A conformally invariant differential operator on Weyl tensor densities

Thomas Branson and A. Rod Gover

Abstract

We derive a tensorial formula for a fourth-order conformally invariant differential operator on conformal 4-manifolds. This operator is applied to algebraic Weyl tensor densities of a certain conformal weight, and takes its values in algebraic Weyl tensor densities of another weight. For oriented manifolds, this operator reverses duality: For example in the Riemannian case, it takes self-dual to anti-self-dual tensors and vice versa. We also examine the place that this operator occupies in known results on the classification of conformally invariant operators, and we examine some related operators.

1 Introduction

Recent work on anomalies in conformal field theory [6] has revealed a potentially important role for a certain conformally invariant linear differential operator D in dimension 4. This operator has order 4, and acts on tensor-densities of the symmetry and trace type of the Weyl conformal curvature tensor. The output of this operator is a tensor-density of a different conformal weight, but also of the symmetry and trace type of the Weyl tensor. Under this operator, self-dual and anti-self-dual Weyl tensor densities are interchanged, in a way reminiscent of the chirality switch effected by the Dirac operator, and the duality switch effected by the the middle-form-density operator $\delta d - d\delta +$(Ricci correction) of [1].

The existence of this operator is probably first due to Eastwood and Rice [9]. Their work constructed a very large class of invariant differential operators on conformal 4-manifolds, and in the process, pioneered an approach now known as the *curved translation principle.* This technique has since been developed significantly, and for conformal manifolds of any dimension $n \geq 3$, many differential operator existence questions can be settled by consulting [11]. (For a recent complete treatment of large classes of invariant operators in the setting of general parabolic geometries, see [5].) However, even given the existence of a particular operator, producing a useful and explicit formula is sometimes a non-trivial matter. In [14, 15] formulas for the operator D, as well as many of the related operators discussed below, are obtained by a rather different construction which uses ideas from a twistor theory. In fact, there are universal formulas which yield D and many of its relatives; a principle of this type is formulated in Theorem 1 below, which may be viewed as an elementary exposition of a class of special cases of the general results of [14, 15]. We discuss the universal formulas and general results in Section 5 below. More recently, universal formulas along these lines have been recovered in an even more general setting in [4], this time via a construction which explicitly uses the normal Cartan connection associated to a parabolic geometry.

In Corollary 3 below, we take the Weyl tensor density operator that motivated the present work and make it even more readily usable, by giving a formula for it in standard abstract index notation. Essentially, this explicitly accomplishes the projections involved in formulas like that of our Theorem 1.

In the construction leading up to Theorem 1, we show how formulas for high-order invariant operators can be built using information about first-order invariant operators; in this case the *Stein-Weiss operators* or *generalized gradients* of [20, 13].

The authors gratefully acknowledge support from US NSF grant INT-9724781.

2 Preliminaries

We shall work for now in the setting of Riemannian conformal geometry. Many of our ultimate conclusions about the existence of invariant operators on tensor-densities and their abstract index formulas will, however, be independent of the metric signature. We shall take stock of this in Section 5 below.

Natural irreducible tensor bundles in oriented n-dimensional Riemannian conformal geometry are labelled by a dominant $\mathrm{SO}(n)$-weight λ and a conformal weight w; we shall write such labels in the form $[w|\lambda]$. The parameter

w is a real number, the *density weight*, and λ is an $\ell := [n/2]$-tuple of integers satisfying the *dominance* condition

$$\begin{array}{ll} \lambda_1 \geq \lambda_2 \geq \cdots \geq |\lambda_\ell| & (n \text{ even}), \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0 & (n \text{ odd}). \end{array} \tag{1}$$

Another important label is the *rho-shift* of $[w|\lambda]$:

$$[[\tilde{w}|\tilde{\lambda}]] = [[w + \tfrac{n}{2}|\tilde{\lambda} + \rho_{\mathfrak{so}(n)}]], \tag{2}$$

where

$$\rho_{\mathfrak{so}(n)} = \left(\tfrac{n-2}{2}, \tfrac{n-4}{2}, \ldots, \tfrac{n-2\ell}{2}\right).$$

We use the extra set of brackets in (2) advisedly, as a reminder of whether we have or have not rho-shifted. The string to the right of the bar in a rho-shifted label is *strictly dominant*, that is the $\geq$ signs in (1) are replaced by $>$ signs.

Let $\mathcal{V}[w|\lambda]$ or $\mathcal{V}[[\tilde{w}|\tilde{\lambda}]]$ denote the bundle with the given label. Then, for example, the conformal Laplacian (Yamabe operator) $L = -\nabla^a\nabla_a + (n-2)R/(4(n-1))$ carries $\mathcal{V}[\frac{2-n}{2}|0,\ldots,0]$ to $\mathcal{V}[\frac{-2-n}{2}|0,\ldots,0]$ in a conformally invariant way: changing the metric g to $\hat{g} = \Omega^2 g$, where Ω is a positive smooth function, has no effect on the operator. If we force the operator to act between bundles of the "wrong" density weights, we get an operator which is *conformally covariant* instead of *invariant*. For example, if we view the Yamabe operator as carrying $\mathcal{V}[0|0,\ldots 0]$ to $\mathcal{V}[0|0,\ldots 0]$, then replacement of g by $\hat{g}$ gives an operator

$$\hat{L}f = \Omega^{-(n+2)/2} L(\Omega^{(n-2)/2} f) \tag{3}$$

on smooth functions f. The concept of conformally covariance (as opposed to invariance) is useful, for example, when one wishes to have a spectrum.

If a metric is specified, i.e. if we are in the setting of Riemannian geometry, irreducible tensor bundles are parameterized simply by the λ above. We shall denote by $\mathcal{V}(\lambda)$ the bundle with the given (non-rho-shifted) label.

There is a chance of having a conformally invariant operator $\mathcal{V}[[w|\lambda]] \to \mathcal{V}[[w'|\lambda']]$ only if the length $\ell+1$ strings (w,λ) and (w',λ') are related by

a permutation and an even number of sign changes,	n even,
a permutation and any number of sign changes,	n odd.

That is, the rho-shifted weights $[[w|\lambda]]$ and $[[w'|\lambda]]$ must be similar under the *affine Weyl group*. (Dually this is equivalent to the corresponding generalized Verma modules having the same *central character* for the enveloping algebra of $\mathfrak{so}(n+2,\mathbb{C})$. See [11] for further details on this and related points here.) Even on round S^n, this is a necessary condition for a nontrivial differential operator invariant under the group of conformal diffeomorphisms. An additional necessary condition is that the pair $[[w|\lambda]]$ and $[[w'|\lambda']]$ have one of the correct relative placements in the *Bernstein-Gelfand-Gelfand* diagram made from the affine Weyl orbit of $[[w|\lambda]]$.

