CONFORMALLY INVARIANT OPERATORS, DIFFERENTIAL FORMS, COHOMOLOGY AND A GENERALISATION OF Q-CURVATURE

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ABSTRACT. We give a geometric construction of a family of conformally invariant differential operators between density-valued differential form bundles and develop their properties (including their ellipticity type in the case of definite conformal signature). The construction is based on the ambient metric of Fefferman-Graham, and its relationship to the tractor bundles for the Cartan normal conformal connection. For each form order, the derivation yields an operator of every even order in odd dimensions, and even order operators up to order n in even dimension n. In the case of unweighted (or *true*) forms as domain, these lead to new conformal complexes which are elliptic in the case of Riemannian signature. In this case we also construct a gauge companion operator which completes each operator to a conformally invariant and (in the case of Riemannian signature) elliptically coercive system. These (operator,gauge) pairs are used to define conformally stable form subspaces which are are candidates for spaces of conformal harmonics. This generalises the $n/2$-form and 0-form cases, in which the harmonics are given by conformally invariant systems. We describe a family of operators on closed forms which generalise in a natural way Branson's Q-curvature and give new conformally invariant global pairings between differential form bundles.

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1. INTRODUCTION

Classically, conformal structure on manifolds is the natural setting for the study of massless particles in Physics. More recently it has proved important in the study of various curvature prescription and extremal metric problems in Riemannian geometry. It also plays major roles in the modern theories of strings and branes, and, via the Fefferman bundle and metric, in complex and CR geometry.

An important idea is that of *conformally invariant differential operators* like the conformal Laplacian and the Maxwell operator. Such operators act on sections of bundles natural for conformal structure, may be defined by universal natural formulae, and depend only on the conformal structure (not, for example, on any choice of metric tensor from the conformal class). Among the most fundamental bundles are differential forms and form-densities. One of our purposes here is to give an explicit geometric construction of all conformally invariant differential operators carrying form-density bundles to form-density bundles which are known to exist in the arbitrarily conformally curved case [24, 36]. Conjecturally, it is only these operators that extend from the conformally flat case to the arbitrarily curved case; some evidence in this direction is given in [34] and [31].

Another purpose is to generalise to a natural and inclusive setting an idea of Eastwood and Singer [22] which might be termed *gauge companion operator*. Roughly speaking, this is an operator G which is invariant when acting on the null space of an invariant operator D, and for which (in Riemannian signature) the invariant system $Du = 0$, $Gu = 0$ is elliptically coercive. The prototypical example is the third-order Eastwood-Singer operator in dimension 4, which completes the ellipticity picture for the second-order Maxwell operator. From another viewpoint, D and G are aspects of a single operator acting between indecomposable bundles natural for conformal structure. Still another objective is to find a natural setting generalising the *Q-curvature*.

Our results are especially revealing and intriguing in the case of the operators on *true forms*, or differential forms of density weight zero, in even dimensions. As mentioned above, the existence of conformally invariant operators between the bundles concerned is not a new result. In fact, for each form order there is, in general, a many parameter family of such invariant operators, all having the same leading symbol. Our construction yields a special class of such operators with remarkable properties. A first point is that the invariant operators, which land in weighted forms, are formally self-adjoint and have factorisations $\delta M d$, where d is the exterior derivative and δ is its formal adjoint with respect to the conformal structure. (Alternatively, in the oriented case, the operators land in true forms of the complementary order, with factorisation dNd.) Despite early glimpses of such factorisations at low order in [5], Theorem 2.10, the existence of factorisations is surprising, and not at all to be expected from the curved translation principle of Eastwood and Rice [21, 20]. It seems probable, in fact, that such factorisations are rare, and perhaps even peculiar to differential form bundles. From these factorisations it follows that the operators may be combined with appropriate sections of the usual de Rham complex and its adjoint to yield new conformally invariant differential complexes. In the Riemannian setting these are elliptic. The operators M are natural only for metric (pseudo-Riemannian) structure *ab initio*, but upon further study yield cohomology maps that depend only on the conformal structure. We show that the operators M at a conformal scale (i.e. a choice of metric within the given conformal class), viewed as operators on closed forms, are natural analogues and generalisations of the Q-curvature.

Moreover, for each k the operator $\delta M_k d$, of order $n-2k$ on k-forms, admits a gauge companion which *interlocks* with the operator at the next even order, in the sense that δM_k is a natural gauge companion operator for $\delta M_{k+1} d$. Specifically, we show that the system $(\delta M_{k+1} d, \delta M_k)$ is, in an appropriate and natural sense, conformally invariant and from this it follows that, in particular, δM_k is conformally invariant on the (conformally invariant) null space of $\delta M_{k+1} d$. In the Riemannian conformal case the conformally invariant system $\delta M_{k+1} d\varphi = 0$, $\delta M_k \varphi = 0$ is elliptically coercive. (Since the leading symbol of $\delta M_{k+1} d$ annihilates $\xi \wedge \Lambda^k M$, it is clear that the system's first equation falls short of ellipticity for $k \geq 0$.) Continuing in the Riemannian signature setting, we show that the conformally invariant null space $\mathcal{H}^k$ ($k \leq n/2 - 1$) of each gauge operator δM_k, as a conformally invariant operator on closed k-forms, is a candidate for a space of conformal harmonics at order k. On compact manifolds $\dim(\mathcal{H}^k)$ is finite and under mild restrictions recovers the k^{th} Betti number. More generally we give estimates bounding the size of each $\mathcal{H}^k$ in terms of the de Rham cohomology and the cohomology of the elliptic complexes mentioned above. The "cohomology maps" mentioned above are, more accurately, conformally invariant maps from $\mathcal{H}^k$ into the vector space dual of $H^k(M)$ (so $H^{n-k}(M)$ on oriented manifolds) and these play a role in obtaining these estimates. These results generalise the conformally invariant *Eastwood-Singer gauge* δM_0 for Maxwell's operator δd in dimension 4 (see [22] and its applications in [23]). Here the corresponding conformally invariant operator $\delta M_0 d$ on true forms is the *Paneitz operator*, discovered independently by Paneitz [41] and Riegert [43] in 1983. The corresponding 4-dimensional Q-curvature, discovered by Branson in 1987 [6, 13], is a pseudo-Riemannian local invariant with the Paneitz operator as its conformal variation. There has been considerable recent interest and progress in understanding Branson's Q-curvatures, especially in low dimensions and on conformally flat structures [17, 18].

The analogue of the 4-dimensional Paneitz operator for higher even dimensions is a dimension-order operator (the *critical GJMS operator*) on true 0-forms, one of the results of a construction of Graham, Jenne, Mason, and Sparling [36]. Our results recover this as the operator $\delta M_0 d$. More generally, our true form results include, and interpolate between, results on the critical GJMS operator and results on what might be called Maxwell's operator in even dimensions, δd on $\Lambda^{n/2-1}T^*M$. For example our operator $\delta M_0 d$ is formally self-adjoint by construction. This gives a new and constructive proof of the existence of a formally self-adjoint conformally invariant power of the Laplacian on functions with factorisation as indicated. Earlier alternative proofs are in [37] and [26]. The higher-dimensional Q-curvature was constructed in [7, 8], modulo precisely this property of the GJMS operators which was put in place later as indicated.

All of the main results, save for some of the operators of order n, are obtained from a single uniform construction derived from the Fefferman-Graham *ambient metric* construction [25], and its relationship, as exposed in [14, 32], to the the theory of a class of bundles natural for conformal structure. These are the so-called *tractor* bundles. Tractor bundles and their normal connections may be viewed as structures associated to the Cartan normal connection [16] but may also be constructed directly [1] by an idea which, in the conformal setting, dates back to Thomas [45]. Penrose's local twistor bundle is an example of a tractor bundle. Here, to avoid unnecessary background, we use the ambient manifold to actually define the tractor bundles required. Given an n-dimensional manifold M with a conformal structure of signature (p, q), the Fefferman-Graham construction associates a formal Taylor series for a metric tensor h of signature $(p + 1, q + 1)$ in a collar $\tilde{M}$ of the conformal metric bundle over M. The metric tensor h is determined by solving the the empty-space Einstein equation $\mathrm{Ric}(h) = 0$, subject to some homogeneity conditions, and with data (given by the conformal structure) on the metric bundle. In this way one can endow $\tilde{M}$ with a metric h, at least formally. A limitation is that in even dimensions, h is only related to the original conformal structure (represented by a tensor density $\boldsymbol{g}$, the *conformal metric*) to a certain finite order. This circumstance itself is intimately tied to Q-curvature: The conformally invariant *Fefferman-Graham (obstruction) tensor*, which governs the breakdown of the relation between h and $\boldsymbol{g}$, is actually the total metric variation of the integral of the Q-curvature [35]. (In dimension 4, this is closely related to the fact that the Bach tensor is the total metric variation of the norm-squared of the Weyl tensor.) There is a certain twisted right action by $\mathbb{R}_+$ on the ambient tangent bundle $T\tilde{M}$, the quotient by which, when restricted to the conformal metric bundle as an embedded submanifold, is a vector bundle on M. Via the characterisation in [16] this bundle, with metric and connection induced from the ambient metric, is identified as the normal standard tractor bundle.

The current work arose in part from a desire to extend the Eastwood-Singer gauge companion idea mentioned above. In [11] the authors showed that the Maxwell operator and its gauge companion are unified via a tractor operator. This led to some extensions. The idea of engaging the ambient construction arose from [32]; our construction of the operators generalising the Q-curvature is a generalisation of the definition of Q given there, and the basic results for weighted 1-forms can be deduced from Theorem 2.5 of that source. What we present here is a considerable development and refinement of these early ideas. In part this is directly inspired by the GJMS operator construction of [36] (see also [39], where a second order operator of [4] between weighted forms is recovered via the ambient manifold). Powers of the ambient form Laplacian are shown to descend to conformally invariant operators on form tractor bundles on M; the latter are exterior powers of the standard tractor bundle. These operators are then composed fore and aft with certain tractor operators to yield invariant operators on the weighted forms of the underlying conformal manifold M. These tractor operators, which we anticipate will be of independent interest, are also defined via the ambient metric and an extensive ambient form calculus is developed and used to define these and establish their properties. (We observe that the basic operators of this ambient calculus generate a copy of the special linear Lie superalgebra $\mathfrak{sl}(2|1)$, although this fact is not used explicitly.) Since the tractor bundle and connection are well understood in terms of the underlying (pseudo-)Riemannian structures (for metrics from the conformal class) there is a straightforward algorithm for expressing our operators in terms of the Levi-Civita connection and its curvature. We note that our form tractor bundles extend to the curved setting the *spannors* and *plyors* of Irving Segal and his collaborators [42, 40]. A feature of our current development is that reducible but indecomposable subquotients of the (weighted) form tractor bundles play an important role. In this setting we construct a first order operator which, for conformal manifolds, generalises the exterior derivative to an operator which is conformally invariant on weighted forms and their generalisations (viz. sections of weighted form tractor subquotient bundles.)

To expand on the conjecture above about conformally invariant differential operators between form-densities: We consider only conformally invariant operators which exist non-trivially in flat conformal classes; the issue for each of these is then whether it generalises to the arbitrarily conformally curved setting. It has long been known [4, 5] that the conformally flat case dictates the principal part

$$\underbrace{\frac{n-2k+2\ell}{2}}_{-u}(\delta d)^{\ell}+\underbrace{\frac{n-2k-2\ell}{2}}_{-w}(d\delta)^{\ell} \tag{1}$$

for an operator of order 2ℓ on k-form densities. Moreover, the source and target density weights must be w, u as given. Note that this recovers the above non-ellipticity remark about true form operators (the $(d\delta)^{\ell}$ coefficient must vanish). On the other hand, it establishes ellipticity in all other cases where we succeed in constructing an operator, since if $w \neq 0$, choosing a metric in the conformal class and composing on either side with $-u^{-1}\delta d - w^{-1}d\delta$ yields $(\delta d+d\delta)^{\ell+1}$.

In the next section we present the main results. Proofs are included in section 2 only if they derive from standard and well known ideas. Otherwise the proofs are delayed until section 4. The results and arguments of that section are based on the ambient calculus developed in section 3 and its interpretation in terms of tractor bundles in section 3.1. In fact in section 4 there are theorems generalising many of the results in section 2 and other theorems of independent value. However these require the technical background of the earlier sections in order to be stated. Section 4.1 is concerned with defining the operators generalising the Q-curvature and proving the theorem giving their properties (Theorem 2.8). In section 6 we show that these operators generalise and relate the definitions of the Q-curvature given recently in [32] and [27], while in section 5 we give a nontriviality result for the maps $\mathcal{H}^k \to H_k(M)$ that they determine. In section 6 we also describe ways of proliferating other operators with transformation laws similar to Q.

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2. The main theorems

Let M be a smooth manifold of dimension $n \geq 3$. A *conformal structure* on M of signature (p,q) (with $p+q=n$) is an equivalence class $[g]$ of smooth pseudo–Riemannian metrics of signature (p,q) on M, with two metrics being equivalent if and only if one is obtained from the other by multiplication with a positive smooth function. Equivalently a conformal structure is a smooth ray subbundle $\mathcal{Q} \subset S^2T^*M$, whose fibre over x consists of the values of g_x for all metrics g in the conformal class. A metric in the conformal class is a section of $\mathcal{Q}$. A conformal structure of signature $(n,0)$ is termed *Riemannian*.

