. We can use this to rewrite the total space of the de Rham complex (now in dimension 4) as

$$\mathcal{E}^* := \mathcal{E} \oplus \mathcal{E}^1 \oplus \mathcal{E}^2 \oplus \mathcal{E}^1[-2] \oplus \mathcal{E}[-4].$$

The invariant pairing (12) then gives a conformally invariant non-degenerate, indefinite inner product on the vector space $\mathcal{E}^*$. This is determined by symmetry, bilinearity and the formulae

$$(\varphi, \psi) = \begin{cases} \int_M \varphi \wedge \star\psi, & \varphi \in \mathcal{E}^k, \psi \in \mathcal{E}^k[2k-4],\ k = 0, 1 \text{ or } 2, \\ 0 & \text{otherwise.} \end{cases}$$

The non-degeneracy of this follows directly from the positive definiteness of each f_k. Note also that for $\varphi, \psi \in \mathcal{E}^2$, we have $\int \varphi \wedge \star\psi = \int \psi \wedge \star\varphi$ since in either case the integrand is $f_2(\varphi, \psi)\epsilon$. Thus there is no conflict with the extension by symmetry. (The restriction to compact M can be lifted by requiring at least one form in the inner product to have compact support, and of course the generalisation to other dimensions is straightforward.)

Invariant operators on a subspace of $\mathcal{E}^*$ will be identified with their trivial extension to an operator on $\mathcal{E}^*$. In this way we define the formal adjoint of operators between subspaces of $\mathcal{E}^*$. For example the formal adjoint of $d : \mathcal{E} \to \mathcal{E}^1$ is the conformally invariant operator $\delta : \mathcal{E}^1[-2] \to \mathcal{E}[-4]$. In this picture the formal self-adjointness of P_4 means that $(\varphi, P_4\psi) = (P_4\varphi, \psi)$ for any $\varphi, \psi \in \mathcal{E}^*$. Of course the only real content of this is just that $(f, P_4h) = (P_4f, h)$ for any $f, h \in \mathcal{E}$.

Proof of the theorem: First note that if, for $\Phi \in \mathcal{E}^1$, $E\Phi = 0$ then clearly $\delta d\Phi = \bar{I}E\Phi = 0$. So $0 = (\Phi, \delta d\Phi) = (d\Phi, d\Phi)$ and hence $d\Phi = 0$, since on 2-forms our inner product agrees with the usual form inner product. So there is a map from $\ker(E)$ to $H^1(M)$ given by mapping the closed form Φ to its class $[\Phi]$ in $H^1(M)$.

On the other hand any closed 1-form Φ satisfies $\bar{I}E\Phi = 0$. (So $\bar{I}E\Phi = 0$ is equivalent to Φ being closed.) To obtain a map from $H^1(M)$ to $\ker(E)$ it remains to verify that there is a unique element Φ' in the class $[\Phi]$ satisfying $\mathsf{S}\Phi' = 0$. Note that $\Phi' = \Phi + df$ for some $f \in \mathcal{E}^0$. So this equation is

$$\mathsf{S}\Phi + \mathsf{S}df = 0. \tag{13}$$

That is $2\mathsf{S}\Phi = P_4f$ since, by (ii) of theorem 4.1, $\mathsf{S}df = -\frac{1}{2}P_4f$ for any function f.

Now since P_4 is elliptic, formally self-adjoint and has $\ker(P_4) \subseteq \ker(d)$ (i.e. in the terminology of section 2, P_4 has harmless kernel) it follows that $\mathrm{image}(P_4)$ is precisely the subspace of $\mathcal{E}[-4]$ orthogonal to the space of locally constant functions (recall $\mathcal{E}[-4]$ pairs with $\mathcal{E}$). Recall that, in any choice of conformal scale, S is of the form δT for a second order operator T on forms. Thus $\mathsf{S}\Phi$ lies in the subspace of $\mathcal{E}[-4]$ orthogonal to locally constant functions. That is in image(P_4). (Note that for $h \in \mathcal{E}$ and $\varphi \in \mathcal{E}^1$ we have, in any choice of conformal scale, $(h, \mathsf{S}\varphi) = (h, \delta T\varphi) = (dh, T\varphi)$. Although $T\varphi$ is not conformally invariant $(dh, T\varphi)$ is conformally invariant.) Thus there is a unique df solving (13). So for any closed 1-form Φ there is a unique element Φ' in the class $[\Phi]$ satisfying $E\Phi' = 0$ and this gives a well defined map $H^1(M) \to \ker(E)$. This clearly inverts the map from $\ker(E)$ to $H^1(M)$ described above. $\square$