Of the differential operators on round S^n that are invariant under the conformal group, all are known to have invariant generalizations to arbitrarily curved manifolds, except the *longest arrows* in even dimensions $n \geq 4$ – operators carrying $[[u|\mu]] \to [[-u|\bar{\mu}]]$, where $u > \mu_1$ and $\bar{\mu} = (\mu_1, \ldots, \mu_{\ell-1}, -\mu_\ell)$. (See [11].) These generalizations need not be unique, but they are differential operator invariants of conformal structure which evaluate to the given (unique up to a constant factor) operator on round S^n. An example of an even-dimensional longest arrow that *does* generalize is the *Paneitz operator* $\mathcal{V}[[2|1,0]] \to \mathcal{V}[[-2|1,0]]$ in dimension 4 (see [18], [19], [10]); or more generally [17], the GJMS operator $P_n : [[\ell|\rho_{\mathfrak{so}(n)}]] \to [[-\ell|\rho_{\mathfrak{so}(n)}]]$ in even dimension n. An example of one which does not generalize [16] is the operator with principal part Δ^3 on scalar densities in S^4; here the labels are $[[3|1,0]] \to [[-3|1,0]]$.

3 A class of fourth-order conformally invariant operators

If λ is an ℓ-tuple, let λ_i be its $i^{\underline{\text{th}}}$ entry. (Recall that ℓ is the integer $[n/2]$.) Let e_i be the ℓ-tuple with 1 in the $i^{\underline{\text{th}}}$ slot and 0 elsewhere. If $\tilde{\lambda} \pm 2e_i$ are strictly dominant $\mathrm{SO}(n)$-weights, then so are $\tilde{\lambda} \pm e_i$ and $\tilde{\lambda}$. Suppose we try to approximate the conformally invariant operator which carries

$$\mathcal{V}[[-(\tilde{\lambda}_i - 2)|\tilde{\lambda} + 2e_i]] \to \mathcal{V}[[-(\tilde{\lambda}_i + 2)|\tilde{\lambda} - 2e_i]] \tag{4}$$

by composing operators

$$\begin{aligned} \mathcal{V}[[-(\tilde{\lambda}_i - 2)|\tilde{\lambda} + 2e_i]] &\to \mathcal{V}[[-(\tilde{\lambda}_i - 1)|\tilde{\lambda} + e_i]] \to \mathcal{V}[[-\tilde{\lambda}_i|\tilde{\lambda}]] \\ &\to \mathcal{V}[[-(\tilde{\lambda}_i + 1)|\tilde{\lambda} - e_i]] \to \mathcal{V}[[-(\tilde{\lambda}_i + 2)|\tilde{\lambda} - 2e_i]]. \end{aligned} \tag{5}$$

This is the unique path composing four first-order Riemannian invariant differential operators. However, it is not a composition of conformally invariant operators, since it is never the case that all five bundles involved are in the same affine Weyl orbit. However, by [13] there are conformally invariant operators

$$\begin{aligned} D_1 : \mathcal{V}[[-(\tilde\lambda_i+1)\,|\,\tilde\lambda+2e_i]] &\to \mathcal{V}[[-(\tilde\lambda_i+2)\,|\,\tilde\lambda+e_i]], \\ D_2 : \mathcal{V}[[-\tilde\lambda_i\,|\,\tilde\lambda+e_i]] &\to \mathcal{V}[[-(\tilde\lambda_i+1)\,|\,\tilde\lambda]], \\ D_3 : \mathcal{V}[[-(\tilde\lambda_i-1)\,|\,\tilde\lambda]] &\to \mathcal{V}[[-\tilde\lambda_i\,|\,\tilde\lambda-e_i]], \\ D_4 : \mathcal{V}[[-(\tilde\lambda_i-2)\,|\,\tilde\lambda-e_i]] &\to \mathcal{V}[[-(\tilde\lambda_i-1)\,|\,\tilde\lambda-2e_i]]. \end{aligned} \tag{6}$$

(See below for tensorial realizations of these operators in the case of the Weyl tensor density problem.) In fact, these are the Stein-Weiss *gradients*, or compressions of the covariant derivative. For example,

$$D_1 = \mathrm{Proj}_{\tilde\lambda+e_i}\nabla|_{\tilde\lambda+2e_i}\,.$$

An invariant operator $D : \mathcal{V}[[w|\tilde\lambda]] \to \mathcal{V}[[w'|\tilde\lambda']]$, when realized as an operator $\mathcal{V}[[a|\tilde\lambda]] \to \mathcal{V}[[b|\tilde\lambda']]$, has conformal variation

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} D_{e^{2\varepsilon\Upsilon}g} =: D'(\Upsilon) = (b-w'-a+w)\Upsilon D - (a-w)[D,\Upsilon].$$

For example, recall (3) above. The Υ in the operator commutator $[D,\Upsilon]$ is an abbreviation for the multiplication operator $\varphi \mapsto \Upsilon\varphi$.

Thus the conformal variation of the composition of the operators in (6) is

$$\begin{aligned} (D_4D_3D_2D_1)'(\Upsilon) = 3D_4D_3D_2[D_1,\Upsilon] + D_4D_3[D_2,\Upsilon]D_1 \\ - D_4[D_3,\Upsilon]D_2D_1 - 3[D_4,\Upsilon]D_3D_2D_1. \end{aligned} \tag{7}$$

One can get a differential operator of homogeneity 4 (i.e. one which is scaled by α^{-4} when the metric is scaled by a constant α^2) from $\mathcal{V}(\lambda+2e_i)$ to $\mathcal{V}(\lambda-2e_i)$ because the cotangent bundle is $\mathrm{SO}(n)$-isomorphic to $\mathcal{V}(e_1)$, and $(\otimes^4\mathcal{V}(e_1))\otimes\mathcal{V}(\lambda+2e_i)$ has a copy of $\mathcal{V}(\lambda-2e_i)$ in its $\mathrm{SO}(n)$ decomposition. In fact, there is just a single copy, and it lives in the subbundle $\mathcal{E}_{(abcd)_0}\otimes\mathcal{V}(\lambda+2e_i)$, where $\mathcal{E}_{(a_1\cdots a_p)_0}$ is the trace-free symmetric part of the fourth tensor power of the cotangent bundle. (To see how this fits into our general notation for tensors and tensor densities, see the beginning of Section 4 below.)