We can view $\mathcal{Q}$ as the total space of a principal bundle $\pi : \mathcal{Q} \to M$ with structure group $\mathbb{R}_+$ and so there are natural line bundles on $(M,[g])$ induced from the irreducible representations of $\mathbb{R}_+$. For $w \in \mathbb{R}$, we write $E[w]$ for the line bundle induced from the representation of weight $-w/2$ on $\mathbb{R}$ (that is $\mathbb{R}_+ \ni x \mapsto x^{-w/2} \in \mathrm{End}(\mathbb{R})$). Thus a section of $E[w]$ corresponds to a real-valued function f on $\mathcal{Q}$ with the homogeneity property $f(x, \Omega^2 g) = \Omega^w f(x, g)$, where Ω is a positive function on M, $x \in M$, and g is a metric from the conformal class $[g]$. We shall write $\mathcal{E}[w]$ for the space of smooth sections of this bundle.

Note that there is a tautological function $\boldsymbol{g}$ on $\mathcal{Q}$ taking values in S^2T^*M, namely the function which assigns to the point $(x, g_x) \in \mathcal{Q}$ the metric g_x at x. This is homogeneous of degree 2 since $\boldsymbol{g}(x, s^2 g_x) = s^2 g_x$. If σ is any non-vanishing function on $\mathcal{Q}$ homogeneous of degree $+1$ then $\sigma^{-2}\boldsymbol{g}$ is independent of the action of $\mathbb{R}_+$ on the fibres of $\mathcal{Q}$, and so $\sigma^{-2}\boldsymbol{g}$ descends to give a metric from the conformal class. Thus $\boldsymbol{g}$ determines and is equivalent to a canonical section of $S^2T^*M \otimes E[2]$ (called the *conformal metric*) that we also denote $\boldsymbol{g}$. This in turn determines a canonical section $\boldsymbol{g}^{-1}$ of $S^2T^*M \otimes E[-2]$ with the property that a single contraction between these gives the identity endomorphism of TM. The conformal metric gives an isomorphism of TM with $T^*M[2]$ that we will view as an identification. We usually also denote by $\sigma \in \mathcal{E}[1]$ the density on M equivalent to the homogeneous function σ on $\mathcal{Q}$. Since $\sigma^{-2}\boldsymbol{g}$ is a metric from the conformal class we term σ a choice of *conformal scale.*

We will use the notation $\mathcal{E}^k$ for the space of smooth sections of $\wedge^k T^*M$, for which we shall sometimes use the alternative notation E^k, and we write $\mathcal{E}^k[w]$ for the smooth sections of the tensor product $\wedge^k T^*M \otimes E[w]$. Some statements about forms or form-densities admit simpler formulations if we allow values of k falling outside the range $0 \leq k \leq n$; by convention these $\wedge^k$ are zero bundles. For $0 \leq k \leq n/2$ we also write $E_k[w]$ (with section space $\mathcal{E}_k[w]$) as a shorthand for $E^k[w + 2k - n]$. This notation is suggested by the duality between the section spaces $\mathcal{E}^k$ and $\mathcal{E}_k$ as follows. For $\varphi \in \mathcal{E}^k$ and $\psi \in \mathcal{E}_k$, with one of these compactly supported, there is the natural conformally invariant global pairing

$$\varphi, \psi \mapsto \langle \varphi, \psi \rangle := \int_M \varphi \cdot \psi \, d\mu_{\boldsymbol{g}}, \tag{2}$$

where $\varphi \cdot \psi \in \mathcal{E}[-n]$ denotes a complete contraction between φ and ψ. This scales so that when M is orientable we have

$$\langle \varphi, \psi \rangle = \int_M \varphi \wedge \star \psi$$

where $\star$ is the conformal Hodge star operator. (On orientable conformal manifolds the bundle of volume densities can be canonically identified with $\mathcal{E}[-n]$ and so the Hodge star operator for each metric from the conformal class induces an isomorphism that we shall also term the Hodge star operator: $\star : \mathcal{E}^k \cong \mathcal{E}^{n-k}[n-2k]$.) The integral is also well-defined if instead $\varphi \in \mathcal{E}_k$ and $\psi \in \mathcal{E}^k$. If we also denote this pairing by $\langle \cdot, \cdot \rangle$, then $\langle \varphi, \psi \rangle = \langle \psi, \varphi \rangle$.

We write δ for the formal adjoint of the exterior derivative with respect to our pairings. That is, for $\varphi \in \mathcal{E}^k$ and $\mu \in \mathcal{E}_{k+1}$, at least one having compact support, we have $\langle d\varphi, \mu \rangle = \langle \varphi, \delta\mu \rangle$. The notation $\mathcal{C}^k$ is used for the space of closed k-forms and $\mathcal{C}_k$ denotes the formal dual space $\mathcal{E}_k / \mathcal{R}(\delta)$.

Our first main theorem concerns the construction and form of a family of natural conformally invariant operators between density valued differential form bundles in dimension $n \geq 3$. We say that P is a *natural differential operator* if P_g can be written as a universal polynomial in covariant derivatives with coefficients depending polynomially on the conformal metric, its inverse, the curvature tensor and its covariant derivatives. The coefficients of natural operators are called *natural tensors*. In the case that they are scalar they are often also called *Riemannian invariants*. We say P is a *conformally invariant differential operator* if it is well defined on conformal structures (i.e. is independent of a choice of conformal scale).

Theorem 2.1. *For each choice of $k \in \{0, 1, \cdots, n\}$, let $\ell \in \{0, 1, 2, \ldots\}$ if n is odd and, if n is even let $\ell \in \{0, 1, \ldots, n/2\}$. Let*

$$w = k + \ell - n/2.$$

On conformal n-manifolds, there is a formally self-adjoint conformally invariant natural differential operator

$$L_k^\ell : \mathcal{E}^k[w] \to \mathcal{E}_k[-w]$$

of order 2ℓ. If n is even, $k \leq n/2$, and $w = 0$, we have

$$L_k := L_k^{n/2-k} = \delta M_k d, \tag{3}$$

where $M_k : \mathcal{C}^{k+1} \to \mathcal{C}_{k+1}$ is a conformally invariant operator.

On Riemannian conformal manifolds the following holds. The operator L_k^ℓ is elliptic if and only if $k \neq n/2 \pm \ell$, and it is positively elliptic if and only if $k \notin [n/2 - \ell, n/2 + \ell]$. For each k the differential operator sequence

$$\mathcal{E}^0 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{k-1} \xrightarrow{d} \mathcal{E}^k \xrightarrow{L_k} \mathcal{E}_k \xrightarrow{\delta} \mathcal{E}_{k-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{E}_0 \tag{4}$$

is an elliptic complex.

The operator $L^1_{n/2-1}$ is, up to a non-zero multiple, the Maxwell operator in even dimension n. $L^0_{n/2}$ is zero. For $k \neq 0$ the existence of conformally invariant differential operators between the spaces $\mathcal{E}^k[w]$ and $\mathcal{E}_k[-w]$ as in the theorem follows from the general theory in [24]. The main point here is the special form (3), together with an explicit construction of these operators. This construction employs tractor calculus and the Fefferman-Graham ambient construction in tandem to generalise the scalar density results in [36] to density valued differential forms. For almost all cases this construction is described in expression (38) based on operators given in (35), (34) and Section 3.2. The main results concerning order and ellipticity are the subject of Proposition 4.4. (Since we are generally only concerned with operators up to a non-vanishing constant multiple, here and throughout the article we say an operator is *positively elliptic* if it has positive *or* negative leading symbol.) Naturality and the form (3) are established in Theorem 4.5 parts (*i*) and (*ii*). There are two classes of exceptional operators not given by expression (38): (a) the operators satisfying $k = \ell + n/2$; (b) The operators of order n when $k \geq 1$. (These classes share the operator of order n on n-forms of weight n.) The operators of type (a) are exactly those which have the spaces $\mathcal{E}_k$, with $k \geq n/2$, as domain, and are treated in Theorem 2.14 below. On k forms with $k \geq n/2$, they are given by $\star L_{n-k} \star$, where L_{n-k} is as in (3) above (and so are never elliptic). In particular the operator of order n on weighted n-forms arises this way. Proposition 4.10 gives a construction of the other operators of order n. The anomalous behaviour of the operators in the two exceptional classes is not unexpected. In the case of operators of type (a) it arises because δ acts invariantly on the domain bundles $\mathcal{E}^k[2k-n]$. This means a certain differential splitting operator (see Proposition 3.12) involved in the general construction (38) fails for these bundles. The failure of the operators of type (b) to arise from (38) is a reflection of the conformal invariance of the Fefferman-Graham obstruction tensor. While we do not elaborate on this point in the current article, in dimension 4 this claim is clear from section 3 of [34].

The complex (4) will be referred to as the k^{th} *de Rham detour complex.* We write $H^k_L(M)$ for the cohomology of this complex at the point $\stackrel{d}{\to} \mathcal{E}^k \stackrel{L_k}{\to}$ and $H^L_k(M)$ for the cohomology at $\stackrel{L_k}{\to} \mathcal{E}_k \stackrel{\delta}{\to}$. In view of the factorisation $L_k = \delta M_k d$, it is immediate that there is a canonical conformally invariant invariant injection $H^k(M) \to H^k_L(M)$ and similarly a canonical surjection $H^L_k(M) \to H_k(M)$. Here, in accordance with our other conventions, by $H_k(M)$ we mean $\mathcal{N}(\delta)/\mathcal{R}(\delta)$ at $\mathcal{E}_k$. (So on oriented manifolds $H_k(M) \cong H^{n-k}(M)$.)

On compact Riemannian conformal manifolds, Hodge theory shows that (2) determines a perfect pairing between the standard de Rham cohomology $H^k(M)$ and $H_k(M)$. That is, via (2) $H_k(M)$ is just the vector space dual of $H^k(M)$. Next observe that since L_k is formally self-adjoint it follows easily from standard Hodge theory that $\dim(H^k_L(M))$ is finite and $\dim(H^k_L(M)) = \dim(H^L_k(M))$. On the other hand it is easily verified that the pairing (2) descends to a well defined pairing between $H^k_L(M)$ and $H^L_k(M)$. We state this in a proposition.

Proposition 2.2. *On compact Riemannian conformal manifolds* (2) *induces an invariant perfect pairing between $H^k_L(M)$ and $H^L_k(M)$.*

Proof: The invariance assertion is clear by construction. If we fix an arbitrary choice of conformal scale then, for each k, the spaces $\mathcal{E}_k$ can be identified with the spaces $\mathcal{E}^k$ and the pairing (2) gives an inner product on each of the spaces $\mathcal{E}^k$. In this setting, using that L_k is formally self-adjoint, the standard Hodge theory of the complex (4) gives

$$\mathcal{R}(d) \oplus \mathcal{R}(L_k) \oplus (\mathcal{N}(\delta) \cap \mathcal{N}(L_k)) \tag{5}$$

as the Hodge decomposition of the space $\mathcal{E}^k$ and also of $\mathcal{E}_k$. For any class $[\varphi'] \in H^k_L(M)$ we can find a representative $\varphi \in \mathcal{N}(\delta) \cap \mathcal{N}(L_k)$ and, via the identification of $\mathcal{E}^k$ with $\mathcal{E}_k$, this is also the preferred representative of a class in $H^L_k(M)$. The pairing of these classes produces (φ, φ), which is positive if $[\varphi']$ is non-zero; this establishes nondegeneracy. $\square$

Remark: The pairing in the proposition induces and is equivalent a symplectic inner product on $H^k_L(M) \oplus H^L_k(M)$, via $\langle (v, v'), (w, w') \rangle = \langle v, w' \rangle - \langle w, v' \rangle$. ‖

To state the next theorem we need one more result.

Proposition 2.3. *On a conformal manifold of dimension n, for each $k \in 0, 1, \cdots, n+1$ there is a natural indecomposable bundle $\mathbb{G}_k$ with a natural* *subbundle isomorphic to E_{k-1}, and corresponding quotient isomorphic to E_k.*

The bundle $\mathbb{G}_k$, its dual and their weighted variants are defined in expression (30) of Section 3.2. In particular $\mathbb{G}_k$ is a subbundle of a certain tractor bundle (see Section 3.1) and arises naturally from the ambient construction. From either picture the properties described in the proposition are immediate. To summarise the composition series of $\mathbb{G}_k$ we will often use the semi-direct sum notation

$$\mathbb{G}_k = E_k \oplus\!\!\!\!+\, E_{k-1},$$

or, on the level of section spaces,

$$\mathcal{G}_k = \mathcal{E}_k \oplus\!\!\!\!+\, \mathcal{E}_{k-1}.$$

For the natural quotient bundle map onto E_k we shall write $q^k : \mathbb{G}_k \to E_k$.