5 The deformation complex

There is an important analogue of the above construction which we believe will have a significant role in the deformation theory of conformal structures. On any manifold let us write $\mathcal{R}_{ab}{}^c{}_d$ or simply $\mathcal{R}$ to denote the bundle of tensors with the same algebraic symmetries as the Riemann tensor of an affine connection. Now consider the second order universal operator mapping metrics to their Weyl curvature tensors, $g \mapsto C(g)$. Linearising about a given metric g leads to an operator on $\mathcal{E}_{(ab)}$ taking values in the bundle $\mathcal{R}$. Notice that the conformal invariance of the Weyl tensor formula means that this operator annihilates the trace part of perturbations, and so the restriction to $\mathcal{E}_{(ab)_0}$ fully captures the operator and also means that the operator yields a well defined operator linearising perturbations of conformal structures. Viewing this as an operator on perturbations of a given conformal metric $\boldsymbol{g}$ gives the operator we will denote $\mathcal{D}_1 : \mathcal{E}_{(ab)_0}[2] \to \mathcal{R}$. Of course the image lies in the totally trace-free (with respect to $\boldsymbol{g}$) part of $\mathcal{R}$, which we shall call $\mathcal{W}$ or $\mathcal{W}_{ab}{}^c{}_d$.

It is well known that, on conformally flat structures, the local kernel of $\mathcal{D}_1$ is the image of the (conformally invariant) conformal Killing operator $\mathcal{D}_0 : \mathcal{E}^a \to \mathcal{E}_{(ab)_0}[2]$ which is given by $t^a \mapsto \nabla_{(a}t_{b)_0}$. We may regard h in $\mathcal{E}_{(ab)_0}[2]$ as a potential for the linearised curvature $\mathcal{D}_1 h$ and the transformations $h \mapsto h + \mathcal{D}_0 t$ as gauge freedom.

Returning to the general setting note that the bundle $\mathcal{W}$ splits into self-dual and anti-self-dual components that we denote $\mathcal{W}^+$ and $\mathcal{W}^-$ respectively. Composing these projections with $\mathcal{D}_1$ yields operators $\mathcal{D}_1^+$ and $\mathcal{D}_1^-$. We will construct further operators as formal adjoints of these. Let $\mathcal{W}^*$ be the direct sum space,

$$\mathcal{W}^* := \mathcal{E}^a \oplus \mathcal{E}_{(ab)_0}[2] \oplus \mathcal{W} \oplus \mathcal{E}_{(ab)_0}[-2] \oplus \mathcal{E}_a[-4].$$

Note that a section $t^a \in \mathcal{E}^a$ can be paired in a conformally invariant way with $w_a \in \mathcal{E}_a[-4]$, $(t, w) := \int_M t^a w_a$. Similarly we have $(h, B) = \int_M h_{ac} B^{ac}$ for $h_{ac} \in \mathcal{E}_{(ac)_0}[2]$, $B_{ac} \in \mathcal{E}_{(ac)_0}[-2]$ and for $U, V \in \mathcal{W}$ there is $\int_M U_{abcd} V^{abcd}$. Setting all other pairings between direct sum components of $\mathcal{W}^*$ to be zero and requiring bilinearity determines a conformally invariant indefinite (but non-degenerate) inner product on $\mathcal{W}^*$ similar to the one on $\mathcal{E}^*$. Also similar to that case, we identify operators on components or subspaces of $\mathcal{W}^*$ with their trivial extension to operators on $\mathcal{W}^*$. It follows immediately that any conformally invariant operator between components of $\mathcal{W}^*$ has a formal adjoint and this is another conformally invariant operator. In particular we have the formal adjoints: $\bar{\mathcal{D}}_1$ of $\mathcal{D}_1$, $\bar{\mathcal{D}}_1{}^{\pm}$ of $\mathcal{D}_1^{\pm}$ and $\bar{\mathcal{D}}_0$ of $\mathcal{D}_0$. Here by $\mathcal{D}_0$ we mean the conformal Killing operator in the general (conformally curved) setting; this is given by the same formula as above.