Note that $\mathcal{E}_{(a_1\cdots a_p)_0} \cong_{\mathrm{SO}(n)} \mathcal{V}(pe_1)$, and we need to drop four units in one of the entries to get from $\lambda+2e_i$ to $\lambda-2e_i$. Thus summands of $\otimes^4\mathcal{V}(e_1)$ which are isomorphic to, for example, $\mathcal{V}(3e_1+e_2)$, cannot contribute. Let us say two indexed expressions A and B are *equivalent*, and write $A \sim B$, if they have the same trace-free symmetric part in their free indices. For example, $\nabla_a \nabla_b \nabla_c \sim \nabla_b \nabla_a \nabla_c$ and $g_{ab} \nabla_c \nabla_d \sim 0$. In particular, if we have a 4-index expression A which gives a differential operator from $\mathcal{V}(\lambda + 2e_i)$ to $\mathcal{V}(\lambda - 2e_i)$ via $\mathcal{A} = \mathrm{Proj}_{\mathcal{V}(\lambda-2e_i)} A|_{\mathcal{V}(\lambda+2e_i)}$, then A may be replaced by any equivalent expression without affecting the value of $\mathcal{A}$. Applying this to the problem at hand, we get from (7) that

$$\begin{aligned}
(D_4 D_3 D_2 D_1)_{abcd} &\sim \nabla_a \nabla_b \nabla_c \nabla_d \,, \\
(D_4 D_3 D_2 D_1)'(\Upsilon)_{abcd} &\sim 3\nabla_a \nabla_b \nabla_c \Upsilon_d + \nabla_a \nabla_b \Upsilon_c \nabla_d - \nabla_a \Upsilon_b \nabla_c \nabla_d \\
&\qquad -3\Upsilon_a \nabla_b \nabla_c \nabla_d \\
&\sim 10\Upsilon_{ab} \nabla_c \nabla_d + 10\Upsilon_{abc} \nabla_d + 3\Upsilon_{abcd}.
\end{aligned}$$

Here and below, we abbreviate $\nabla_b \nabla_a \Upsilon$ as Υ_{ab}, and similarly for other strings of derivatives of Υ. This already tells us that the composition $D_4 D_3 D_2 D_1$ is invariant under the conformal transformation group of S^n, since the infinitesimal conformal factors of that group (the homogeneous coordinate functions) have vanishing trace-free symmetrized covariant derivatives of order 2 and higher.

In dealing with conformal variation, it is often convenient to decompose the Riemann curvature tensor into the Weyl tensor $C^a{}_{bcd}$ and a trace renormalization P_{ab} of the Ricci tensor:

$$R_{abcd} = C_{abcd} + 2g_{c[a}\mathsf{P}_{b]d} + 2g_{d[b}\mathsf{P}_{a]c} \,.$$

Part of the convenience of P_{ab} derives from its conformal variational formula $(\mathsf{P}_{ab})'(\Upsilon) = -\Upsilon_{ab}$. Together with the above, this suggests trying to correct by adding (inside the compression $\mathrm{Proj}_{\mathcal{V}(\lambda-2e_i)} \cdot |_{\mathcal{V}(\lambda+2e_i)}$)

$$10\mathsf{P}_{ab} \nabla_c \nabla_d + 10(\nabla_a \mathsf{P}_{bc}) \nabla_d + 3(\nabla_a \nabla_b \mathsf{P}_{cd}). \tag{8}$$

To compute the conformal variation of this, first note that

$$\begin{aligned}
(\nabla_a \mathsf{P}_{bc})'(\Upsilon) &\sim -\Upsilon_{abc} - 4\Upsilon_a \mathsf{P}_{bc}, \\
(\nabla_a \nabla_b \mathsf{P}_{cd})'(\Upsilon) &\sim -\Upsilon_{abcd} - 4\Upsilon_{ab} \mathsf{P}_{cd} - 10\Upsilon_a \nabla_b \mathsf{P}_{cd} \,.
\end{aligned}$$

In addition, the $\nabla\nabla$ and ∇ on the right of compositions in (8) may be replaced by (respectively) $D_2 D_1$ and D_1. The new expressions are not the same, but they are equivalent. The upshot is that the conformal variation of

$$\nabla_a \nabla_b \nabla_c \nabla_d + 10\mathsf{P}_{ab} \nabla_c \nabla_d + 10(\nabla_a \mathsf{P}_{bc}) \nabla_d + 3(\nabla_a \nabla_b \mathsf{P}_{cd})$$

is equivalent to $18\Upsilon_{ab}\mathsf{P}_{cd}$. But this is equivalent to the conformal variation of $-9\mathsf{P}_{ab}\mathsf{P}_{cd}$.

If we wish to move in the other direction, from $\mathcal{V}(\lambda - 2e_i)$ to $\mathcal{V}(\lambda + 2e_i)$, the calculation is similar; we just have to change a few signs.

Theorem 1 *Suppose $\lambda + 2e_i$ and $\lambda - 2e_i$ are dominant SO(n)-weights. Then the differential operators*

$$\begin{aligned}\mathrm{Proj}_{\mathcal{V}(\lambda \mp 2e_i)}(&\nabla_a\nabla_b\nabla_c\nabla_d + 10\mathsf{P}_{ab}\nabla_c\nabla_d + 10(\nabla_a\mathsf{P}_{bc})\nabla_d \\ &+ 3(\nabla_a\nabla_b\mathsf{P}_{cd}) + 9\mathsf{P}_{ab}\mathsf{P}_{cd})|_{\mathcal{V}(\lambda\pm 2e_i)}\end{aligned} \tag{9}$$

are conformally invariant $\mathcal{V}[\mp\tilde{\lambda}_i + 2 - \frac{n}{2}|\lambda \pm 2e_i] \to \mathcal{V}[\mp\tilde{\lambda}_i - 2 - \frac{n}{2}|\lambda \mp 2e_i]$. In particular, in dimension 4, if $\lambda_1 \geq |\lambda_2| + 2$, then there are invariant operators $\mathcal{V}[\mp(\lambda_1 + 1)|\lambda_1 \pm 2, \lambda_2] \to \mathcal{V}[\mp(\lambda_1 + 1) - 4|\lambda_1 \mp 2, \lambda_2]$ and $\mathcal{V}[\mp\lambda_2|\lambda_1, \lambda_2 \pm 2] \to \mathcal{V}[\mp\lambda_2 - 4|\lambda_1, \lambda_2 \mp 2]$. As a special case of this in dimension 4 (with $\lambda = (2, 0)$), there are invariant operators $\mathcal{V}[0|2, \pm 2] \to \mathcal{V}[-4|2, \mp 2]$.

4 Tensorial realizations

Let us now consider tensorial realizations. Let $\mathcal{E}[w]$ be the bundle of w-densities; this is a realization of $\mathcal{V}[w|0, \ldots, 0]$. Tensor bundles will be denoted by adorning the symbol $\mathcal{E}$ with the index configuration of their sections; thus the tangent bundle is $\mathcal{E}^a$ and the cotangent bundle is $\mathcal{E}_a$. Standard symmetry type notation will also be used, as for example when we spoke of $\mathcal{E}_{(abcd)_0}$ above, or as in the example of the exterior 2-form bundle $\mathcal{E}_{[ab]}$. The tensor product with a density bundle will be abbreviated, for example, by $\mathcal{E}_{[ab]}[w] := \mathcal{E}[w] \otimes \mathcal{E}_{[ab]}$. Because the conformal metric is an element of $\mathcal{E}_{(ab)}[2]$, the raising and lowering of indices has an effect on the weight. For example, $\mathcal{E}^a \cong_{\mathrm{CO}(n)} \mathcal{E}_a[2] \cong_{\mathrm{CO}(n)} \mathcal{V}[1|1, 0, \ldots, 0]$, and $\mathcal{E}_a \cong_{\mathrm{CO}(n)} \mathcal{V}[-1|1, 0, \ldots, 0]$, where CO$(n)$ denotes the extension of the structure group SO(n) by pointwise scalings.