From Theorem 2.1, on Riemannian conformal manifolds the operators L_k^ℓ are elliptic except when operating on unweighted forms, that is when the dimension is even and $\ell = n/2 - k$. The next theorem asserts that these operators $L_k : \mathcal{E}^k \to \mathcal{E}_k$ admit special *gauge companion* operators G_k so that the pairs (L_k, G_k) are graded injectively elliptic and, in an appropriate sense, conformally invariant. A general definition of graded injective ellipticity is possible along the lines of [19]. A definition more focused on our present purposes is as follows. Let $P : V \to W$ be a natural differential operator between bundles which are natural for conformal structure, and suppose that a choice of scale g naturally splits V as $V_1 \oplus \cdots \oplus V_r$ and W as $W_1 \oplus \cdots \oplus W_s$. Let $P_i^j : V_j \to W_i$ be the block decomposition with respect to this splitting. Then P is *graded injectively elliptic* if there is a positive integer m and there are differential operators $\overline{P}_k^i : W_i \to V_k$, natural for Riemannian structure, with

$$\sum_{i=1}^{s} \overline{P}_k^i P_i^j = \delta_k^j \Delta^m + (\text{order } < 2m) \tag{6}$$

at any Riemannian metric. The various P_j^i will generally have different orders. Graded injective ellipticity in the above sense implies, for an appropriately natural operator, any reasonable graded injective ellipticity property that might be formulated in the setting of more general partial differential operators. An injectively elliptic P has finite-dimensional kernel on a compact manifold, since the operator $\overline{P}P$ described in (6) does.

By the same token, we can speak of *graded surjectively elliptic operators.* In the notation above, P is such if there are natural differential operators $\underline{P}_j^k : W_k \to V_j$ with

$$\sum_{j=1}^{r} P_i^j \underline{P}_j^k = \delta_i^k \Delta^p + (\text{order } < 2p) \tag{7}$$

for some p.

In other signatures, we can still consider the properties (6) and (7), but now the right hand side will involve powers of the (pseudo-)Laplacian for that signature. In general signature, we will say that an operator P is *quasi-Laplacian* if it is a right factor of a power Laplacian in the sense of (6). Of course, for indefinite metric signatures, the quasi-Laplacian property does not naturally relate to ellipticity considerations, but rather to properties of hyperbolicity or ultra-hyperbolicity. Most of the following theorem takes place in the setting of arbitrary metric signature.

Theorem 2.4. *On conformal manifolds of even dimension n, for each $k = 0, 1, \cdots, n/2$ there is a natural conformally invariant differential operator*

$$\mathbb{L}_k : \mathcal{E}^k \to \mathcal{G}_k$$

with the following properties.
(i) $q^k \mathbb{L}_k = L_k$. *In particular $q^k \mathbb{L}_k$ is trivial on the null space of L_k.*
(ii) $\mathbb{L}_k$ *determines a conformally invariant operator $G_k : \mathcal{N}(L_k) \to \mathcal{E}_{k-1}$, which satisfies*

$$(n-2k+4)G_k d = L_{k-1}.$$

(iii) $G_k = \delta \tilde{M}_{k-1}$, *where $\tilde{M}_{k-1} : \mathcal{N}(L_k) \to \mathcal{E}_k/\mathcal{N}(\delta)$ is conformally invariant.*
(iv) For each $k \le n/2 - 1$ the operator $\mathbb{L}_k$ is quasi-Laplacian, and thus in the case of Riemannian signature it is a graded injectively elliptic operator.

The operator $\mathbb{L}_k$ is the $\ell = n/2 - k$ special case of the operator $\mathbb{L}^\ell_k$ defined by equation (38) below. This will actually make the equation $q^k \mathbb{L}^\ell_k = L^\ell_k$ the definition of L^ℓ_k. That these are conformally invariant operators is established in part *(i)* of Theorem 4.5. Part *(ii)* above is proved in part *(iii)* of Theorem 4.5. This and the fact that the L_k are formally self-adjoint enable us to conclude the result (3) in Theorem 2.1. Part *(iii)* above is part *(iv)* of Theorem 4.5, and finally the result *(iv)* above is exactly Proposition 4.6. Note that, from parts *(iv,v)* of Theorem 4.5, $L_{n/2} = 0$ and $G_{n/2}$ is a non-zero multiple of δ. At the other extreme of k we note that q^0 is the identity on $\mathcal{G}_0 \cong \mathcal{E}_0$ so $\mathbb{L}_0 = L_0$ and G_0 is the zero operator.

Note that the injective ellipticity of $\mathbb{L}_k$ for conformal Riemannian structures implies that it has a finite-dimensional conformally invariant null space $\mathcal{H}^k_{\mathbb{L}}$ for compact M, and thus this null space is a candidate for a space of "conformal harmonics". Since the exterior derivative is a right factor of L_k *ab initio*, the space $\mathcal{H}^k := \mathcal{N}(G_k : \mathcal{C}^k \to \mathcal{E}_{k-1})$ is contained in $\mathcal{H}^k_{\mathbb{L}}$ and leads to simpler results which we discuss first. First we state a proposition giving a relationship between $\mathcal{H}^k$ and $H^k(M)$ in the general case.

Proposition 2.5. *On even dimensional conformal manifolds there is a canonical exact sequence of vector space homomorphisms*

$$0 \to H^{k-1}(M) \to H^{k-1}_L(M) \to \mathcal{H}^k \to H^k(M) \qquad \text{for } k = 1, \cdots, n/2 - 1,$$

where $\mathcal{H}^k \to H^k(M)$ is the map taking $w \in \mathcal{H}^k \subset \mathcal{C}^k$ to its equivalence class in $H^k(M)$. Thus in the compact Riemannian case we have

$$\dim H^{k-1}(M) + \dim \mathcal{H}^k \le \dim H^k(M) + \dim H^{k-1}_L(M).$$

Proof: The theorem is clear for $k = 0$ since by their definitions both $\mathcal{H}^0$ and $H^0(M)$ are the space of locally constant functions. Suppose now $k \ge 1$. If $w \in \mathcal{H}^k$ is mapped to the class of 0 in $H^k(M)$ then w is exact. Since in addition $G_k w = 0$, it follows from Theorem 2.4 part *(ii)* that $w = du$ for

$u \in \mathcal{N}(L_{k-1})$. On the other hand recall that by Theorem 2.1, d is a right factor of L_{k-1} and so $\mathcal{C}^{k-1} \subseteq \mathcal{N}(L_{k-1})$. Thus there is an exact sequence

$$0 \to \mathcal{C}^{k-1} \to \mathcal{N}(L_{k-1}) \to \mathcal{H}^k \to H^k(M)$$

where the map $\mathcal{N}(L_{k-1}) \to \mathcal{H}^k$ is the restriction of exterior differentiation. Since $H^{k-1}(M)$ is the image of $\mathcal{C}^{k-1}$ under the composition $\mathcal{C}^{k-1} \to \mathcal{N}(L_{k-1}) \to H_L^{k-1}(M)$, the sequence in the lemma is constructed. $\square$

Remark: By a very similar argument one shows that there is also a canonical exact sequence of vector space homomorphisms

$$0 \to H^{k-1}(M) \to H_L^{k-1}(M) \to \mathcal{H}_{\mathbb{L}}^k \to H_L^k(M)$$

giving $\dim H^{k-1}(M) + \dim \mathcal{H}_{\mathbb{L}}^k \leq \dim H_L^k(M) + \dim H_L^{k-1}(M)$. ▮

By the proposition, the map $\mathcal{H}^k \to H^k(M)$ is injective if and only if $H^{k-1}(M) = H_L^{k-1}(M)$. In fact in this case it is an isomorphism.

Theorem 2.6. *On compact conformal Riemannian manifolds of even dimension n, $\mathcal{H}^{n/2}$ is isomorphic to $H^{n/2}(M)$. In addition, for each $k = 0, 1, \cdots, n/2 - 1$, if $H^{k-1}(M) = H_L^{k-1}(M)$ then the conformally invariant null space $\mathcal{H}^k$ of G_k, acting on $\mathcal{C}^k$, is naturally isomorphic, as a vector space, to $H^k(M)$.*

Proof: The first statement is immediate, since $G_{n/2}$ is a non-zero constant multiple of δ.

For $k < n/2$, as in the proof of Proposition 2.2, at an arbitrary choice of conformal scale we have the Hodge decomposition (5) of $\mathcal{E}^k$ or $\mathcal{E}_k$. But under the assumption of the theorem we have $H^{k-1}(M) = H_L^{k-1}(M)$ and so $\mathcal{N}(\delta) \cap \mathcal{N}(L_{k-1})$ is the usual space of de Rham harmonics $\mathcal{N}(\delta) \cap \mathcal{N}(d)$. Since by the usual de Rham Hodge decomposition $\mathcal{R}(\delta)$ is the intersection of $\mathcal{E}^k$ with the L^2 orthocomplement of $\mathcal{R}(d) \oplus (\mathcal{N}(\delta) \cap \mathcal{N}(d))$, it follows that

$$\mathcal{R}(L_{k-1}) = \mathcal{R}(\delta).$$

Now recall from Theorem 2.4 part *(iii)* that $G_k = \delta \tilde{M}_{k-1}$ on $\mathcal{C}^k$. From this and the last display it is immediate that given any equivalence class $[w]$ in $H^k(M)$ there exists $u \in \mathcal{E}^{k-1}$ solving the equation

$$(n - 2k + 4) G_k w + L_{k-1} u = 0.$$

By Theorem 2.4 part *(ii)* this gives the solution $w' = w + du$ to the problem of finding $w' \in [w]$ satisfying $G_k w' = 0$. Thus the injective map $\mathcal{H}^k \to H^k(M)$ is also surjective. $\square$

Since in general we would expect that $H_L^k(M) = H^k(M)$, it is worth noting the obvious corollary.

Corollary 2.7. *For each $k = 0, 1, \cdots, n/2 - 1$, if $H^{k-1}(M) = H_L^{k-1}(M)$ and $H^k(M) = H_L^k(M)$, then $\mathcal{H}_{\mathbb{L}}^k := \mathcal{N}(\mathbb{L}_k)$ is naturally isomorphic, as a vector space, to $H^k(M)$.*

Note that if $k = n/2 - 1$ then L_k is the usual Maxwell operator; thus the condition $H^k(M) = H_L^k(M)$ is automatically satisfied and $\mathcal{H}_{\mathbb{L}}^k = \mathcal{H}_L^k(M)$.

Finally in this section we show the operators M_k and $\tilde{M}_k$ above are related to an operator on forms which in an appropriate sense generalises Branson's

Q-curvature. Each operator L_{k-1} of Theorem 2.1 evidently only determines M_{k-1} as an operator $\mathcal{C}^k \to \mathcal{E}_k/\mathcal{N}(\delta)$, while $\tilde{M}_{k-1}$ is similarly fixed only as an operator $\mathcal{N}(L_k) \to \mathcal{E}_k/\mathcal{N}(\delta)$. One might hope that these conformally invariant operators are the restrictions of some conformally invariant operator or operators $\mathcal{E}^k \to \mathcal{E}_k$. In fact this is impossible. The invariant differential operators on the standard conformally flat model are classified via the bijective relationship with generalised Verma module homomorphisms and the classification of the latter in [3]. (See [24] and references therein for a summary of the relevant representation theoretic results and details on how these correspond dually to a classification of invariant differential operators.) From this well known classification, it is clear that on such structures the L_k are (up to constant multiples) unique. Since $L_k d = 0 = \delta L_k$ these are not candidates for the M_{k-1} or the $\tilde{M}_{k-1}$. Expression (41) of Section 4.1 defines, on even dimensional conformal manifolds, for each choice of conformal scale σ and for $k = 0, 1, \cdots n/2$, an operator

$$Q_k^\sigma : \mathcal{E}^k \to \mathcal{E}_k.$$

Parts (*i,ii*) of the next theorem assert that the operators M_{k-1} and $\tilde{M}_{k-1}$ are (up to a multiple) restrictions of Q_k^σ. Parts (*iii–v*) show that as operators on closed forms the Q_k^σ generalise Branson's Q-curvature.

We should point out that the construction (41) defines each Q_k^σ as a composition of tractor operators arising from the ambient construction, but via (35) and the theory of Section 3.2, this may be readily re-expressed as a universal polynomial expression in the covariant derivative ∇ and the Riemann curvature R.

Theorem 2.8. (*i*) *In a conformal scale σ the operator*

$$Q_k^\sigma : \mathcal{E}^k \to \mathcal{E}_k$$

is formally self-adjoint.
(*ii*) *As an operator on $\mathcal{N}(L_k)$, δQ_k^σ is conformally invariant and*

$$\delta Q_k^\sigma = G_k.$$

(*iii*) *Operating on $\mathcal{E}^{k-1}$, we have*

$$2(\ell + 2)\delta Q_k^\sigma d = L_{k-1}.$$

(*iv*) *As an operator on closed k-forms*

$$Q_k^\sigma : \mathcal{C}^k \to \mathcal{E}_k,$$

Q_k^σ has the conformal transformation law

$$Q_k^{\hat{\sigma}} u = Q_k^\sigma u + L_k(\Upsilon u)$$

where Υ is a smooth function and $\hat{\sigma} = e^{-\Upsilon}\sigma$.
(*v*). *$Q_0^\sigma 1$ is the Branson Q-curvature.*

This theorem is proved in the last part of Section 4.1. Note that from part (*ii*) above we see that acting on $\mathcal{N}(L_k)$, Q_k^σ gives the operator $\tilde{M}_{k-1}$ of Theorem 2.4. On $\mathcal{C}^k \subseteq \mathcal{N}(L_k)$, we have $(n - 2k + 4)Q_k^\sigma = M_{k-1}$. That this is conformally invariant as an operator $\mathcal{C}^k \to \mathcal{C}_k$ is immediate from part (*iv*) since $L_k : \mathcal{E}^k \to \mathcal{R}(\delta) \subset \mathcal{E}_k$. From our observations concerning M_k after Corollary 2.7 it is clear that, as an operator on $\mathcal{E}^k$ with $k < n/2$, Q_k^σ is **not** of the form (conformally invariant operator) $+ \delta U + V d$, where U and V are differential operators. Using the tools we develop below, it is easy to verify that $Q^{\sigma}_{n/2}$ is a multiple of the identity.