Summarising the situation we have the sequence and operators indicated by the solid arrows in the following diagram.

$$\begin{tikzcd}
 & & \mathcal{W}^+ \arrow[dr] & & \\
\mathcal{E}^a \arrow[r] \arrow[rrrr, dashed, bend right=60] & \mathcal{E}_{(ab)_0} \arrow[ur] \arrow[dr] \arrow[rr, bend left=80] & & \mathcal{E}_{(ab)_0}[-2] \arrow[r] & \mathcal{E}_a[-4] \\
 & & \mathcal{W}^- \arrow[ur] & &
\end{tikzcd} \tag{14}$$

The operator $\mathcal{E}_{(ab)_0}[2] \to \mathcal{E}_{(ab)_0}[-2]$ is defined here to be $\bar{\mathcal{D}}_1 \mathcal{D}_1$. Now on the conformal sphere case some now well known representation theory can produce a similar sequence of operators – see for example [14]. From that theory we also know that in that setting of the flat model several things are true: There is also a conformally invariant operator $L : \mathcal{E}^a \to \mathcal{E}_a[-4]$ as indicated by the long arrow in the diagram. All the operators in the diagram are unique (up to scale) and the diagram (including L) gives a complete set of the conformally invariant differential operators with the bundles concerned as domain or range bundles. In fact the existence of the differential operators is purely a local issue and these results all carry over to the general conformally flat setting. Let us write B for the operator $\bar{\mathcal{D}}_1\mathcal{D}_1$ on a conformally flat compact manifold. Note that the formal adjoint of L is also a non-trivial conformally invariant operator $\mathcal{E}^a \to \mathcal{E}_a[-4]$. Thus by uniqueness L is formally self-adjoint.

We are now in a similar setting to the Maxwell problem considered above. In fact the situation here is still somewhat more complicated and this affects the overall progress we will make below. What we will show is how to construct a formally-self-adjoint conformally invariant curved analogue of B and a conformally invariant elliptically coercive extension of this. Then in the conformally flat case we will use this to isolate the subspace of $\mathcal{E}_{(ab)_0}[2]$ corresponding to the first cohomology of the complex $\mathcal{E}^a \to \mathcal{E}_{(ab)_0} \to \mathcal{W}$.

A well known conformal invariant is the Bach tensor. In terms of the Weyl tensor this is the trace-free symmetric tensor of weight -2 given by $B_{ab} := \nabla^c\nabla^d C_{acbd} + \mathsf{P}^{cd}C_{acbd}$. Arguing as above it is straightforward to conclude that the linearisation of this is a conformally invariant operator on $\mathcal{E}_{(ab)_0}[2]$. Since B_{ab} is trace-free it follows that perturbations on a conformal manifold with vanishing Bach tensor yield an operator $\mathcal{E}_{(ab)_0}[2] \to \mathcal{E}_{(ab)_0}[-2]$. Clearly then in the conformally flat case B agrees with this linearisation of the Bach tensor and so we will refer to B as the Bach operator.

5.1 The extended Bach operator

Let us once again return to arbitrary dimension $n \geq 3$. Recall that $\mathcal{E}_{\bar{A}}[-1]$ has the composition series $\mathcal{E}_{\bar{A}}[-1] = \mathcal{E} \oplus\!\!\!\!+ \mathcal{E}_a$. So tensoring with $\mathcal{E}_b$ we have $\mathcal{E}_{\bar{A}b}[-1] = \mathcal{E}_b \oplus\!\!\!\!+ \mathcal{E}_{ab}$. Thus there is a canonical bundle injection $\mathcal{E}_{[ab]} \oplus \mathcal{E}[2] \to \mathcal{E}_{\bar{A}b}[-1]$ and we define $\mathcal{F}_{\bar{A}b}[-1]$ to be the quotient. Tensoring now with $\mathcal{E}[w+2]$ gives $\mathcal{F}_{\bar{A}b}[w+1]$ which has the composition series

$$\mathcal{F}_{\bar{A}b}[w+1] = \mathcal{E}_b[w+2] \oplus\!\!\!\!+ \mathcal{E}_{(ab)_0}[w+2].$$