The Weyl tensor $C^a{}_{bcd}$ of the metric is conformally invariant, and thus is a section of $\mathcal{W}^a{}_{bcd}$, where we use $\mathcal{W}$ to denote curvature symmetries and the absence of traces. By the above remarks on the tangent and cotangent bundles, $C^a{}_{bcd}$ also lives in a copy of

$$\mathcal{V}[1|1, 0, \ldots, 0] \otimes (\otimes^3 \mathcal{V}[-1|1, 0, \ldots, 0]).$$

Thus the Weyl tensor is a section of a direct sum of irreducibles bundles having the form $\mathcal{V}[-2|\lambda]$. In fact, it is a section of $\mathcal{V}[-2|2, 2, 0, \ldots, 0]$ if

$n \geq 5$, and to $\mathcal{V}[-2|2,2] \oplus \mathcal{V}[-2|2,-2]$ if $n=4$. The two summands in dimension 4 correspond to the two dualities, or eigenvalues of the Hodge $\star$ applied in the cd index pair; these will be denoted $\mathcal{W}_\pm$. Algebraic Weyl tensor-densities are obtained by tensoring with density bundles. Examples that are relevant for what follows are

$$\mathcal{W}^a{}_b{}^c{}_d \cong_{\mathrm{CO}(n)} \mathcal{W}^a{}_{bcd}[2] \quad \text{and} \quad \mathcal{W}_{abcd} \cong_{\mathrm{CO}(n)} \mathcal{W}^a{}_{bcd}[-2].$$

The following is just the last conclusion of Theorem 1 stated in these terms:

Corollary 2 *If $n=4$, formula (9) for $\lambda=(2,0)$ gives conformally invariant operators $D_\pm$ from $(\mathcal{W}_\pm)^a{}_b{}^c{}_d$ to $(\mathcal{W}_\mp)_{abcd}$.*

The following will make it clear that there is a unified tensorial formula for D_+ and D_-, so that one need not actually accomplish the decomposition into self-dual and anti-self-dual parts in order to apply the formula for the operator. That is, the tensorial formula we shall give is really one for the operator which is, in block form,

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}. \tag{10}$$

To get our tensorial realization, choose a metric g. If Q_{abcd} is a differentio-tensorial expression, for example $\nabla_a\nabla_b\nabla_c\nabla_d$ or $\mathsf{P}_{ab}\mathsf{P}_{cd}$, and Y is an algebraic Weyl tensor, we define

$$\begin{aligned}(Q \bullet Y)_{abcd} &= Q_{(acef)_0} Y^e{}_b{}^f{}_d - Q_{(adef)_0} Y^e{}_b{}^f{}_c \\ &\qquad + Q_{(bdef)_0} Y^e{}_a{}^f{}_c - Q_{(bcef)_0} Y^e{}_a{}^f{}_d.\end{aligned}$$

We claim that $Q\bullet$ is a nonzero $\mathrm{SO}(n)$-equivariant action of $\mathcal{E}_{(abcd)_0}$ on algebraic Weyl tensors interchanging the self-dual and anti-self-dual summands.

First note that $Q\bullet$ propagates the curvature symmetries: if Y satisfies

$$Y_{abcd} = Y_{cdab} = Y_{[ab]cd} = -Y_{acdb} - Y_{adbc}\,,$$

then $Q \bullet Y$ behaves similarly. The statements on trace and duality follow from the fact that

$$\begin{aligned}&\mathcal{V}(4,0) \otimes \mathcal{V}(2,\pm 2) \cong_{SO(4)} \\ &\mathcal{V}(2,\mp 2) \oplus \mathcal{V}(3,\mp 1) \oplus \mathcal{V}(4,0) \oplus \mathcal{V}(5,\pm 1) \oplus \mathcal{V}(6,\pm 2).\end{aligned} \tag{11}$$

In particular, the bundles of algebraic Weyl tensors on the right and left sides have opposite duality. Traces of $Q \bullet Y$ would need to land in $\mathcal{V}(2,0) \oplus \mathcal{V}(1,1) \oplus \mathcal{V}(1,-1) \oplus \mathcal{V}(0,0)$, none of whose summands occur on the right in (11). And in fact, it is easily computed that the ac-trace, and thus any trace, of $(Q \bullet Y)_{abcd}$ vanishes.

To show that $Q\bullet$ is nonzero, let ξ be a one-form, and let

$$\begin{aligned}
X_{abcd} &= \xi_{(a}\xi_b\xi_c\xi_{d)_0} \\
&= \xi_a\xi_b\xi_c\xi_d \\
&\qquad -\tfrac{1}{8}(\xi_a\xi_b g_{cd} + \xi_a\xi_c g_{bd} + \xi_a\xi_d g_{bc} + \xi_b\xi_c g_{ad} + \xi_b\xi_d g_{ac} + \xi_c\xi_d g_{ab})|\xi|^2 \\
&\qquad +\tfrac{1}{48}(g_{ab}g_{cd} + g_{ac}g_{bd} + g_{ad}g_{bc})|\xi|^4,
\end{aligned}$$

where $|\xi|^2 = \xi^a\xi_a$. Direct calculation shows that

$$(X \bullet Y)^{abcd}(X \bullet Y)_{abcd} = \tfrac{1}{16}|\xi|^8 Y^{abcd}Y_{abcd}\,, \tag{12}$$

where $|\xi|^2 := \xi^a\xi_a$. This shows that $X \bullet Y$ is nonzero if ξ and Y are. (This calculation is quite special to dimension 4; in higher dimensions, the dependence on ξ is not just through $|\xi|^2$.)

In fact, the computation of $X \bullet Y$ is exactly that of the leading symbol of the operator D of (10), and (12) shows that the leading symbol of D^*D is $|\xi|^8/16$. In other words, D^*D has principal part $\Delta^4/16$, where $\Delta = -\nabla^a\nabla_a$.

A more concrete workout of the duality issue can be obtained by writing $X\bullet$ (for X as just above) in terms of the exterior and interior multiplication $\varepsilon(\xi)$ and $\iota(\xi)$ of differential forms by a one-form ξ. Here we use the fact that an algebraic Weyl tensor density is (among other things) a $(\Lambda^2 \otimes \Lambda^2)$-density, and the fact that for ξ is a one-form, the Hodge $\star$ anticommutes with $\iota(\xi)\varepsilon(\xi) - \varepsilon(\xi)\iota(\xi)$.