A celebrated property of the Q-curvature is that its integral is conformally invariant. In the next result we observe that there is a somewhat stronger invariance result, in that one can integrate invariantly against the null space $\mathcal{N}(L_0)$. This property is generalised by the operators Q^{σ}_k.

Theorem 2.9. *(i) As an operator between $\mathcal{C}^k$ and $\mathcal{E}_k/\mathcal{R}(L_k)$, Q^{σ}_k is conformally invariant. Further restricting to $\mathcal{H}^k \subset \mathcal{C}^k$, we obtain a conformally invariant operator*

$$Q_k : \mathcal{H}^k \to H^L_k(M),$$

where $\mathcal{H}^k := \mathcal{N}(G_k : \mathcal{C}^k \to \mathcal{E}_{k-1})$.
(ii) On compact manifolds, Q^{σ}_k gives a conformally invariant pairing between $\mathcal{N}(L_k)$ and $\mathcal{C}^k$ by

$$(u, w) \mapsto \langle u, Q_k w \rangle$$

for $w \in \mathcal{C}^k$ and $u \in \mathcal{N}(L_k)$. On compact conformal manifolds, the same integral formula determines a pairing between $H^k_L(M)$ and $\mathcal{H}^k$ by taking $w \in \mathcal{H}^k$ and u any representative of $[u] \in H^L_k(M)$.

Proof: The first statement is a trivial consequence of part *(iii)* of the previous theorem. Now suppose that $u \in \mathcal{H}^k$. By part *(ii)* of Theorem 2.8, $\delta Q^{\sigma}_k u = G_k u = 0$. So the conformally invariant map $Q^{\sigma}_k : \mathcal{C}^k \to \mathcal{E}_k/\mathcal{R}(L_k)$ descends to a well-defined map $Q_k : \mathcal{H}^k \to H^L_k(M)$. (In view of the conformal invariance we omit the argument σ from Q_k here.) This establishes part *(i)*. For part *(ii)*, the first statement follows from the pairing (2), the first statement in part *(i)* and the fact that L_k is formally self-adjoint. The second can be deduced from this, or follows immediately from part *(i)* and the earlier observation that (2) induces a conformally invariant pairing between $H^k_L(M)$ and $H^L_k(M)$. $\square$

Recall that, from the factorisation $L_k = \delta M_k d$, there is a natural (conformally invariant) surjection $\mathcal{E}_k/\mathcal{R}(L_k) \to \mathcal{C}_k$ inducing the map $H^L_k(M) \to H_k(M)$. Thus from part *(i)* above, Q^{σ}_k induces a conformally invariant map $Q_k : \mathcal{C}^k \to \mathcal{C}_k$. Since $\mathcal{C}^k \subseteq \mathcal{N}(L_k)$, Q^{σ}_k induces a conformally invariant pairing of $\mathcal{C}^k$ with itself by restriction of the pairing in part *(ii)* above. (In the compact Riemannian setting the latter is equivalent to the map $Q_k : \mathcal{C}^k \to \mathcal{C}_k$.) Similarly the composition of the displayed map in part *(i)* with $H^L_k(M) \to H_k(M)$ gives a conformally invariant map into de Rham cohomology, $\mathcal{H}^k \to H_k(M)$, and the pairing just described descends to a conformally invariant pairing between $H^k(M)$ and $\mathcal{H}^k$. We summarise this in:

Corollary 2.10. *Q_k induces conformally invariant maps $\mathcal{C}^k \to \mathcal{C}_k$ and $\mathcal{H}^k \to H_k(M)$. In the compact setting, Q_k induces conformally invariant pairings of $\mathcal{C}^k$ with itself, and of $H^k(M)$ and $\mathcal{H}^k$.*

As an application of these results we can now show that on compact Riemannian manifolds $\dim \mathcal{H}^k \geq \dim H^k(M)$. Combining the map $\mathcal{H}^k \to$

$H_k(M)$ just discussed with the composition of $H^k_L(M) \to H^k(M)$ and the map $Q_k : \mathcal{H}^k \to H^L_k(M)$ of Theorem 2.9 part (*i*), we obtain a conformally invariant map $I : \mathcal{H}^k \to H^k(M) \oplus H_k(M)$ given by

$$\omega \mapsto ([\omega], [Q^\sigma_k \omega]).$$

Clearly ω is killed by this map if and only if ω is both exact and $Q^\sigma_k\omega$ is in $\mathcal{R}(\delta)$. Thus the space

$$\mathcal{B}^k := \{d\varphi \mid Q^\sigma_k d\varphi \in \mathcal{R}(\delta)\}$$

is conformally invariant and

$$I : \mathcal{H}^k/\mathcal{B}^k \to H^k(M) \oplus H_k(M)$$

is injective. It turns out that the domain space here has the same dimension as $H^k(M)$. To show this we explicitly identify a space which, via the pairing (2), is its vector space dual. Let

$$\mathcal{H}_k := \{\xi \in \mathcal{E}_k \mid \delta\xi = \delta Q^\sigma_k \eta \text{ for some } \eta \in \mathcal{C}^k\},$$

and

$$\mathcal{B}_k := \{\xi \in \mathcal{E}_k \mid \xi - Q^\sigma_k d\varphi \in \mathcal{R}(\delta) \text{ for some } \varphi \in \mathcal{E}^{k-1}\}.$$

The space $\mathcal{H}_k$ is conformally stable since from Theorem 2.8 part (*ii*), $\delta Q^\sigma_k = G_k$ is conformally invariant on $\mathcal{N}(L_k) \supseteq \mathcal{C}^k$. The conformal invariance of $\mathcal{B}_k$ is immediate from part (*iii*) of the same theorem.

Next let us fix a conformal scale σ. Then we have the map

$$P : \mathcal{C}^k \oplus \mathcal{N}(\delta) \to \mathcal{H}_k \quad \text{given by} \quad (\eta, \xi) \mapsto \xi - Q^\sigma_k \eta.$$

This is surjective since for $\xi \in \mathcal{H}^k$ there is $\eta \in \mathcal{C}^k$ such that $\delta\xi = \delta Q^\sigma_k\eta$, and so $(-\eta, \xi - Q^\sigma_k\eta)$ is a pre-image of ξ. Now if $\eta = d\varphi$ and $\xi = \delta\rho$ then $P(\eta,\xi) = \delta\rho - Q^\sigma_k d\varphi$ which is in $\mathcal{B}_k$. Thus P descends to a well-defined surjective map $H^k(M) \oplus H_k(M) \to \mathcal{H}_k/\mathcal{B}_k$. It is clear that $\ker(P) = \mathrm{Im}(I)$ so, in summary, in a conformal scale we have an exact sequence

$$0 \to \mathcal{H}^k/\mathcal{B}^k \xrightarrow{I} H^k(M) \oplus H_k(M) \xrightarrow{P} \mathcal{H}_k/\mathcal{B}_k \to 0, \tag{8}$$

from which it is clear that $\dim(\mathcal{H}_k/\mathcal{B}_k)$ is finite. The following result shows that $\dim(\mathcal{H}_k/\mathcal{B}_k) = \dim(\mathcal{H}^k/\mathcal{B}^k) = b^k := \dim H^k(M)$.

Theorem 2.11. *For compact even dimensional Riemannian conformal manifolds and each $k = 0, 1, \cdots, n/2 - 1$, the conformally invariant pairing between $\mathcal{H}^k$ and $\mathcal{H}_k(M)$ given by the restriction of (2) descends to a well-defined conformally invariant perfect pairing of $\mathcal{H}^k/\mathcal{B}^k$ with $\mathcal{H}_k/\mathcal{B}_k$.*

Proof: The conformal invariance of the pairing between $\mathcal{H}^k$ and $\mathcal{H}_k$ is immediate from the invariance of the pairing (2). It is clear that if this descends as claimed then the result is an invariant pairing. Note that for $k = 0$ we have by construction $\mathcal{H}^0 = H^0(M)$ and $\mathcal{H}_0 = H_0(M)$ and so the result holds trivially. Henceforth we assume $k \geq 1$ and fix a conformal scale σ.

Note that if $d\varphi \in \mathcal{B}^k$ and $\xi \in \mathcal{H}_k$ we have

$$\langle d\varphi, \xi\rangle = \langle \varphi, \delta\xi\rangle = \langle \varphi, \delta Q^\sigma_k\eta\rangle = \langle Q^\sigma_k d\varphi, \eta\rangle = \langle \delta\mu, \eta\rangle = \langle \mu, d\eta\rangle = 0,$$

where we have used that Q_k^σ is formally self-adjoint. Thus via the pairing, $\mathcal{B}^k$ annihilates $\mathcal{H}_k$. Now suppose that $\omega \in \mathcal{H}^k$ and $\xi \in \mathcal{B}_k$. Then there is a pair $(\varphi, \rho) \in \mathcal{E}^{k-1} \oplus \mathcal{E}_{k+1}$ so that

$$\langle \omega, \xi \rangle = \langle \omega, Q_k d\varphi - \delta\rho \rangle = \langle \delta Q_k^\sigma \omega, \varphi \rangle - \langle d\omega, \rho \rangle = 0,$$

since $w \in \mathcal{H}^k$ means that $d\omega = 0$ and $\delta Q_k^\sigma \omega = G_k \omega = 0$. So $\mathcal{B}_k$ annihilates $\mathcal{H}^k$ and $\langle \cdot, \cdot \rangle$ descends to a bilinear function on $(\mathcal{H}^k/\mathcal{B}^k) \times (\mathcal{H}_k/\mathcal{B}_k)$ as claimed. It remains to show the pairing is perfect. Suppose that $\omega \in \mathcal{H}^k$ satisfies $\langle \omega, \xi \rangle = 0$ for all $\xi \in \mathcal{H}_k$. Then in particular $\langle \omega, \xi \rangle = 0$ for any $\xi \in \mathcal{N}(\delta) \subset \mathcal{E}_k$. Thus by Poincaré duality $\omega = d\varphi$ for some $\varphi \in \mathcal{E}^{k-1}$. Now consider $\langle \eta, Q_k \omega \rangle$ where $\eta \in \mathcal{C}^k$. Note that $Q_k \eta \in \mathcal{H}_k$ and so, by the assumption, $\langle \omega, Q_k^\sigma \eta \rangle = 0$. But Q_k^σ is formally self-adjoint and this implies $\langle \eta, Q_k^\sigma \omega \rangle = 0$. Again by Poincaré duality, since η is an arbitrary element of $\mathcal{C}^k$, this implies $Q_k^\sigma \omega \in \mathcal{R}(\delta)$. So the pair $(\omega, Q_k^\sigma \omega)$ vanishes in $H^k(M) \oplus H_k(M)$.

Finally suppose that $\xi \in \mathcal{H}_k$ satisfies $\langle \omega, \xi \rangle = 0$ for all $\omega \in \mathcal{H}^k$. Observe that by standard de Rham Hodge theory $d\omega = 0 \Leftrightarrow \Delta^\ell d\omega = 0$ where Δ is the form Laplacian $\delta d + d\delta$ and $\ell = n/2 - k$. This gives an alternative description of $\mathcal{H}^k$,

$$\mathcal{H}^k = \{\omega \in \mathcal{E}^k \mid \Delta^\ell d\omega = 0 \ \text{ and } \ \delta Q_k^\sigma \omega = 0\}.$$

So by the assumption ξ is orthogonal to the kernel in $\mathcal{E}^k$ of the operator pair $(\delta Q_k^\sigma, \Delta^\ell d) : \mathcal{E}^k \to \mathcal{E}_{k-1} \oplus \mathcal{E}_{k+1}$ (ignoring conformal weights as we may since we have fixed a conformal scale). Put another way, ξ is orthogonal to kernel of the adjoint of the operator

$$(Q_k^\sigma d, \delta\Delta^\ell) : \mathcal{E}^{k-1} \oplus \mathcal{E}^{k+1} \to \mathcal{E}^k, \qquad \begin{pmatrix} \varphi \\ \mu \end{pmatrix} \mapsto Q_k^\sigma d\varphi + \delta\Delta^\ell \mu. \tag{9}$$

But from Theorem 2.8 part *(iii)* it follows that the leading term of $Q_k^\sigma d$ is, up to a non-vanishing constant multiple, $(d\delta)^\ell d = \Delta^\ell d$. Thus there is a number α so that

$$\left(Q_k^\sigma d, \delta\Delta^\ell\right) \begin{pmatrix} \alpha\delta \\ d \end{pmatrix} = \Delta^{\ell+1} + \mathrm{LOT} : \mathcal{E}^k \to \mathcal{E}^k.$$

(Here and below "LOT" is an abbreviation for "lower order terms".) Thus the operator (9) is surjectively elliptic, and in the resulting Hodge decomposition

$$\mathcal{E}^k = \mathcal{R}(Q_k^\sigma d, \delta\Delta^\ell) \oplus \mathcal{N}\begin{pmatrix} \delta Q_k^\sigma \\ \Delta^\ell d \end{pmatrix},$$

ξ must be in the first summand. Thus $\xi \in \mathcal{B}_k$. $\square$

By the finite dimensionality of the spaces $\mathcal{H}^k/\mathcal{B}^k$ and $\mathcal{H}_k/\mathcal{B}_k$, the duality established above and the exact sequence (8) we have the following.