We define $\mathbf{I}_{\bar{A}}{}^a : \mathcal{E}_{(ab)_0}[w+2] \to \mathcal{F}_{\bar{A}b}[w+1]$ to be the obvious inclusion. Thus, for example, for $h_{ab} \in \mathcal{E}_{(ab)_0}[w+2]$, we have $[\mathbf{I}_{\bar{A}}{}^a h_{ab}]_g = (0, h_{ab})$. In the alternative notation,

$$\mathbf{I}_{\bar{A}}{}^a h_{ab} = Z_{\bar{A}}{}^a h_{ab}\,.$$

Clearly there is a bundle surjection $\mathcal{E}_{Ab}[w+1] \to \mathcal{F}_{\bar{A}b}[w+1]$. For $w \neq -2, -n, -n/2$ there differential splitting operator (cf. (10)) $\mathbf{D}_B{}^{\bar{B}} : \mathcal{F}_{\bar{B}a}[w+1] \to \mathcal{E}_{Ba}[w+1]$ given by $(Y_{\bar{B}}\sigma_a + Z_{\bar{B}}{}^b s_{ba}) \mapsto (Y_B\sigma_a + Z_B{}^b S_{ba} + X_B\rho_a)$ where

$$S_{ba} = s_{ba} + \frac{\nabla_{[b}\sigma_{a]}}{w+2} + \frac{\boldsymbol{g}_{ba}\nabla^c\sigma_c}{n(n+w)}$$

and

$$\rho_a = -\frac{1}{n+w}\left[\nabla^b s_{ba} + \tfrac{1}{2}\mathsf{J}\sigma_a + \frac{n-2}{2n}\mathsf{P}_a{}^b\sigma_b\right] + \frac{n-2}{2(n+2w)(n+w)(w+2)}\left[\frac{(n+2w+4)}{n}(\nabla_a\nabla^c\sigma_c + (n+w)\mathsf{P}_a{}^b\sigma_b) - (\nabla^c\nabla_c\sigma_a + (w+1)\mathsf{J}\sigma_a)\right].$$

Both of these operators exist and the corresponding formulae are valid if the fields concerned take values in other tractor bundles. This is trivial for $\mathbf{I}_{\bar{A}}{}^a$ and straightforward to verify for $\mathbf{D}_B{}^{\bar{B}}$. We do not need this here for these operators, but we shall for their formal adjoints, for which it follows automatically. The formal adjoint of $\mathbf{D}_B{}^{\bar{B}}$, for example, yields a conformally invariant operator $\bar{\mathbf{D}}^{\bar{C}B} : \mathcal{E}^{\Psi}_{Ba}[1+\underline{w}] \to \bar{\mathcal{F}}^{\bar{C}}_{\Psi a}[1+\underline{w}]$ where $\underline{w} = -n-w$ and $\mathcal{E}^{\Psi} = \mathcal{E}_{\Psi}$ indicates any tractor bundle (tensored here into $\mathcal{E}_{Ba}[1+\underline{w}]$ and $\bar{\mathcal{F}}^{\bar{C}}{}_a[1+\underline{w}]$). It is straightforward to calculate explicit formulae for the formal adjoints.

Finally, we need the operator

$$\mathbb{D}^{\beta l} : \mathcal{E}^{\Psi}_l[w+1] \to \mathcal{E}^{\beta}_{\Psi}[w-1]$$

and its formal adjoint, where again $\mathcal{E}^{\Psi} = \mathcal{E}_{\Psi}$ indicates any tractor bundle. Omitting the Ψ, the operator sends h_l to $\mathbb{D}^{\beta l}h_l$, where

$$[\mathbb{D}^{\beta l}h_l]_g = \mathbb{Y}^{\beta}{}_l h^l + \mathbb{Z}^{\beta}{}_k{}^l \mu_l{}^k + \mathbb{W}^{\beta}\varphi + \mathbb{X}^{\beta k}\rho_k\,.$$

Here

$$\begin{aligned}
\varphi &= \frac{1}{n+w-1}\nabla_k h^k\,,\\
\mu_{kj} &= \frac{1}{w+1}(\nabla_k h_j - \nabla_j h_k),\\
\rho_j &= \frac{1}{(w+1)(n+2w-2)}\Big[\nabla^l\nabla_l h_j + (w)\mathsf{J}h_j\\
&\qquad -\frac{n+2w}{n+w-1}(\nabla_j\nabla^q h_q + (n+w-1)\mathsf{P}_j{}^q h_q)\Big].
\end{aligned}$$