We now have a tensorial realization of the compression
$\mathrm{Proj}_{\mathcal{V}(\lambda\mp 2e_i)} \cdot |_{\mathcal{V}(\lambda\pm 2e_i)}$, and may conclude:

Corollary 3 *The operator*

$$
\begin{aligned}
Y^a{}_b{}^c{}_d \mapsto \quad & \{\nabla_{(a}\nabla_c\nabla_e\nabla_{f)_0} + 10\mathsf{P}_{(ac}\nabla_e\nabla_{f)_0} + 10(\nabla_{(a}\mathsf{P}_{ce})\nabla_{f)_0} \\
& +3(\nabla_{(a}\nabla_c\mathsf{P}_{ef)_0}) + 9\mathsf{P}_{(ac}\mathsf{P}_{ef)_0}\}Y^e{}_b{}^f{}_d \\
- \quad & \{\nabla_{(a}\nabla_d\nabla_e\nabla_{f)_0} + 10\mathsf{P}_{(ad}\nabla_e\nabla_{f)_0} + 10(\nabla_{(a}\mathsf{P}_{de})\nabla_{f)_0} \\
& +3(\nabla_{(a}\nabla_d\mathsf{P}_{ef)_0}) + 9\mathsf{P}_{(ad}\mathsf{P}_{ef)_0}\}Y^e{}_b{}^f{}_c \\
+ \quad & \{\nabla_{(b}\nabla_d\nabla_e\nabla_{f)_0} + 10\mathsf{P}_{(bd}\nabla_e\nabla_{f)_0} + 10(\nabla_{(b}\mathsf{P}_{de})\nabla_{f)_0} \\
& +3(\nabla_{(b}\nabla_d\mathsf{P}_{ef)_0}) + 9\mathsf{P}_{(bd}\mathsf{P}_{ef)_0}\}Y^e{}_a{}^f{}_c \\
- \quad & \{\nabla_{(b}\nabla_c\nabla_e\nabla_{f)_0} + 10\mathsf{P}_{(bc}\nabla_e\nabla_{f)_0} + 10(\nabla_{(b}\mathsf{P}_{ce})\nabla_{f)_0} \\
& +3(\nabla_{(b}\nabla_c\mathsf{P}_{ef)_0}) + 9\mathsf{P}_{(bc}\mathsf{P}_{ef)_0}\}Y^e{}_a{}^f{}_d
\end{aligned}
\tag{13}
$$

is conformally invariant $\mathcal{W}^a{}_b{}^c{}_d$ *to* $\mathcal{W}_{abcd}$ *, and carries the subbundle* $(\mathcal{W}_\pm)^a{}_b{}^c{}_d$ *to* $(\mathcal{W}_\mp)_{abcd}$ *.*

There has also been some interest in tensorial realizations of the first-order operators D_i of (6). Note that by the result of Fegan [13], any $\mathrm{SO}(n)$-invariant first-order operator between irreducible $\mathrm{SO}(n)$-bundles is a compression of the covariant derivative (i.e. has the form $\mathrm{Proj}_{\mathcal{V}(\mu)}\nabla|_{\mathcal{V}(\lambda)}$), and "promotes" to a conformally covariant operator $\mathcal{V}[w|\lambda] \to \mathcal{V}[w-1|\mu]$, for a unique w which is computable from λ and μ. With this in mind, our task in writing down the D_i reduces to writing nonzero $\mathrm{SO}(n)$-invariant first-order operators that move between the bundles advertised.

The first may be realized as a divergence:

$$D_1 : Y_{abcd} \mapsto \eta_{bcd} = \nabla^a Y_{abcd}\,.$$

If we start in $\mathcal{V}(2, 2\varepsilon)$, where $\varepsilon = \pm 1$, this lands us in the bundle $\mathcal{V}(2,\varepsilon)$, which has a realization as the totally trace-free tensors $\eta_{bcd} = \eta_{b[cd]}$ which have duality ε in the $[cd]$ indices, and satisfy the Bianchi-like identity $\eta_{bcd} + \eta_{cdb} + \eta_{dbc} = 0$. Let us denote this symmetry type (as a Riemannian bundle) by $\mathcal{A}_{bcd} \cong_{\mathrm{SO}(n)} \mathcal{V}(2,1) \oplus \mathcal{V}(2,-1)$. We then switch to an alternative realization $\mathcal{A}'_{abc}$ of $\mathcal{V}(2,1)\oplus\mathcal{V}(2,-1)$ as the totally trace-free 3-tensors $\eta'_{cab} = \eta'_{c(ab)}$ also satisfying a Bianchi-like identity. The $\mathrm{SO}(n)$-equivariant map between the two realizations is $\eta \mapsto \eta'$, where

$$\eta'_{cab} := \eta_{abc} + \eta_{bac}\,.$$

(We have not bothered to normalize this map to an isometry, as we are just working up to nonzero multiples.) Applying a divergence in the first argument, we have D_2:

$$\eta'_{cab} \mapsto \alpha_{ab} = \nabla^c \eta'_{cab}\,;$$

this lands us in $\mathcal{E}_{(ab)_0}$.

To get (by D_3) to the bundle $\mathcal{V}(2,-\varepsilon)$, say $D_3\alpha = \beta$, we first take

$$\begin{aligned}\beta''_{cab} := \;& \tfrac{2}{3}\nabla_c\alpha_{ab} - \tfrac{1}{3}\nabla_a\alpha_{bc} - \tfrac{1}{3}\nabla_b\alpha_{ca} \\ & -\tfrac{1}{9}(g_{ca}\nabla^e\alpha_{eb} + g_{cb}\nabla^e\alpha_{ae}) + \tfrac{2}{9}g_{ab}\nabla^e\alpha_{ec}\,.\end{aligned} \tag{14}$$

This lands us in $\mathcal{A}'_{cab}$; To pick out the $\mathcal{V}(2,-\varepsilon)$ summand, we go to the alternate realization $\mathcal{A}_{bcd}$:

$$\beta'_{cab} = \beta''_{abc} - \beta''_{bac}\,. \tag{15}$$

Let β_{cab} be the $(-\varepsilon)$-dual part of β'_{cab} in the $[ab]$ indices. This process, $\alpha \mapsto \beta'' \mapsto \beta' \mapsto \beta$, is the operator D_3 .

Finally, we need D_4 to get us to the $(-\varepsilon)$-dual tensors with Weyl symmetry and trace type. To accomplish this, we first take the map

$$\beta_{dab} \mapsto \bar{Z}_{cdab} := \nabla_{[c}\beta_{d]ab} + \nabla_{[a}\beta_{b]cd}. \tag{16}$$

The result of this process clearly satisfies the identities $\bar{Z}_{cdab} = \bar{Z}_{cd[ab]} = \bar{Z}_{abcd}$, and a short computation shows that in addition, $\bar{Z}_{cdab} + \bar{Z}_{cabd} + \bar{Z}_{cbda} = 0$. Thus $\bar{Z}$ has curvature symmetries. It is not, however, totally trace-free, though its double traces $\bar{Z}^{ab}{}_{ab}$ do vanish (using the fact that β is totally trace-free). The tensor

$$Z_{cdab} := \bar{Z}_{cdab} - \frac{1}{2}(\bar{Z}^e{}_{deb}g_{ca} + \bar{Z}_c{}^e{}_{ae}g_{db} + \bar{Z}^e{}_{dae}g_{cb} + \bar{Z}_c{}^e{}_{eb}g_{da}) \tag{17}$$

is totally trace-free and enjoys curvature symmetries; i.e. it has Weyl symmetry and trace type. Since $\mathcal{V}(1,0) \otimes \mathcal{V}(2,-\varepsilon)$ has a $\mathcal{V}(2,-2\varepsilon)$ summand but no $\mathcal{V}(2,2\varepsilon)$ summand, Z has duality $-\varepsilon$.