Corollary 2.12. *For any compact even dimensional Riemannian conformal manifold and $k = 0, 1, \cdots, n/2 - 1$ we have*

$$\dim(\mathcal{H}^k/\mathcal{B}^k) = b^k = \dim(\mathcal{H}_k/\mathcal{B}_k).$$

Thus we have

$$\dim \mathcal{H}^k = b^k + \dim \mathcal{B}^k \quad \text{and} \quad \dim \mathcal{B}^k \leq \dim(H_L^{k-1}(M)/H^{k-1}(M)).$$

Note that the final conclusion is clear, since from the definition of $\mathcal{B}^k$ and part *(iii)* of Theorem 2.8, $d\varphi \in \mathcal{B}^k \Rightarrow L_{k-1}\varphi = 0$.

From Theorem 2.11 we see that via (2) $\mathcal{H}_k/\mathcal{B}_k$ may identified with the vector space dual to $\mathcal{H}^k/\mathcal{B}^k$. It is straightforward to verify that via (2) $H_k(M)$ may be similarly identified with the vector space dual to $H^k(M)$. Thus we have the following result.

Corollary 2.13. *For any even dimensional compact Riemannian conformal manifold and* $k = 0, 1, \cdots, n/2 - 1$, *we have*

$$\begin{aligned} &\mathcal{H}^k/\mathcal{B}^k \to H^k(M) \text{ is injective} \quad \Leftrightarrow \quad \mathcal{H}^k/\mathcal{B}^k \to H^k(M) \text{ is surjective} \\ \Leftrightarrow \quad &H_k(M) \to \mathcal{H}_k/\mathcal{B}_k \text{ is injective} \quad \Leftrightarrow \quad H_k(M) \to \mathcal{H}_k/\mathcal{B}_k \text{ is surjective.} \end{aligned}$$

Theorem 2.11, Corollary 2.12 and Corollary 2.13 generalise, respectively, Theorem 4.1, Corollary 4.2 and Corollary 4.3, of Eastwood and Singer [23], which deal with 1-forms in dimension 4. Our treatment of these last three results has been heavily influenced by their development of that case. For each k, the (equivalent) conditions of Corollary 2.13 constitute some conformally invariant condition on the Riemannian conformal manifold, or Riemannian manifold, that we term $(k-1)$-*regularity*. This generalises the notion of 'regular' for $k = 1$, $n = 4$ discussed in [23]. Similarly the conformally invariant hypothesis $H^k(M) = H^k_L(M)$ of Theorem 2.6, which we term *strong* k-*regularity*, generalises the dimension 4 notion of 'strong regularity' in [44].

Of course if $H^{k+1}(M)$ vanishes, then clearly M is k-regular; this is the trivial case. We expect that k-regularity, for each k, should hold generically in some appropriate sense for compact conformal Riemannian manifolds. Note that in $\mathcal{E}^k$ we have the subspace inequality

$$\mathcal{B}^{k+1} = \{d\varphi \mid Q_{k+1}d\varphi \in \mathcal{R}(\delta)\} \subseteq \{d\varphi \mid L_k\varphi = (n-2k+2)\delta Q_{k+1}d\varphi = 0\}.$$

By Proposition 2.5, k-regularity is the case of equality in this inequality, while strong k-regularity is the assertion that the space on the right side vanishes. In particular, strong k-regularity implies k-regularity. In dimension 4, [23] shows that Einstein manifolds are 0-regular, and also gives an example of a manifold which fails to be strongly 0-regular.

2.1. **Extensions to the theory and non-orientable manifolds.** No assumptions have been made above concerning the orientability of M. In the general case that M may have non-orientable components there is further information to be extracted via an extension to the theory. We present this rather concisely since at one level this extension arises rather simply from the machinery described above and given in the following sections.

In the case that M is orientable the conformal Hodge star operator gives an isomorphism $\star : \mathcal{E}^k[w] \overset{\cong}{\to} \mathcal{E}^{n-k}[w+n-2k] = \mathcal{E}_{n-k}[w]$. Since up to a sign $\star\star$ is the identity, it follows that there are operators $L^{n-k}_{\ell,\star} : \mathcal{E}_{n-k}[w] \to \mathcal{E}^{n-k}[-w]$ where

$$L^{n-k}_{\ell,\star} = \star L^{\ell}_{k} \star \quad \text{locally.} \tag{10}$$

We have written "locally" since we also define the operators $L^{n-k}_{\ell,\star}$ on non-orientable manifolds by requiring that (10) hold on every orientable neighbourhood, for any choice of orientation on a given neighbourhood. Since the local choice of orientation only affects the sign of $\star$ it is clear that each $L^{n-k}_{\ell,\star}$ is well defined. When M is orientable the operators $L^{n-k}_{\ell,\star}$ are, by construction, equivalent to the L^{ℓ}_{k}. Otherwise there need not be an isomorphism between $\mathcal{E}^k$ and $\mathcal{E}_{n-k}$ and so these are new formally self-adjoint conformally invariant operators. These are non-trivial and natural for k and ℓ as in Theorem 2.1. When $w := k+\ell-n/2 = 0$ we use the alternative notation $L^{n-k}_{\star} := L^{n-k}_{\ell,\star}$ and we have the following result.

Theorem 2.14. *If n is even and $w=0$, we have that*

$$L^{n-k}_{\star} = dM^{n-k}_{\star}\delta,$$

where, up to a constant multiple, $M^{n-k}_{\star}$ is given locally (and in a choice of conformal scale σ) by $\star Q^{\sigma}_{k+1}\star$.

On Riemannian conformal manifolds the following holds. The operator $L^{n-k}_{\ell,\star}$ is elliptic if and only if $k \neq n/2 \pm \ell$, and it is positively elliptic if and only if $k \notin [n/2-\ell, n/2+\ell]$. For each k the differential operator sequence

$$\mathcal{E}_n \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{E}_{n-k+1} \xrightarrow{\delta} \mathcal{E}_{n-k} \xrightarrow{L^{n-k}_{\star}} \mathcal{E}^{n-k} \xrightarrow{d} \mathcal{E}^{n-k+1} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{n} \tag{11}$$

is an elliptic complex.

Proof: On orientable neighbourhoods we have $L^{n-k}_{\star} = 2(\ell+1)\star\delta Q^{\sigma}_{k+1}d\star$, from Theorem 2.8. Recalling also that as operators on k'-forms we have $\star d = (-1)^{k'+1}\delta\star$ and $d\star = (-1)^{k'}\star\delta$ the first result is proved. The remaining facts follow immediately from the corresponding results for the L_k. $\square$

That the k^{th} *de Rham codetour complex* (11) is not equivalent to the detour complex (4) is already clear by taking the cohomology at $0 \to \mathcal{E}_n \xrightarrow{\delta}$.

Proposition 2.15. *On compact Riemannian conformal manifolds, $\dim H_n(M)$ is the number of oriented components in M.*

Proof: Suppose that, on a connected component M' of M, φ is a non-zero section of $\mathcal{E}_n$ annihilated by δ. Pick a metric g from the conformal class and identify $\mathcal{E}_n$ with $\mathcal{E}^n$ on M'. Consider an arbitrary orientable neighbourhood U in M', let V^g be a choice of volume form on U consistent with g, and observe that $\varphi = aV^g$ for some function a. Since $\delta V^g = 0$ and V^g is co-closed, it follows that $\iota(da)V^g = 0$. Thus a is constant on U. Now V^g is preserved by the Levi-Civita connection and so φ is covariantly constant on U. Since U was arbitrary, it follows that φ is covariantly constant and so nowhere vanishing on M'. Thus M' is orientable. $\square$

As a result, $\dim H_n(M)$ is in general less than $\dim H^0(M)$. Note that from the perfect pairing between $H_n(M)$ and $H^n(M)$ the proof above recovers the well known result that $\dim H^n(M)$ is the number of connected components.

The main point now is that there is a theory for the operators $L^{n-k}_{\star}$ which closely parallels the theory for the operators L_k. For example for $k \leq n/2-1$ each $L^{n-k}_{\star}$ has an extension to a quasi-Laplacian operator $\mathbb{L}^{n-k}_{\star} : \mathcal{E}_{n-k} \to \mathcal{G}^{n-k}_{\star}$ where $\mathcal{G}^{n-k}_{\star}$ is the section space of a bundle with a composition series $E^{n-k} \oplus E^{n-k+1}$. There is an analogue for Theorem 2.4. In this, for example, (ignoring non-zero constant scalar multiples) the analogue of G_k is $dQ^{\sigma,\star}_{n-k}$ where, for each choice of conformal scale σ, $Q^{\sigma,\star}_{n-k}$ is the unique operator given on oriented neighbourhoods by $\star Q^{\sigma}_{k}\star$. $Q^{\sigma,\star}_{n-k}$ is another type of Q-operator and satisfies the obvious analogue of Theorem 2.8 parts (*i–iv*). The existence of the bundle $\mathcal{G}^{n-k}_{\star}$, and the other local results we have mentioned here, follow trivially from earlier results since the operators $L^{n-k}_{\star}$ and L_k are, by construction, locally equivalent. On the other hand it is also straightforward to directly develop these results using the calculus in the later sections and some straightforward variations on the constructions there. See in particular the remarks on pages 30 and 39 where the key tools are described.

Of course when M is not orientable there is no result corresponding to part (*v*) of Theorem 2.8 and this is an important distinction between the cases at a global level. There are obvious analogues for all the cohomological theorems. We may define $H^{L_\star}_{n-k}(M)$ and $H^{n-k}_{L_\star}(M)$ for the cohomology of (11) at, respectively, the bundles $\mathcal{E}_{n-k}$ and $\mathcal{E}^{n-k}$. On the other hand we may define the space $\mathcal{H}^{\star}_{n-k}$ of conformal harmonics to be the null space of $dQ^{\sigma,\star}_{n-k}$ as an operator on $\mathcal{N}(\delta : \mathcal{E}_{n-k} \to \mathcal{E}_{n-k+1})$. These satisfy analogues of Proposition 2.2, Proposition 2.5, Theorem 2.6, Theorem 2.11 and the corollaries of the latter. The Q-operator $Q^{\sigma,\star}_{n-k}$ satisfies the analogue of Theorem 2.9.

3. The ambient construction and tractor calculus

The basic relationship between the Fefferman-Graham ambient metric construction and tractor calculus is described in [14]. We review this briefly and establish our notation before developing an exterior calculus for the ambient manifold and for tractor fields.

Let $\pi : \mathcal{Q} \to M$ be a conformal structure of signature (p, q). Let us use ρ to denote the $\mathbb{R}_+$ action on $\mathcal{Q}$ given by $\rho(s)(x, g_x) = (x, s^2 g_x)$. An *ambient manifold* is a smooth $(n + 2)$-manifold $\tilde{M}$ endowed with a free $\mathbb{R}_+$–action ρ and an $\mathbb{R}_+$–equivariant embedding $i : \mathcal{Q} \to \tilde{M}$. We write $\boldsymbol{X} \in \mathfrak{X}(\tilde{M})$ for the fundamental field generating the $\mathbb{R}_+$–action, that is for $f \in C^\infty(\tilde{M})$ and $u \in \tilde{M}$ we have $\boldsymbol{X} f(u) = (d/dt) f(\rho(e^t)u)|_{t=0}$.

If $i : \mathcal{Q} \to \tilde{M}$ is an ambient manifold, then an *ambient metric* is a pseudo–Riemannian metric $\boldsymbol{h}$ of signature $(p+1, q+1)$ on $\tilde{M}$ such that the following conditions hold:

(i) The metric $\boldsymbol{h}$ is homogeneous of degree 2 with respect to the $\mathbb{R}_+$–action, i.e. if $\mathcal{L}_{\boldsymbol{X}}$ denotes the Lie derivative by $\boldsymbol{X}$, then we have $\mathcal{L}_{\boldsymbol{X}} \boldsymbol{h} = 2\boldsymbol{h}$. (I.e. $\boldsymbol{X}$ is a homothetic vector field for h.)

(ii) For $u = (x, g_x) \in \mathcal{Q}$ and $\xi, \eta \in T_u\mathcal{Q}$, we have $\boldsymbol{h}(i_*\xi, i_*\eta) = g_x(\pi_*\xi, \pi_*\eta)$.