The formal adjoint of this, expressed in terms of $\underline{w} = -n-w$, is an operator

$$\bar{\mathbb{D}}_{\beta a} : \mathcal{E}^{\beta}[\underline{w}+1] \to \mathcal{E}_a[\underline{w}+1].$$

Setting n=4 and $w=0$ we obtain the following:

Theorem 5.1 *The operator*

$$(F^{\bar{B}}{}_d{}^{ab} := \bar{\mathbf{D}}^{\bar{B}A}\bar{\mathbb{D}}_{d\beta}\Box\mathbb{D}^{\beta b}\mathbf{D}_A{}^{\bar{A}}\mathbf{I}_{\bar{A}}{}^a) : \mathcal{E}_{(ab)_0}[2] \to \mathcal{F}^{\bar{B}}{}_d[-3]$$

is a conformally invariant elliptically coercive operator such that

$$(\mathsf{B}_{cd}{}^{ab} := \bar{\mathbf{I}}_{c\bar{B}}\bar{\mathbf{D}}^{\bar{B}A}\bar{\mathbb{D}}_{d\beta}\Box\mathbb{D}^{\beta b}\mathbf{D}_A{}^{\bar{A}}\mathbf{I}_{\bar{A}}{}^a) : \mathcal{E}_{(ab)_0}[2] \to \mathcal{E}_{(cd)_0}[-2]$$

is a formally self-adjoint curved analogue of the operator B.

Proof: Conformal invariance is clear and the last displayed operator is formally self-adjoint by construction. The final claim follows from the uniqueness of B and an argument which completely parallels the corresponding point in the Maxwell case. It remains to establish elliptic coercivity.

We will show the operator F, when evaluated in any scale, is a factor of that scale's $(\nabla^a\nabla_a)^4$, modulo lower order terms; this will establish elliptic coercivity. Suppose a scale is chosen and we decompose the bundles in the usual way; then direct computation with the above formulas yields the principal parts

$$\begin{aligned}(\mathbf{I}F = \mathsf{B}) : h_{ab} \mapsto H_{ab} := &-\tfrac{1}{2}\nabla^j\nabla_j\nabla^i\nabla_i h_{ab} + \nabla^j\nabla_j\nabla^i\nabla_{(a}h_{b)i} \\ &- \tfrac{1}{3}\nabla_a\nabla_b\nabla^j\nabla^i h_{ij} - \tfrac{1}{6}\nabla^k\nabla_k\nabla^j\nabla^i h_{ij}g_{ab}\end{aligned}$$

and

$$\mathsf{G} : h_{ab} \mapsto \eta_a := -\tfrac{1}{6}\nabla_a\nabla^k\nabla_k\nabla^j\nabla^i h_{ij} + \tfrac{1}{8}\nabla^k\nabla_k\nabla^j\nabla_j\nabla^i h_{ia}$$

for the operator and gauge-part respectively. A useful way to express these principal parts is in terms of the conformal Killing operator $\mathcal{D}_0$ given above (using the scale to identify vector fields and one-forms):

$$\begin{aligned}\mathsf{B} &= -\tfrac{1}{3}(\mathcal{D}_0\mathcal{D}_0^* + \Delta)(\mathcal{D}_0\mathcal{D}_0^* + \tfrac{3}{2}\Delta) + (\text{lower order}), \\ \mathsf{G} &= \mathcal{D}_0^*\left(\tfrac{7}{18}(\mathcal{D}_0\mathcal{D}_0^*)^2 + \tfrac{47}{36}\mathcal{D}_0\mathcal{D}_0^*\Delta + \tfrac{25}{24}\Delta^2 + (\text{lower order})\right).\end{aligned}$$