5 Epilogue: BGG diagrams, other metric signatures, and standard operators

BGG diagrams of tensors in 4-dimensional Riemannian conformal geometry are parameterized by similarity classes of integral rho-shifted weights $[[a|b,c]]$

which are strictly dominant after the bar. (Recall (1) and the remarks immediately following.) Here we are speaking only of *tensorial* BGG diagrams; to include those that depend on spin structure, we just need to admit properly half-integral $[[a|b,c]]$. (See below for a discussion of how the Dirac operator fits into this picture.) *Regular* BGG diagrams correspond to similarity classes of cardinality 6, and are in one-to-one correspondence with triples a, b, c of integers with $a > b > |c|$. These appear as follows:

$$\begin{tikzcd}
 & & \mathcal{V}[[c|a,b]] \arrow[dr] & & \\
\mathcal{V}[[a|b,c]] \arrow[r] \arrow[rrrr, bend right] & \mathcal{V}[[b|a,c] \arrow[ur] \arrow[dr] \arrow[rr, bend left] & & \mathcal{V}[[-b|a,-c]] \arrow[r] & \mathcal{V}[[-a|b,-c]] \\
 & & \mathcal{V}[[-c|a,-b]] \arrow[ur] & &
\end{tikzcd}$$

For example, the de Rham complex extends to a BGG diagram with $a = 2$, $b = 1$, and $c = 0$. All compositions in this diagram vanish on S^4, except for one linear combination of the two compositions around the diamond; we represent this composition by the shorter rectangular arrow. For the de Rham diagram, this surviving composition is the Maxwell operator $d \star d$ on vector potentials, and the longest arrow is the *Paneitz operator* mentioned above in Section 2.

Singular BGG diagrams (for 4-dimensional conformal geometry) correspond to similarity classes of cardinality 2. Each gives rise to a single (nonzero and non-identity) operator. If $a > |c|$, we have an operator $\mathcal{V}[[a|a,c]] \to \mathcal{V}[[-a|a,-c]]$, and if $a > c > 0$, we have the operators $\mathcal{V}[[c|a,\pm c]] \to \mathcal{V}[[-c|a,\mp c]]$. (In the last case the $\pm$ sign parameterizes two similarity classes.) Our operators on Weyl tensor densities, $\mathcal{V}[[2|3,\pm 2]] \to \mathcal{V}[[-2|3,\mp 2]]$, are of this final type. In higher even dimension n, the cardinality of a similarity class of bundles is either $n + 2$ (the regular case) or 2 (the singular case). A regular diagram just extends the 4-dimensional one above in the obvious way, with conformal weights decreasing as one moves to the right.

All operators have arbitrarily curved conformally invariant generalizations, except for some of the longest arrows in regular diagrams. For example, the Paneitz operator is conformally invariant in the arbitrarily curved case, but the operator $\mathcal{V}[[3|1,0]] \to \mathcal{V}[[-3|1,0]]$ is known not to have an arbitrarily curved generalization [16].

All differential operators invariant under the conformal group of round S^n are captured in BGG diagrams (when one includes long arrows). In particular, consider homogeneous combinations D of $\nabla\cdots\nabla$ terms whose index combinatorics are such that $D:\mathcal{V}(\lambda)\to\mathcal{V}(\mu)$ for some λ,μ – i.e. combinations that pass between irreducible Riemannian bundles. One might harbor the naive hope that any such combination could be completed to a conformally invariant differential operator by first assigning appropriate conformal weights, and then adding lower-order terms. This must fail in general, since for a given $\nabla\cdots\nabla$ expression to have any chance, it must be (in the round S^n case) the principal part of an operator in a BGG diagram. If the expression passes this test, it may still fail in the conformally curved case, if its position in the round BGG was that of the longest arrow.

If we wish to speak of tensor-spinor bundles, we just need to add bundles with proper half-integer entries to the above discussion. For example, the Dirac operator carries $\mathcal{V}[[\frac{1}{2}|\frac{3}{2},\pm\frac{1}{2}]]\to\mathcal{V}[[-\frac{1}{2}|\frac{3}{2},\mp\frac{1}{2}]]$, and so is much like our Weyl tensor density operators. The operator $\mathcal{V}[[1|2,\pm1]]\to\mathcal{V}[[-1|2,\mp1]]$ is the form-density operator of [1] in the case of 2-forms in 4 dimensions; this interchanges the two dualities: $(\mathcal{E}_\pm)_{[ab]}[1]\to(\mathcal{E}_\mp)_{[ab]}[-1]$.

The operator of Theorem 1 may occur in regular BGG diagrams. For example, one of the simplest operators we could construct from the theorem carries scalar densities to trace-free symmetric 4-tensor densities, $\mathcal{V}[[5|1,0]]\to\mathcal{V}[[1|5,0]]$; that is, $\mathcal{E}[3]\to\mathcal{E}_{(abcd)_0}[3]$. This is the first arrow in the BGG diagram above with $a=5$, $b=1$, $c=0$. In tensor notation, the operator is

$$\begin{aligned} f\mapsto\ &(\nabla_{(a}\nabla_b\nabla_c\nabla_{d)_0}+10\mathsf{P}_{(ab}\nabla_c\nabla_{d)_0}+10(\nabla_{(a}\mathsf{P}_{bc})\nabla_{d)_0}\\ &+3(\nabla_{(a}\nabla_b\mathsf{P}_{cd)_0})+9\mathsf{P}_{(ab}\mathsf{P}_{cd)_0})f. \end{aligned}$$

We could also take trace-free symmetric 4-tensor densities to scalar densities: $\mathcal{V}[[-1|5,0]]\to\mathcal{V}[[-5|1,0]]$ or $\mathcal{E}_{(abcd)_0}[1]\to\mathcal{E}[-7]$; this is in fact the formal adjoint of the operator just above, and is also the final arrow in the same BGG diagram. A tensorial realization is

$$\begin{aligned} \varphi_{abcd}\mapsto\ &(\nabla^a\nabla^b\nabla^c\nabla^d+10\mathsf{P}^{ab}\nabla^c\nabla^d+10(\nabla^a\mathsf{P}^{bc})\nabla^d\\ &+3(\nabla^a\nabla^b\mathsf{P}^{cd})+9\mathsf{P}^{ab}\mathsf{P}^{cd})\varphi_{abcd}\,. \end{aligned}$$

In fact, this operator is contained in a class of conformally invariant operators, the

$$\mathcal{E}_{(a_1\dots a_k)_0}[k-p-n+1]\to\mathcal{E}_{(b_1\dots b_p)_0}[p-k-n+1]\ \ \text{with } k>p, \tag{18}$$

that plays a featured role in the recent work of Dolan, Nappi, and Witten [7].