To simplify the notation we will usually identify $\mathcal{Q}$ with its image in $\tilde{M}$ and suppress the embedding map i. To link the geometry of the ambient manifold to the underlying conformal structure on M one requires further conditions. In [25] Fefferman and Graham treat the problem of constructing a formal power series solution along $\mathcal{Q}$ for the Goursat problem of finding an ambient metric $\boldsymbol{h}$ satisfying (i) and (ii) and the condition that it be Ricci flat, i.e. $\mathrm{Ric}(\boldsymbol{h}) = 0$. A key result is Theorem 2.1 of their paper: If n is odd, then up to a $\mathbb{R}_+$-equivariant diffeomorphism fixing $\mathcal{Q}$, there is a unique power series solution for $\boldsymbol{h}$ satisfying (i), (ii) and $\mathrm{Ric}(\boldsymbol{h}) = 0$. If n is even, then up to a $\mathbb{R}_+$-equivariant diffeomorphism fixing $\mathcal{Q}$ and the addition of terms vanishing to order $n/2$, there is a unique power series solution for $\boldsymbol{h}$ satisfying

(i), (ii);
$\mathrm{Ric}(\boldsymbol{h})$ vanishes to order $n/2 - 2$ along $\mathcal{Q}$;
tangential components of $\mathrm{Ric}(\boldsymbol{h})$ vanish to order $n/2 - 1$ along $\mathcal{Q}$.

It turns out that in metrics satisfying these conditions $Q := \boldsymbol{h}(\boldsymbol{X}, \boldsymbol{X})$ is a defining function for $\mathcal{Q}$ and $2\boldsymbol{h}(\boldsymbol{X}, \cdot) = dQ$ to all orders in odd dimensions and up to the addition of terms vanishing to order $n/2$ in even dimensions. It is straightforward to show [32, 36] that one can extend the solution slightly in even dimensions to obtain

$$\text{(iii)} \quad \mathrm{Ric}(\boldsymbol{h}) = 0 \begin{cases} \text{to all orders if } n \text{ is odd,} \\ \text{up to the addition of terms vanishing} \\ \qquad \text{to order } n/2 - 1 \text{ if } n \text{ is even,} \end{cases}$$

with (i), (ii) and $\boldsymbol{h}(\boldsymbol{X}, \cdot) = \frac{1}{2} dQ$ to all orders in both dimension parities. Henceforth, unless otherwise indicated, the term ambient metric will mean an ambient manifold with metric satisfying all these conditions. We should point out that we only use the existence part of the Fefferman-Graham construction. The uniqueness of the operators we will construct is a consequence of the fact that they can be uniquely expressed in terms of the underlying conformal structure as we shall later explain. Finally we note that if M is locally conformally flat then there is a canonical solution, to all orders, to the ambient metric problem. This is the flat ambient metric. This is forced by (i–iii) in odd dimensions (see (26) and the proof of Lemma 3.6). But in even dimensions this extends the solution. When discussing the conformally flat case we assume this solution.

We write $\boldsymbol{\nabla}$ for the ambient Levi-Civita connection determined by $\boldsymbol{h}$ and use upper case abstract indices $A, B, \cdots$ for tensors on $\tilde{M}$. For example, if v^B is a vector field on $\tilde{M}$, then the ambient Riemann tensor will be denoted $\boldsymbol{R}_{AB}{}^{C}{}_{D}$ and defined by $[\boldsymbol{\nabla}_A, \boldsymbol{\nabla}_B] v^C = \boldsymbol{R}_{AB}{}^{C}{}_{D} v^D$. In this notation the ambient metric is denoted $\boldsymbol{h}_{AB}$ and with its inverse this is used to raise and lower indices in the usual way. We will not normally distinguish tensors related in this way even in index free notation; the meaning should be clear from the context. Thus for example we shall use $\boldsymbol{X}$ to mean both the Euler vector field $\boldsymbol{X}^A$ and the 1-form $\boldsymbol{X}_A = \boldsymbol{h}_{AB} \boldsymbol{X}^B$.

The condition $\mathcal{L}_{\boldsymbol{X}} \boldsymbol{h} = 2\boldsymbol{h}$ is equivalent to the statement that the symmetric part of $\boldsymbol{\nabla X}$ is $\boldsymbol{h}$. On the other hand, since $\boldsymbol{X}$ is exact, $\boldsymbol{\nabla X}$ is symmetric. Thus

$$\boldsymbol{\nabla X} = \boldsymbol{h}, \tag{12}$$

which in turn implies

$$\boldsymbol{X} \lrcorner \boldsymbol{R} = 0. \tag{13}$$

Equalities without qualification, as here, indicate that the results hold to all orders or identically on the ambient manifold.

3.1. **Tractor bundles.** Let $\tilde{\mathcal{E}}(w)$ denote the space of functions on $\tilde{M}$ which are homogeneous of degree $w \in \mathbb{R}$ with respect to the action ρ. That is $f \in \tilde{\mathcal{E}}(w)$ means that $\boldsymbol{X} f = wf$. Similarly a tensor field F on $\tilde{M}$ is said to be *homogeneous of degree* w if $\rho(s)^* F = s^w F$ or equivalently $\mathcal{L}_{\boldsymbol{X}} F = wF$. Just as sections of $\mathcal{E}[w]$ are equivalent to functions in $\tilde{\mathcal{E}}(w)|_{\mathcal{Q}}$ we will see that the restriction of homogeneous tensor fields to $\mathcal{Q}$ have interpretations on M.

On the ambient tangent bundle $T\tilde{M}$ we define an action of $\mathbb{R}_+$ by $s{\cdot}\xi := s^{-1}\rho(s)_*\xi$. The sections of $T\tilde{M}$ which are fixed by this action are those which are homogeneous of degree -1. Let us denote by $\boldsymbol{\mathcal{T}}$ the space of such sections and write $\boldsymbol{\mathcal{T}}(w)$ for sections in $\boldsymbol{\mathcal{T}} \otimes \tilde{\mathcal{E}}(w)$, where the $\otimes$ here indicates a tensor product over $\tilde{\mathcal{E}}(0)$. Along $\mathcal{Q}$ the $\mathbb{R}_+$ action on $T\tilde{M}$ is compatible with the $\mathbb{R}_+$ action on $\mathcal{Q}$, so defining $\mathbb{T}$ to be the quotient $(T\tilde{M}|_{\mathcal{Q}})/\mathbb{R}_+$, yields a rank $n+2$ vector bundle $\mathbb{T}$ over $\mathcal{Q}/\mathbb{R}_+ = M$. By construction, sections of $p : \mathbb{T} \to M$ are equivalent to sections from $\boldsymbol{\mathcal{T}}|_{\mathcal{Q}}$. We write $\mathcal{T}$ to denote the space of such sections.

Since the ambient metric $\boldsymbol{h}$ is homogeneous of degree 2 it follows that for vector fields ξ and η on $\tilde{M}$ which are homogeneous of degree -1, the function $\boldsymbol{h}(\xi, \eta)$ is homogeneous of degree 0 and thus descends to a smooth function on M. Hence $\boldsymbol{h}$ descends to a smooth bundle metric h of signature $(p+1, q+1)$ on $\mathbb{T}$.

Next we show that the space $\boldsymbol{\mathcal{T}}$ has a filtration reflecting the geometry of $\tilde{M}$. First observe that for $\varphi \in \tilde{\mathcal{E}}(-1)$, $\varphi \boldsymbol{X} \in \boldsymbol{\mathcal{T}}$. Restricting to $\mathcal{Q}$ this determines a canonical inclusion $E[-1] \hookrightarrow \mathbb{T}$ with image denoted by $\mathbb{V}$. Since $\boldsymbol{X}$ generates the fibres of $\pi : \mathcal{Q} \to M$ the smooth distinguished line subbundle $\mathbb{V} \subset \mathbb{T}$ reflects the inclusion of the vertical bundle in $T\tilde{M}|_{\mathcal{Q}}$. We write X for the canonical section in $\mathcal{T}[1]$ giving this inclusion. We define $\mathbb{F}$ to be the orthogonal complement of $\mathbb{V}$ with respect to h. Since $Q = \boldsymbol{h}(\boldsymbol{X}, \boldsymbol{X})$ is a defining function for $\mathcal{Q}$ it follows that X is null and so $\mathbb{V} \subset \mathbb{F}$. Clearly $\mathbb{F}$ is a smooth rank $n+1$ subbundle of $\mathbb{T}$. Thus $\mathbb{T}/\mathbb{F}$ is a line bundle and it is immediate from the definition of $\mathbb{F}$ that there is a canonical isomorphism $E[1] \cong \mathbb{T}/\mathbb{F}$ arising from the map $\mathbb{T} \to E[1]$ given by $V \mapsto h(X, V)$. Now recall $2\boldsymbol{h}(\boldsymbol{X}, \cdot) = dQ$, so the sections of $\boldsymbol{\mathcal{T}}$ corresponding to sections of $\mathbb{F}$ are just those that take values in $T\mathcal{Q} \subset T\tilde{M}|_{\mathcal{Q}}$. Finally we note that if $\tilde{\xi}$ and $\tilde{\xi}'$ are two lifts to $\mathcal{Q}$ of $\xi \in \mathfrak{X}(M)$ then they are sections of $T\mathcal{Q}$ which are homogeneous of degree 0 and with difference $\tilde{\xi} - \tilde{\xi}'$ taking values in the vertical subbundle. Since $\pi : \mathcal{Q} \to M$ is a submersion it follows immediately that $\mathbb{F}[1]/\mathbb{V}[1] \cong TM \cong T^*M[2]$ (where recall by our conventions $\mathbb{F}[1]$ means $\mathbb{F} \otimes E[1]$ etc.). Tensoring this with $E[-1]$ and combining this observation with our earlier results we can summarise the filtration of $\mathbb{T}$ by the composition series

$$\mathbb{T} = E[1] \oplus\!\!\!\!\!\!+\ T^*M[1] \oplus\!\!\!\!\!\!+\ E[-1]. \tag{14}$$

Next we show that the Levi-Civita connection $\boldsymbol{\nabla}$ of $\boldsymbol{h}$ determines a linear connection on $\mathbb{T}$. Since $\boldsymbol{\nabla}$ preserves $\boldsymbol{h}$ it follows easily that if $U \in \boldsymbol{\mathcal{T}}(w)$ and $V \in \boldsymbol{\mathcal{T}}(w')$ then $\boldsymbol{\nabla}_U V \in \boldsymbol{\mathcal{T}}(w + w' - 1)$. The connection $\boldsymbol{\nabla}$ is torsion free so $\boldsymbol{\nabla}_{\boldsymbol{X}} U - \boldsymbol{\nabla}_U \boldsymbol{X} - [\boldsymbol{X}, U] = 0$ for any tangent vector field U. Now $\boldsymbol{\nabla}_U \boldsymbol{X} = U$, so this simplifies to $\boldsymbol{\nabla}_{\boldsymbol{X}} U = [\boldsymbol{X}, U] - U$. If now $U \in \boldsymbol{\mathcal{T}}$, or equivalently $[\boldsymbol{X}, U] = -U$, then $\boldsymbol{\nabla}_{\boldsymbol{X}} U = 0$. The converse is clear and it follows that sections of $\boldsymbol{\mathcal{T}}$ may be characterised as those sections of $T\tilde{M}$ which are covariantly parallel along the integral curves of $\boldsymbol{X}$ (which on $\mathcal{Q}$ are exactly the fibres of π). These two results imply that $\boldsymbol{\nabla}$ determines a connection ∇ on $\mathbb{T}$. For $U \in \mathcal{T}$, let $\tilde{U}$ be the corresponding section of $\boldsymbol{\mathcal{T}}|_{\mathcal{Q}}$. Similarly a tangent vector field ξ on M has a lift to a field $\tilde{\xi} \in \boldsymbol{\mathcal{T}}(1)$, on $\mathcal{Q}$, which is everywhere tangent to $\mathcal{Q}$. This is unique up to adding $f\boldsymbol{X}$, where $f \in \tilde{\mathcal{E}}(0)$. We extend $\tilde{U}$ and $\tilde{\xi}$ smoothly and homogeneously to fields on $\tilde{M}$. Then we can form $\boldsymbol{\nabla}_{\tilde{\xi}} \tilde{U}$; this is clearly independent of the extensions. Since $\boldsymbol{\nabla}_{\boldsymbol{X}} \tilde{U} = 0$, the section $\boldsymbol{\nabla}_{\tilde{\xi}} \tilde{U}$ is also independent of the choice of $\tilde{\xi}$ as a lift of ξ. Finally, $\boldsymbol{\nabla}_{\tilde{\xi}} \tilde{U}$ is a section of $\boldsymbol{\mathcal{T}}(0)$ and so determines a section $\nabla_\xi U$ of $\mathbb{T}$ which only depends on U and ξ. It is easily verified that this defines a covariant derivative on $\mathbb{T}$ which, by construction, is compatible with the bundle metric h.