(Here we are using $\mathcal{D}_0^*$ for the formal adjoint of $\mathcal{D}_0$. Elsewhere we have used overbars to indicate formal adjoints. The point is here we mean this in the usual Riemannian sense and we want to distinguish this from the conformally invariant $\bar{\mathcal{D}}_0$.) The elliptic deficiency in B is clear from the fact that the sixth-order symbol of the operator

$$(\mathcal{D}_0\mathcal{D}_0^* + \Delta)(\mathcal{D}_0\mathcal{D}_0^* + \tfrac{3}{2}\Delta)\mathcal{D}_0\mathcal{D}_0^*$$

vanishes. This shows that the leading symbol of $(\mathcal{D}_0\mathcal{D}_0^* + \Delta)(\mathcal{D}_0\mathcal{D}_0^* + \frac{3}{2}\Delta)$ annihilates the (non-trivial) range of the leading symbol of $\mathcal{D}_0\mathcal{D}_0^*$, and so cannot be invertible. On the other hand,

$$(a_1(\mathcal{D}_0\mathcal{D}_0^*)^2 + a_2\mathcal{D}_0\mathcal{D}_0^*\Delta + a_3\Delta^2, \mathcal{D}_0(a_4\mathcal{D}_0\mathcal{D}_0^* + a_5\Delta)) \begin{pmatrix} \mathsf{B} \\ \mathsf{G} \end{pmatrix} = \Delta^4 + (\text{lower order})$$

for $(a_1, a_2, a_3, a_4, a_5) = -(56, \frac{340}{3}, 2, 48, 56)$. $\square$

Remark 5.2 When image$(\mathcal{D}_0) \subset \ker(\mathsf{B})$ (as, for example, in the next section), the operator $\mathsf{G}\mathcal{D}_0$ will be conformally invariant, so it is of interest to examine it more closely. In general it has the form

$$\begin{aligned} \mathsf{G}\,\mathcal{D}_0 &= \mathcal{D}_0^* \left(\tfrac{7}{18}(\mathcal{D}_0\mathcal{D}_0^*)^2 + \tfrac{47}{36}\mathcal{D}_0\mathcal{D}_0^*\Delta + \tfrac{25}{24}\Delta^2 + (\text{lower order})\right)\mathcal{D}_0 \\ &= \tfrac{7}{18}(\mathcal{D}_0^*\mathcal{D}_0)^3 + \tfrac{47}{36}(\mathcal{D}_0^*\mathcal{D}_0)^2\Delta + \tfrac{25}{24}\mathcal{D}_0^*\mathcal{D}_0\Delta^2 + (\text{lower order}). \end{aligned}$$

Since

$$\mathcal{D}_0^*\mathcal{D}_0 = \delta d + 2\tfrac{n-1}{n}d\delta + (\text{lower order})$$

in dimension n, we have that

$$\begin{aligned} \mathsf{G}\,\mathcal{D}_0 &= \tfrac{1}{8}(\delta d)^3 - \tfrac{1}{16}(d\delta)^3 + (\text{lower order}) \\ &= (\tfrac{1}{8}\delta d - \tfrac{1}{16}d\delta)\Delta^2 + (\text{lower order}) \qquad (n = 4). \end{aligned} \tag{15}$$

These coefficients check with [3], Remark 3.30, which shows that an order $2p$ invariant operator $\mathcal{E}_{[a_1\cdots a_k]}[w] \to \mathcal{E}_{[a_1\cdots a_k]}[w']$ in the conformally flat case must have $w = -(n-2k-2p)/2$ and $w' = -(n-2k+2p)/2$, and must take the form $w'(\delta d)^p + w(d\delta)^p + (\text{lower order})$ up to a constant factor. The operator in (15) is elliptic, though not positively so. That is, its leading symbol is invertible but not positive definite. To check invertibility, just note that if $a \neq 0 \neq b$, then $(a^{-1}\delta d + b^{-1}d\delta)(a\delta d + bd\delta) = \Delta^2 + (\text{lower order})$. In particular,

$$(8\delta d - 16 d\delta)\mathsf{G}\,\mathcal{D}_0 = \Delta^4 + (\text{lower order}).$$