For any specified operator order p, [14] provides an analogue of Theorem 1 (where $p = 4$), and an elementary proof along the lines of that of Section 3 above is possible. Among other things, this allows one to write the lower-order terms of the operators (18). The $p = 1$ theorem is the result of Fegan mentioned above. The first of these involve compressing the expressions

$$\begin{array}{cr} \nabla & (p=1) \\ \nabla\nabla + \mathsf{P} & (p=2) \\ \nabla\nabla\nabla + 4\mathsf{P}\nabla + 2(\nabla\mathsf{P}) & (p=3) \\ \nabla\nabla\nabla\nabla + 10\mathsf{P}\nabla\nabla + 10(\nabla\mathsf{P})\nabla + 3(\nabla\nabla\mathsf{P}) + 9\mathsf{PP} & (p=4) \end{array}$$

Section 5 of [14] also gives the analogous expressions for $p = 5, 6, 7$. As in Theorem 1, these same expressions turn up in other dimensions [15, 4]. Things can be made to look more symmetric if we write expressions in which terms act on everything to their right; for example $\nabla^4 + 4\nabla\mathsf{P}\nabla + 3(\nabla\nabla\mathsf{P} + \mathsf{P}\nabla\nabla) + 9\mathsf{PP}$ for the fourth-order operator.

For example, back in dimension 4, we can get a conformally invariant operator $\mathcal{V}[[2|3,0]] \to \mathcal{V}[[0|3,\pm 2]]$; that is, from trace-free symmetric 2-tensors with the index configuration $\alpha^a{}_b$ to $(\mathcal{W}_\pm)^a{}_{bcd}$. These operators appear in the *gravitational diagram*; that is, the regular 4-dimensional BGG diagram with $a = 3$, $b = 2$, and $c = 0$, and may be interpreted as linearized Weyl curvature operators applicable to a trace-free metric perturbation.

In fact, the index combinatorics are given above in (14-17), and the operators are the self-dual and anti-self-dual projections of the expression (setting $\mathsf{J} := \mathsf{P}^a{}_a$):

$$\begin{array}{rcl} \alpha_{ab} \mapsto & & \alpha_{ac|(bd)} - \alpha_{ad|(bc)} - \alpha_{bc|(ad)} + \alpha_{bd|(ac)} \\ & + & \frac{1}{2} g_{ac}\big(-\alpha_{bd|e}{}^e + \alpha_b{}^e{}_{|(de)} + \alpha_d{}^e{}_{|(be)}\big) - \frac{1}{2} g_{ad}\big(-\alpha_{bc|e}{}^e + \alpha_b{}^e{}_{|(ce)} + \alpha_c{}^e{}_{|(be)}\big) \\ & - & \frac{1}{2} g_{bc}\big(-\alpha_{ad|e}{}^e + \alpha_a{}^e{}_{|(de)} + \alpha_d{}^e{}_{|(ae)}\big) + \frac{1}{2} g_{bd}\big(-\alpha_{ac|e}{}^e + \alpha_a{}^e{}_{|(ce)} + \alpha_c{}^e{}_{|(ae)}\big) \\ & + & \frac{1}{3}(g_{ad}g_{bc} - g_{ac}g_{bd})\alpha^{ef}{}_{|ef} + \mathsf{P}_{bd}\alpha_{ac} - \mathsf{P}_{bc}\alpha_{ad} - \mathsf{P}_{ad}\alpha_{bc} + \mathsf{P}_{ac}\alpha_{bd} \\ & + & \frac{1}{2} g_{ac}(-\mathsf{J}\alpha_{bd} + \mathsf{P}_{de}\alpha_b{}^e + \mathsf{P}_{be}\alpha_d{}^e) - \frac{1}{2} g_{ad}(-\mathsf{J}\alpha_{bc} + \mathsf{P}_{ce}\alpha_b{}^e + \mathsf{P}_{be}\alpha_c{}^e) \\ & - & \frac{1}{2} g_{bc}(-\mathsf{J}\alpha_{ad} + \mathsf{P}_{de}\alpha_a{}^e + \mathsf{P}_{ae}\alpha_d{}^e) + \frac{1}{2} g_{bd}(-\mathsf{J}\alpha_{ac} + \mathsf{P}_{ce}\alpha_a{}^e + \mathsf{P}_{ae}\alpha_c{}^e) \\ & + & \frac{1}{3}(g_{ad}g_{bc} - g_{ac}g_{bd})\mathsf{P}_{ef}\alpha^{ef}. \end{array}$$

Going the other way, we can get operators $\mathcal{V}[[0|3,\pm 2]] \to \mathcal{V}[[-2|3,0]]$ by

$$Y_{abcd} \mapsto \nabla^b\nabla^d Y_{abcd} + \mathsf{P}^{bd} Y_{abcd}\,.$$

Recalling the discussion of Section 4, the self-dual and anti-self-dual parts of the Weyl tensor of the conformal structure live in $\mathcal{V}[-2|2,\pm 2] = \mathcal{V}[[0|3,\pm 2]]$, so these operators may be applied to $(C_\pm)^a{}_{bcd}$. The result of applying to the full Weyl tensor is called the *Bach tensor*:

$$\mathcal{B}^a{}_c := \nabla^b \nabla^d C^a{}_{bcd} + \mathsf{P}^{bd} C^a{}_{bcd}\,. \tag{19}$$

In fact, by the uniqueness of the Bach tensor as a natural conformally invariant section of $\mathcal{E}_{(ab)_0}[-2]$ in dimension 4, together with the universality of the calculation and the possibility of orientation reversal, we must recover $\frac{1}{2}\mathcal{B}$ upon application of the above operator to either of $C_\pm$.

There is also an interesting second-order operator on Weyl tensor densities, developed in detail in [12]. This carries $\mathcal{W}^a{}_{bcd}[3-\frac{n}{2}]$ to $\mathcal{W}^a{}_{bcd}[1-\frac{n}{2}]$ in general dimension $n \geq 4$. In dimension 6, it acts on Weyl tensors $\mathcal{W}^a{}_{bcd}$, and is a composition of first-order operators. Some material on this operator also appears in [2], Sec. 3.d.

Though we have proceeded throughout under the assumption of Riemannian metric signature, the question of conformal invariance of abstract index tensor expressions is signature independent. Thus we also have a result, in dimension 4, for Lorentzian and signature $(2,2)$ conformal structures. One only needs to note that the self-dual vs. anti-self-dual split becomes, for Lorentz signature, an $\sqrt{-1}$-dual vs. $-\sqrt{-1}$-dual split. (In general, on p-forms in dimension n and a signature with q minus signs, $\star\star = (-1)^{p(n-p)+q}$.)