The ambient metric is conformally invariant; no choice of metric from the conformal class on M is involved in solving the ambient metric problem. Thus the bundle, metric and connection $(\mathbb{T}, h, \nabla)$ are by construction conformally invariant. On the other hand the ambient metric is not unique. Nevertheless it is straightforward to verify that ∇ satisfies the required non-degeneracy condition and curvature normalisation condition that lead to the following result.

Proposition 3.1. *The bundle and connection pair $(\mathbb{T}, \nabla)$, induced by $\boldsymbol{h}$, is a normal standard (tractor bundle, connection) pair.*

This is proved in [14]. (In fact it is shown there that to obtain the normal standard tractor bundle and connection it is sufficient to replace property (iii) of the ambient metric with the weaker condition that the tangential components of $\mathrm{Ric}(\boldsymbol{h})$ vanish along $\mathcal{Q}$.) From a standard tractor bundle and connection it is straightforward to construct a Cartan bundle $\mathcal{G}$ and connection ω from which the tractor bundle and connection arise as associated structures (via the defining representation of $\mathrm{SO}(p+1, q+1)$). The notion of normality of a tractor connection is equivalent to that on Cartan structure (see [16]). So although the ambient metric is not unique the induced tractor bundle structure $(\mathbb{T}, h, \nabla)$ is equivalent to a normal Cartan connection, and so is unique up to bundle isomorphisms preserving the filtration structure of $\mathbb{T}$, and preserving h and ∇.

In particular this means that given a choice of metric g from the conformal class the structure $(\mathbb{T}, h, \nabla)$ can be expressed in terms of T^*M, g and the Levi-Civita connection for g (which is also denoted ∇) by explicit formulae which we give below. In an abstract index notation TM is denoted E^a and E_a means T^*M; we write $\mathcal{E}^a$ and $\mathcal{E}_a$ for the corresponding section spaces. (We use the early part of the alphabet for abstract indices. In view of this and context $\mathcal{E}^a$ should not be confused with the space of k-forms $\mathcal{E}^k$). Similarly the section spaces of the tractor bundle and its dual can also be denoted $\mathcal{T}^A$ and $\mathcal{T}_A$. It is often convenient choose a metric g from the conformal class which determines [1, 15] a canonical splitting of the composition series (14). Via this the semi-direct sums $\oplus\!\!\!\!+$ in that series get replaced by direct sums $\oplus$, and we introduce g-dependent sections

$Z^{Ab} \in \mathcal{T}^{Ab}[-1]$ and $Y^A \in \mathcal{T}^A[-1]$ that describe this decomposition of $\mathbb{T}$ into the direct sum $\mathbb{T}^A = E[1] \oplus E_a[1] \oplus E[-1]$. A section $V \in \mathcal{T}$ then corresponds to a triple (σ, μ, ρ) of sections from the direct sum according to $V^A = Y^A\sigma + Z^{Ab}\mu_b + X^A\rho$, and in these terms the tractor metric is given by $h(V, V) = \boldsymbol{g}^{ab}\mu_a\mu_b + 2\sigma\rho$. The sections Y and Z are defined in terms of the Levi-Civita connection, and have ambient space equivalents which will be partially described below. If $\hat{Y}^A$ and $\hat{Z}^A{}_b$ are the corresponding quantities in terms of the metric $\hat{g} = e^{2\omega}g$ then we have

$$\hat{Z}^{Ab} = Z^{Ab} + \Upsilon^b X^A, \quad \hat{Y}^A = Y^A - \Upsilon_b Z^{Ab} - \tfrac{1}{2}\Upsilon_b\Upsilon^b X^A,$$

where $\Upsilon := d\omega$. In terms of this splitting for g the tractor connection is given by

$$\nabla_a X_A = Z_{Aa}\,, \quad \nabla_a Z_{Ab} = -\mathsf{P}_{ab}X_A - Y_A\boldsymbol{g}_{ab}\,, \quad \nabla_a Y_A = \mathsf{P}_{ab}Z_A{}^b, \tag{15}$$

(see [1, 32]) where P_{ab} is a trace adjustment of a constant multiple of $\mathrm{Ric}(g)$ known as the Schouten (or Rho) tensor. (Note that in (15), ∇ is the coupled tractor–Levi-Civita connection.)

The bundle of k-form tractors $\mathbb{T}^k$ is the $k^{\underline{\text{th}}}$ exterior power of the bundle of standard tractors. This has a composition series which, in terms of section spaces, is given by

$$\mathcal{T}^k = \Lambda^k\mathcal{T} \cong \mathcal{E}^{k-1}[k] \mathbin{\oplus\!\!\!\!\!+} \{\mathcal{E}^k[k] \oplus \mathcal{E}^{k-2}[k-2]\} \mathbin{\oplus\!\!\!\!\!+} \mathcal{E}^{k-1}[k-2]. \tag{16}$$

Given a choice of metric g from the conformal class there is a splitting of this composition series corresponding to the splitting of $\mathcal{T}$ as mentioned above. Relative to this, a typical k-form tractor field F corresponds to a 4-tuple $(\sigma, \mu, \varphi, \rho)$ of sections of the direct sum (obtained by replacing each $\mathbin{\oplus\!\!\!\!\!+}$ with $\oplus$ in (16)) and we write

$$F = \mathbb{Y}^k\!\cdot\!\sigma + \mathbb{Z}^k\!\cdot\!\mu + \mathbb{W}^k\!\cdot\!\varphi + \mathbb{X}^k\!\cdot\!\rho,$$

where '$\cdot$' is the usual pointwise form inner product in the tensor arguments,

$$\varphi\!\cdot\!\psi = \frac{1}{p!}\varphi^{a_1\cdots a_p}\psi_{a_1\cdots a_p} \quad \text{for } p\text{-forms},$$

and for $k > 1$, if $\underline{\wedge}$ is the wedge product in the tractor arguments,

$$\mathbb{Z}^k = Z\underline{\wedge}\mathbb{Z}^{k-1}, \quad \mathbb{X}^k = X\underline{\wedge}\mathbb{Z}^{k-1}, \quad \mathbb{Y}^k = Y\underline{\wedge}\mathbb{Z}^{k-1}, \quad \mathbb{W}^k = Y\underline{\wedge}X\underline{\wedge}\mathbb{Z}^{k-2}. \tag{17}$$

(By convention, $\mathbb{Z}^0 = 1$ and $\mathbb{Z}^{-1} = 0$.) Note that because Z is vector valued, $Z\underline{\wedge}Z$ does not vanish, though expressions like $X\underline{\wedge}X$ and $Y\underline{\wedge}Y$ do vanish. The form tractor bundles $\mathbb{T}^k$ are non-zero for $k = 0, \ldots, n+2$; $\mathbb{Z}^k$ vanishes for $k \geq n+1$; $\mathbb{W}$ vanishes for $k \leq 1$; and $\mathbb{X}^k$, $\mathbb{Y}^k$ vanish for $k = 0, n+2$. $\mathbb{X}^k$ is an invariant section, while $\mathbb{Y}^k$ depends on a choice of scale. $\mathbb{Z}^k$ depends on a choice of scale unless $k = 0$; $\mathbb{W}$ depends on a choice of scale unless $k = n+2$.

An invariant metric on $\mathbb{T}^k$ is

$$\langle(\nu, \mu, \varphi, \rho), (\tilde{\nu}, \tilde{\mu}, \tilde{\varphi}, \tilde{\rho})\rangle = \nu\!\cdot\!\tilde{\rho} + \rho\!\cdot\!\tilde{\nu} + \mu\!\cdot\!\tilde{\mu} - \varphi\!\cdot\!\tilde{\varphi}. \tag{18}$$

In fact, this is the restriction of the ambient k-form metric

$$\Phi \bullet \Psi := \frac{1}{k!}\Phi^{A_1\cdots A_k}\Psi_{A_1\cdots A_k}, \tag{19}$$

in the sense that after restricting to homogeneous ambient k-forms along $\mathcal{Q}$ and identifying these with k-form tractors we obtain a $\bullet$ on the latter. Choosing a conformal scale we observe that

$$(\mathbb{Y}^k \cdot \nu + \mathbb{Z}^k \cdot \mu + \mathbb{W}^k \cdot \varphi + \mathbb{X}^k \cdot \rho) \bullet (\mathbb{Y}^k \cdot \tilde{\nu} + \mathbb{Z}^k \cdot \tilde{\mu} + \mathbb{W}^k \cdot \tilde{\varphi} + \mathbb{X}^k \cdot \tilde{\rho})$$

reduces to the expression in (18). This follows in turn from the formulae

$$Z^{Ab} Z_{Ac} = \delta^b{}_c, \quad X^A Y_A = 1,$$

with all other quadratic contractions of X, Y, Z vanishing, together with formula (21) below for the $\triangle$ with a tractor-one-form.

If α and β are one-forms and φ is a form (or if these objects are form-densities), let

$$E(\alpha \otimes \beta)\varphi := \alpha \otimes \varepsilon(\beta)\varphi, \quad I(\alpha \otimes \beta)\varphi := \alpha \otimes \iota(\beta)\varphi,$$

and extend from simple tensors $\alpha \otimes \beta$ to arbitrary 2-tensors by linearity. The formulae for the covariant derivatives of X, Y, Z at a scale imply that

$$\begin{aligned} \nabla \mathbb{X} &= -E(\boldsymbol{g})\mathbb{W} + I(\boldsymbol{g})\mathbb{Z}, & \nabla \mathbb{Z} &= -E(\mathsf{P})\mathbb{X} - E(\boldsymbol{g})\mathbb{Y}, \\ \nabla \mathbb{W} &= I(\mathsf{P})\mathbb{X} - I(\boldsymbol{g})\mathbb{Y}, & \nabla \mathbb{Y} &= I(\mathsf{P})\mathbb{Z} + E(\mathsf{P})\mathbb{W}, \end{aligned} \tag{20}$$

where we have suppressed the superscript k. Under a change of scale $\hat{g} = e^{2\omega} g$, the behaviour of X, Y, Z gives

$$\begin{aligned} \hat{\mathbb{X}} &= \mathbb{X}, \\ \hat{\mathbb{Z}} &= \mathbb{Z} + \varepsilon(\Upsilon)\mathbb{X}, \\ \hat{\mathbb{W}} &= \mathbb{W} - \iota(\Upsilon)\mathbb{X}, \\ \hat{\mathbb{Y}} &= \mathbb{Y} - \iota(\Upsilon)\mathbb{Z} - \varepsilon(\Upsilon)\mathbb{W} + \tfrac{1}{2}(\varepsilon(\Upsilon)\iota(\Upsilon) - \iota(\Upsilon)\varepsilon(\Upsilon))\mathbb{X}, \end{aligned}$$

where again $\Upsilon = d\omega$.

3.2. **Exterior calculus on the ambient manifold.** Let $\boldsymbol{d}$ be the exterior derivative on the ambient manifold $\tilde{M}$, and let $\boldsymbol{\delta}$ be its formal adjoint with respect to the usual form metric, which is derived in turn from the ambient metric $\boldsymbol{h}$. If u is a one-form on $\tilde{M}$, we have exterior multiplication $\varepsilon(u)$ and its formal adjoint, the interior multiplication $\iota(u)$. Using the ambient metric (19) to raise and lower indices the conventions are as follows. Exterior and interior multiplication by a 1-form ω are given by

$$\begin{aligned} (\varepsilon(\omega)\varphi)_{A_0 \cdots A_k} &= (k+1)\omega_{[A_0}\varphi_{A_1 \cdots A_k]}\,, \\ (\iota(\omega)\varphi)_{A_2 \cdots A_k} &= \omega^{A_1}\varphi_{A_1 \cdots A_k}\,. \end{aligned} \tag{21}$$

We extend the notation for interior and exterior multiplication in an obvious way to operators which increase the rank by one. For example since the ambient connection is symmetric we have $\boldsymbol{d}\varphi = \varepsilon(\boldsymbol{\nabla})\varphi$ and $\boldsymbol{\delta}\varphi = -\iota(\boldsymbol{\nabla})\varphi$. These notations and conventions are also used for form tractors, and for forms and form-densities on the underlying conformal manifold.