5.2 Application: The moduli space of conformally flat deformations

Recall that on a conformally flat manifold B is the operator $\bar{\mathcal{D}}_1\mathcal{D}_1$. It follows from the uniqueness of the operators in the pattern (14) that B is twice the composition around either edge of the diamond. Thus $(\bar{\mathcal{D}_1}^+, -\bar{\mathcal{D}_1}^-)$ annihilates the image of $(\mathcal{D}_1^+, \mathcal{D}_1^-)$ and so there is a conformally invariant resolution

$$0 \to \mathbb{W} \to \mathcal{E}^a \to \mathcal{E}_{(ab)_0}[2] \to \mathcal{W} \to \mathcal{E}_{(ab)_0}[-2] \to \mathcal{E}_a[-4] \to 0$$

where $\mathbb{W}$ is the space of conformal Killing vectors. We will write $\mathcal{D}_2$ for the operator $\mathcal{W} \to \mathcal{E}_{(ab)_0}[-2]$ and the last operator is just $\bar{\mathcal{D}}_0$, the formal adjoint of $\mathcal{D}_0$. (In fact on conformally flat structures conformal Killing vectors correspond to parallel adjoint tractors and for the conformal sphere case $\mathbb{W}$ is isomorphic to $\mathfrak{so}(n+1,1)$. This will be discussed elsewhere [9].)

Since by definition $\bar{\mathcal{D}}_1$ is the formal adjoint of $\mathcal{D}_1$ the second cohomology of the resolution is, according to standard Hodge theory, isomorphic to $\ker(\mathcal{D}_2) \cap \ker(\bar{\mathcal{D}}_1)$, that is the space of harmonics in $\mathcal{W}$. As the operators $\mathcal{D}_i$ are conformally invariant this subspace is conformally invariant. This is analogous to the de Rham setting above. Also in a parallel to that case we will see that the gauge extension of the Bach operator is related to the first cohomology. First we note that the latter has nice interpretation.

We observed already that $\mathcal{E}_{(ab)_0}[2]$ is the space of infinitesimal conformal metric deformations. Thus the kernel of the map $\mathcal{D}_1 : \mathcal{E}_{(ab)_0}[2] \to \mathcal{W}$ consists of deformations preserving conformal flatness. On the other hand the image of the conformal Killing operator $\mathcal{D}_0 : \mathcal{E}^a \to \mathcal{E}_{(ab)_0}[2]$ is the subspace of deformations coming from infinitesimal diffeomorphisms. Thus the first cohomology of the complex is the formal tangent space to the moduli space of conformally flat structures. Toward the question of integrability of deformations, Calderbank and Diemer [6] have shown that if the second cohomology vanishes then all deformations can be formally integrated (to a power series).

Before we prove the main result let us observe that we are once again fully in the setting of figure 2. By the uniqueness of B it is clear that it is recovered, in the conformally flat setting, by the operator $\mathsf{B} = \bar{\mathbf{I}}F$ from theorem 5.1. Now since $\mathcal{D}_1\mathcal{D}_0 = 0$ and $B = \bar{\mathcal{D}}_1\mathcal{D}_1$ it is immediate that $B\mathcal{D}_0 = 0$. Thus the theorem gives F as an elliptically coercive gauge extension of B. Then G is a corresponding gauge operator in the sense of figure 2. Since the image of $\mathcal{D}_0$ is in the null space of B it follows that $\mathsf{G}\mathcal{D}_0 : \mathcal{E}^a \to \mathcal{E}_a[-4]$ is conformally invariant. Using the ellipticity of $\Box$ and arguments similar to those used in the proof of theorem 4.1 it is clear that this is non-trivial. In fact we have already verified explicitly in remark 5.2 that this is elliptic. So up to scale $\mathsf{G}\mathcal{D}_0$ must agree with L. Let us henceforth take L to be this elliptic operator. By construction here we have that L factors through $\mathcal{D}_0$. On the other hand we have already observed that it is formally-self-adjoint. Thus $L = \bar{\mathcal{D}}_0 U \mathcal{D}_0$ for some operator U. It follows easily that on $\ker(B)$, G has the form $\bar{\mathcal{D}}_0 N$ for some operator N.

Recall that we say the long operator $L : \mathcal{E}^a \to \mathcal{E}_a[-4]$ has harmless kernel if $\ker(L) \subset \ker(\mathcal{D}_0)$.