In dimension 4 the operators in Theorem 1 are essentially a subfamily of the so-called *standard operators* constructed in [14] (see also [8]). There, on a complex holomophic conformal spin manifold $\mathcal{M}$, conformally invariant operators are proliferated as direct images of a class of natural operators on the total space of the bundle of null directions of $\mathcal{M}$. Once the operators are constructed in this way it is clear that the same formulae yield conformally invariant operators on a real conformal 4-manifold of any signature. In [14], irreducible holomorphic bundles are described in terms of weights on Dynkin diagrams as in $\mathcal{O}(\overset{a}{\bullet}{-}\overset{b}{\times}{-}\overset{c}{\bullet})$ and the order of a differential operator $\mathcal{O}(\overset{a}{\bullet}{-}\overset{b}{\times}{-}\overset{c}{\bullet}) \to \mathcal{O}(\overset{d}{\bullet}{-}\overset{e}{\times}{-}\overset{f}{\bullet})$ is the difference $\frac{d+2e+f}{2} - \frac{a+2b+c}{2}$. Note in particular that D_{abcd} from [14] yields the formula (9). From that source we see D_{abcd} will yield fourth order conformally invariant differential operators $\mathcal{O}(\overset{a}{\bullet}{-}\overset{3}{\times}{-}\overset{c}{\bullet}) \to \mathcal{O}(\overset{a+4}{\bullet}{-}\overset{-5}{\times}{-}\overset{c+4}{\bullet})$, $\mathcal{O}(\overset{a}{\bullet}{-}\overset{-a+2}{\times}{-}\overset{c}{\bullet}) \to \mathcal{O}(\overset{a-4}{\bullet}{-}\overset{-a-2}{\times}{-}\overset{c+4}{\bullet})$, $\mathcal{O}(\overset{a}{\bullet}{-}\overset{-c+2}{\times}{-}\overset{c}{\bullet}) \to \mathcal{O}(\overset{a+4}{\bullet}{-}\overset{-c-2}{\times}{-}\overset{c-4}{\bullet})$, $\mathcal{O}(\overset{a}{\bullet}{-}\overset{1-a-c}{\times}{-}\overset{c}{\bullet}) \to \mathcal{O}(\overset{a-4}{\bullet}{-}\overset{1-a-c}{\times}{-}\overset{c-4}{\bullet})$ where the integers over the uncrossed nodes must be non-negative. For integers

a, b, c with a, c non-negative, the representation $\overset{a}{\bullet}\!\!-\!\!\!-\!\!\overset{b}{\times}\!\!-\!\!\!-\!\!\overset{c}{\bullet}$ corresponds to $[\frac{a+2b+c}{2}|\frac{a+c}{2}, \frac{c-a}{2}]$ in our current notation. Thus these four classes of operator are respectively the operators $\mathcal{V}[[\tilde{\lambda}_1 + 2|\tilde{\lambda}_1 - 2, \tilde{\lambda}_2]] \to \mathcal{V}[[\tilde{\lambda}_1 - 2|\tilde{\lambda}_1 + 2, \tilde{\lambda}_2]]$, $\mathcal{V}[[\tilde{\lambda}_2 + 2|\tilde{\lambda}_1, \tilde{\lambda}_2 - 2]] \to \mathcal{V}[[\tilde{\lambda}_2 - 2|\tilde{\lambda}_1, \tilde{\lambda}_2 + 2]]$, $\mathcal{V}[[-\tilde{\lambda}_2 + 2|\tilde{\lambda}_1, \tilde{\lambda}_2 + 2]] \to \mathcal{V}[[-\tilde{\lambda}_2 - 2|\tilde{\lambda}_1, \tilde{\lambda}_2 - 2]]$, and, $\mathcal{V}[[-\tilde{\lambda}_1 + 2|\tilde{\lambda}_1 + 2, \tilde{\lambda}_2]] \to \mathcal{V}[[-\tilde{\lambda}_1 - 2|\tilde{\lambda}_1 - 2, \tilde{\lambda}_2]]$ of the theorem.

In other even dimensions the analogue of this construction [15] again yields all the operators of the theorem including the formula (9), but in odd dimensions the operator is missed whenever it occurs as the middle operator in the BGG pattern.

References

[1] T. Branson, *Conformally covariant equations on differential forms*, Commun. Partial Diff. Eq. **7** (1982) 392–431.

[2] T. Branson, *Second-order conformal covariants*, University of Copenhagen Mathematical Institute preprint series, nos. 2,3 (1989). Archived at ftp://ftp.math.uiowa.edu/pub/branson/Copenhagen_notes_1989/zthree.ps.

[3] T. Branson and A.R. Gover, *Gauge for metric perturbations*, in preparation.

[4] A. Čap, J. Slovák, V. Souček, *Invariant operators on manifolds with almost Hermitian symmetric structures, III. Standard operators* Diff. Geom. Appl. **12** No. 1 (2000) 51–84.

[5] A. Čap, J. Slovák, V. Souček, *Bernstein-Gelfand-Gelfand sequences*, Ann. Math. **154** (2001) 97–113.

[6] S. Deser, *Closed form effective conformal anomaly actions in $D \geq 4$*, Phys. Lett. B **479** (2000) 315–320.

[7] L. Dolan, C.R. Nappi, and E. Witten, *Conformal operators for partially massless states*. arXiv:hep-th/0109096.

[8] M.G. Eastwood, *The Penrose transform for curved ambitwistor space*, Quart. J. Math. Oxford **39** (1988), 427–441.

[9] M.G. Eastwood and J.W. Rice, Conformally invariant differential operators on Minkowski space and their curved analogues, *Commun. Math. Phys.* **109** (1987), 207–228. Erratum, *Commun. Math. Phys.* **144** (1992), 213.

[10] M.G. Eastwood and M. Singer, *A conformally invariant Maxwell gauge*, Phys. Lett. **107A** (1985) 73–74.

[11] M.G. Eastwood and J. Slovák, *Semiholonomic Verma modules*, J. Algebra **197** (1997) 424–448.

[12] J. Erdmenger, *Conformally covariant differential operators: properties and applications*, Class. Quant. Grav. **14** (1997) 2061–2084.

[13] H. Fegan, *Conformally invariant first order differential operators*, Quart. J. Math. Oxford **27** (1976) 371–378.

[14] A.R. Gover, *Conformally invariant operators of standard type*, Quart. J. Math. Oxford **40** (1989), 197–207.

[15] A.R. Gover, *A geometric construction of conformally invariant operators* D.Phil. Thesis Oxford (1989).

[16] C.R. Graham, *Conformally invariant powers of the Laplacian, II: Nonexistence*, J. London Math. Soc. **46** (1992) 566–576.

[17] C.R. Graham, R. Jenne, L. Mason and G. Sparling, *Conformally invariant powers of the Laplacian, I: existence*, J. Lond. Math. Soc. **46** (1992), 557–565.

[18] S. Paneitz, *A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds*, preprint, 1983.

[19] R. Riegert, *A non-local action for the trace anomaly*, Phys. Lett. **134B** (1984) 56–60.

[20] E. Stein and G. Weiss, *Generalization of the Cauchy-Riemann equations and representations of the rotation group*, Amer. J. Math. **90** (1968) 163–196.