Theorem 5.3 *Suppose M is a conformally flat compact oriented manifold and that L has harmless kernel. Then* $\ker(F : \mathcal{E}_{(ab)_0}[2] \to \mathcal{F}^{\bar{A}}{}_b[-3])$ *is a conformally invariant subspace of* $\mathcal{E}_{(ab)_0}[2]$ *isomorphic to the first cohomology of the deformation complex.*

Proof: The argument is formally almost identical to the proof of theorem 4.2. Suppose h is in the null space of F. Then clearly $\mathbf{I}F = B$ annihilates h. But since this has the form $B = \bar{\mathcal{D}}_1 \mathcal{D}_1$ it follows that $(\mathcal{D}_1 h, \mathcal{D}_1 h) = 0$. The inner product is definite on $\mathcal{W}$ so we have $h \in \ker(\mathcal{D}_1)$. So there is a map from $\ker(F)$ to the first cohomology given simply by $h \mapsto [h]$.

In a parallel to the de Rham case, to invert this map we establish that there is a unique element $h' \in \ker(F)$ in the class $[h]$. This time we have $h' = h + \mathcal{D}_0 t$ for some tangent vector field t. The class is a subspace of $\ker(B)$ so this boils down to solving $\mathsf{G}h + \mathsf{G}\mathcal{D}_0 t = 0$ or in other words

$$\mathsf{G}h = -Lt. \tag{16}$$

Now since L is formally-self-adjoint and elliptic with harmless kernel it follows that image(L) is just the subspace in $\mathcal{E}_a[-4]$ orthogonal to the space of conformal Killing vectors in $\mathcal{E}^a$. Since $\mathsf{G}h$ has the form $\bar{\mathcal{D}}_0 N h$ it is immediate that this lies in this image and so (16) is solvable and determines $\mathcal{D}_0 t$ uniquely as required. $\square$

In a choice of scale, $[Fh]_g$ has the form (Bh, G). Thus the conformally invariant space $\ker(F)$ is recovered in any choice of scale by $\ker(B) \cap \ker(\mathsf{G})$.

6 Duality and final remarks

Note that the H^3 of the deformation complex is naturally the vector space dual of H^1. We have the conformally invariant map $H^3 \to (H^1)^*$ given by $U \mapsto (U, \cdot)$, for U any representative of H^3. That this is bijective follows easily by choosing a metric from conformal class and using standard Hodge theory arguments. An analogous argument shows $H^4 = (H^0)^*$. H^0 is the cohomology at $\mathcal{E}^a$ of the complex $0 \to \mathcal{E}^a \to \mathcal{E}_{(ab)_0}[2] \to \cdots$. That is it is the vector space $\mathbb{W}$ from above. Then H^4 is invariantly realised as $\mathbb{W}^*$ by using the conformally invariant inner product to pair sections of $\mathcal{E}_a[-4]$ against conformal Killing vectors. From this final point we note that the deformation resolution from the previous section could be adjusted in a natural and conformally invariant way to the "duality-symmetric" sequence

$$0 \to \mathbb{W} \to \mathcal{E}^a \to \mathcal{E}_{(ab)_0}[2] \to \mathcal{W} \to \mathcal{E}_{(ab)_0}[-2] \to \mathcal{E}_a[-4] \to \mathbb{W}^* \to 0.$$

Of course similar remarks apply to the de Rham resolution. By the invariant inner product of that case there is a conformally invariant interpretation of $H^{4-i}(M)$ as the vector space dual of $H^i(M)$.

Some further remarks: Firstly the operator F in theorem 5.1 is not unique. There are many ways to modify the formula. (For example, in an appropriate sense one can swap the order of $\mathbb{D}$ and $\mathbf{D}$.) Part of the motivation for the approach taken was to make the operator B formally-self-adjoint by construction. We have computed a "conventional" tensorial expression for the operator of the theorem which differs from a constant multiple of the linearised Bach operator by a (nontrivial) conformally invariant lower order operator. When one leaves the friendly confines of the de Rham complex, there is much more "room" for different tractor constructions of invariant operators to differ below the leading order. At the date of writing, the precise meaning of the difference operator described above is not clear.

On structures with vanishing Bach tensor the linearised Bach operator annihilates the image of the Killing operator. It may be that this is a good setting to generalise the ideas of Section 5.

We have already mentioned that much of the story told here extends to other dimensions and [4, 5] describe this. In fact there is a large class of so called Bernstein-Gelfand-Gelfand sequences (as in e.g. [6] and references therein) for which many of these ideas extend. We mean here in the conformal setting but also to some extent for other similar (i.e. parabolic) geometries.

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