Building polynomially on $\boldsymbol{d}, \boldsymbol{\delta}, \varepsilon(\boldsymbol{X}), \iota(\boldsymbol{X})$, we get several more differential operators. In particular, we get the *form Laplacian* $\boldsymbol{\Delta} := \boldsymbol{\delta}\boldsymbol{d} + \boldsymbol{d}\boldsymbol{\delta}$, and $Q = \iota(\boldsymbol{X})\varepsilon(\boldsymbol{X}) + \varepsilon(\boldsymbol{X})\iota(\boldsymbol{X})$. We also obtain the Lie derivative with respect to $\boldsymbol{X}$, and its formal adjoint as operators on forms:

$$\mathcal{L}_{\boldsymbol{X}} = \iota(\boldsymbol{X})\boldsymbol{d} + \boldsymbol{d}\iota(\boldsymbol{X}), \quad \mathcal{L}^*_{\boldsymbol{X}} = \boldsymbol{\delta}\varepsilon(\boldsymbol{X}) + \varepsilon(\boldsymbol{X})\boldsymbol{\delta}. \tag{22}$$

In general, given a bundle, we shall use the notation $\Gamma(\cdot)$ for its smooth section space, if this space has not been given another name. The subspace of $\Gamma(\wedge^k T^*\tilde{M})$ consisting of ambient k-forms F satisfying $\boldsymbol{\nabla_X} F = wF$ for a given $w \in \mathbb{R}$ will be denoted $\boldsymbol{\mathcal{T}}^k(w)$. We say such forms are (homogeneous) of *weight* w. From the definitions in the previous section, it is straightforward to verify that

$$\boldsymbol{\mathcal{T}}^k(w) = (\Lambda^k \boldsymbol{\mathcal{T}}) \otimes \tilde{\mathcal{E}}(w)$$

where the tensor (and exterior) products are over $\tilde{\mathcal{E}}(0)$. Given its weight, the degree of F is dependent on its order. In general from (12) we have the identities

$$\begin{aligned} \mathcal{L}_{\boldsymbol{X}} &= \boldsymbol{\nabla_X} + \mathbf{p}, \\ \mathcal{L}_{\boldsymbol{X}} - \mathcal{L}^*_{\boldsymbol{X}} &= 2\boldsymbol{\nabla_X} + n + 2, \end{aligned} \tag{23}$$

where $\mathbf{p}$ is the operator that multiplies by $p - q$ the part of a tensor with rank (q,p) (i.e. q indices up and p down).

Considering further commutators and anticommutators we note that the eight operators in Tables 1 and 2 generate an isomorphic copy $\mathfrak{g}$ of the linear Lie superalgebra $\mathfrak{sl}(2|1)$. This decomposes into the -1, 0 and 1 eigenspaces of $Z := (\mathcal{L}_{\boldsymbol{X}} + \mathcal{L}^*_{\boldsymbol{X}})/2$: $\mathfrak{g} = \mathfrak{g}_1^- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1^+$. The odd part of $\mathfrak{g}$ is $\mathfrak{g}_1 = \mathfrak{g}_1^- \oplus \mathfrak{g}_1^+$, $\mathfrak{g}_1^-$ is spanned by $\boldsymbol{\delta}$ and $\iota(\boldsymbol{X})$, $\mathfrak{g}_1^+$ is spanned by $\boldsymbol{d}$ and $\varepsilon(\boldsymbol{X})$, and via anticommutators these generate $\mathfrak{g}_0$, that is $\{\mathfrak{g}_1, \mathfrak{g}_1\} = \mathfrak{g}_0$. Denoting by E_{ij} $(i,j = 1,2,3)$ the standard matrix units in $\mathbb{C}^3 \times \mathbb{C}^3$ (or $\mathbb{R}^3 \times \mathbb{R}^3$) one family of Lie superalgebra isomorphisms from $\mathfrak{g}$ to the defining representation of $\mathfrak{sl}(2|1)$ is given by

$$\begin{array}{llll} \iota(\boldsymbol{X}) \mapsto -\frac{2}{c}E_{31}, & \boldsymbol{\delta} \mapsto \frac{2}{c}E_{32}, & \boldsymbol{d} \mapsto cE_{13}, & \varepsilon(\boldsymbol{X}) \mapsto -cE_{23}, \\ Q \mapsto 2E_{21}, & \boldsymbol{\not\Delta} \mapsto 2E_{12}, & \mathcal{L}_{\boldsymbol{X}} \mapsto -2E_{11} - 2E_{33}, & \mathcal{L}^*_{\boldsymbol{X}} \mapsto -2E_{22} - 2E_{33}, \end{array}$$

where c is a non-zero complex number (or real number if we work over $\mathbb{R}$). Observe that the even part $\mathfrak{g}_0$ of $\mathfrak{g}$ is isomorphic to $\mathfrak{u}(2)$. Since Z is central in $\mathfrak{g}_0$, it is clear that the decomposition of $\mathfrak{g}_1^- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1^+$ is $\mathfrak{g}_0$-equivariant and we note from the isomorphism that $\mathfrak{g}_1^-$ and $\mathfrak{g}_1^+$ are dual $\mathfrak{g}_0$-modules. In the 0-form-density case, the $\mathfrak{su}(2)$ subalgebra played a role in [36].

$\{\cdot,\cdot\}$	$\boldsymbol{d}$	$\boldsymbol{\delta}$	$\varepsilon(\boldsymbol{X})$	$\iota(\boldsymbol{X})$
$\boldsymbol{d}$	0	$\boldsymbol{\not\Delta}$	0	$\mathcal{L}_{\boldsymbol{X}}$
$\boldsymbol{\delta}$	$\boldsymbol{\not\Delta}$	0	$\mathcal{L}^*_{\boldsymbol{X}}$	0
$\varepsilon(\boldsymbol{X})$	0	$\mathcal{L}^*_{\boldsymbol{X}}$	0	Q
$\iota(\boldsymbol{X})$	$\mathcal{L}_{\boldsymbol{X}}$	0	Q	0

TABLE 1. Anticommutators $\{\mathfrak{g}_1, \mathfrak{g}_1\}$

The relations in these tables all follow from $Q = \boldsymbol{h}(\boldsymbol{X}, \boldsymbol{X})$, $dQ = 2\boldsymbol{X}$, (12), and the usual identities of exterior calculus on pseudo-Riemannian manifolds (e.g. (22)). In particular, they hold in all dimensions and to all orders.

Of particular interest are differential operators P on ambient form bundles, or subquotients thereof, which act *tangentially* along $\mathcal{Q}$, in the sense that $PQ = QP'$ for some operator P' (or equivalently $[P,Q] = QP''$ for

$[\cdot,\cdot]$	$\boldsymbol{d}$	$\boldsymbol{\delta}$	$\varepsilon(\boldsymbol{X})$	$\iota(\boldsymbol{X})$	$\not{\boldsymbol{\Delta}}$	$\mathcal{L}_{\boldsymbol{X}}$	$\mathcal{L}^*_{\boldsymbol{X}}$	Q
$\not{\boldsymbol{\Delta}}$	0	0	$-2\boldsymbol{d}$	$2\boldsymbol{\delta}$	0	$2\not{\boldsymbol{\Delta}}$	$-2\not{\boldsymbol{\Delta}}$	$-2\mathcal{K}_{\boldsymbol{X}}$
$\mathcal{L}_{\boldsymbol{X}}$	0	$-2\boldsymbol{\delta}$	$2\varepsilon(\boldsymbol{X})$	0	$-2\not{\boldsymbol{\Delta}}$	0	0	$2Q$
$\mathcal{L}^*_{\boldsymbol{X}}$	$2\boldsymbol{d}$	0	0	$-2\iota(\boldsymbol{X})$	$2\not{\boldsymbol{\Delta}}$	0	0	$-2Q$
Q	$-2\varepsilon(\boldsymbol{X})$	$2\iota(\boldsymbol{X})$	0	0	$2\mathcal{K}_{\boldsymbol{X}}$	$-2Q$	$2Q$	0

TABLE 2. Commutators $[\mathfrak{g}_0,\mathfrak{g}_1]$ and $[\mathfrak{g}_0,\mathfrak{g}_0]$, where $\mathcal{K}_{\boldsymbol{X}} := \mathcal{L}_{\boldsymbol{X}} - \mathcal{L}^*_{\boldsymbol{X}}$

some P''). Note that compositions of tangential operators are tangential. If tangential operators are suitably homogeneous then they descend to operators on on M. Such P are *fragile* if they are tangential only when acting on sections F of some weight w; a fragile operator descends to an operator which is invariant for a particular weight. The *robust* P are those which are tangential when acting on arbitrary smooth sections; these descend to operators which are invariant for any weight.

An example of a fragile tangential operator is given by:

Proposition 3.2. $\not{\boldsymbol{\Delta}}^m : \boldsymbol{\mathcal{T}}^k(m-n/2) \to \boldsymbol{\mathcal{T}}^k(-m-n/2)$ *is tangential.*

Proof: We need to calculate $\not{\boldsymbol{\Delta}}^m(Qf)$ for Qf of homogeneity $m-n/2$ (i.e. for f of homogeneity $m-2-n/2$). Without any homogeneity assumption, we have

$$[\not{\boldsymbol{\Delta}}^m, Q] = \sum_{p=0}^{m-1} \not{\boldsymbol{\Delta}}^{m-1-p}[\not{\boldsymbol{\Delta}}, Q]\not{\boldsymbol{\Delta}}^p, \tag{24}$$

and $[\not{\boldsymbol{\Delta}}, Q] = 2(\mathcal{L}^*_{\boldsymbol{X}} - \mathcal{L}_{\boldsymbol{X}})$, from Table 2. Thus letting (24) act on $\boldsymbol{\mathcal{T}}^k(w)$, the $p^{\underline{\text{th}}}$ term on the right acts as $-2[2(w-2p)+n+2]\not{\boldsymbol{\Delta}}^{m-1}$, so that $[\not{\boldsymbol{\Delta}}^m, Q]$ acts as $-2m(2w-2m+n+4)\not{\boldsymbol{\Delta}}^{m-1}$. This vanishes identically if and only if $w = m-2-n/2$, so that $\not{\boldsymbol{\Delta}}^m$ is tangential on $\boldsymbol{\mathcal{T}}^k(m-n/2)$, as desired. □

Remark: We should point out that results along these lines are not peculiar to forms or the form Laplacian. Recall that when acting on $\boldsymbol{\mathcal{T}}^k(w)$, $2(\mathcal{L}^*_{\boldsymbol{X}} - \mathcal{L}_{\boldsymbol{X}}) = -2(2\boldsymbol{\nabla}_{\boldsymbol{X}} + n + 2)$. On the other hand from (12) one calculates that for the ambient Bochner Laplacian $\boldsymbol{\Delta} := \boldsymbol{\nabla}^*\boldsymbol{\nabla} = -\boldsymbol{\nabla}^A\boldsymbol{\nabla}_A$ we have

$$[\boldsymbol{\Delta}, Q] = -2(2\boldsymbol{\nabla}_{\boldsymbol{X}} + n + 2), \tag{25}$$

as an operator on *any ambient tensor*. Thus by essentially the same argument as above we conclude that $\boldsymbol{\Delta}^m$ is tangential on arbitrary tensors of weight w. ‖

Simple examples of robust tangential operators are given in the following proposition:

Proposition 3.3. *The operators*

$$\varepsilon(\not{\boldsymbol{D}}) := \boldsymbol{d}(n + 2\boldsymbol{\nabla}_{\boldsymbol{X}} - 2) + \varepsilon(\boldsymbol{X})\not{\boldsymbol{\Delta}}, \quad \iota(\not{\boldsymbol{D}}) := -\boldsymbol{\delta}(n + 2\boldsymbol{\nabla}_{\boldsymbol{X}} - 2) + \iota(\boldsymbol{X})\not{\boldsymbol{\Delta}}$$

act tangentially along $\mathcal{Q}$.

Proof: Using Table 2 one calculates $[Q,\varepsilon(\boldsymbol{D\!\!\!/})] = -4Q\boldsymbol{d}$ and $[Q,\iota(\boldsymbol{D\!\!\!/})] = 4Q\boldsymbol{\delta}$. $\square$

Note that in view of the relations $[\boldsymbol{\nabla}_{\boldsymbol{X}},\varepsilon(\boldsymbol{D\!\!\!/})] = -\varepsilon(\boldsymbol{D\!\!\!/})$ and $[\boldsymbol{\nabla}_{\boldsymbol{X}},\iota(\boldsymbol{D\!\!\!/})] = -\iota(\boldsymbol{D\!\!\!/})$, these operators lower weight by 1 and by construction $\varepsilon(\boldsymbol{D\!\!\!/})$ raises form order by 1 while $\iota(\boldsymbol{D\!\!\!/})$ lowers it by 1. In summary:

$$\varepsilon(\boldsymbol{D\!\!\!/}) : \boldsymbol{\mathcal{T}}^k(w) \to \boldsymbol{\mathcal{T}}^{k+1}(w-1), \quad \iota(\boldsymbol{D\!\!\!/}) : \boldsymbol{\mathcal{T}}^k(w) \to \boldsymbol{\mathcal{T}}^{k-1}(w-1).$$

These satisfy identities as follows.

Proposition 3.4. *On $\tilde{M}$,*

$$\iota(\boldsymbol{D\!\!\!/})\iota(\boldsymbol{D\!\!\!/}) = 0, \quad \varepsilon(\boldsymbol{D\!\!\!/})\varepsilon(\boldsymbol{D\!\!\!/}) = 0, \quad \iota(\boldsymbol{D\!\!\!/})\varepsilon(\boldsymbol{D\!\!\!/}) + \varepsilon(\boldsymbol{D\!\!\!/})\iota(\boldsymbol{D\!\!\!/}) = Q\boldsymbol{\Delta\!\!\!/}^2.$$

Proof: These formulae follow from (23) and Tables 1 and 2. $\square$

3.3. **Form tractors and invariant operators.** Recall that we write $\mathcal{T}^k$ for the space of *k-form tractor* fields. That is $\mathcal{T}^k$ is the space of sections of $\mathbb{T}^k$. $\mathcal{T}^k[w]$ denotes the space of *weighted* k-form tractors of weight w; this is the space of sections of $\mathbb{T}^k \otimes E[w]$. (Naturally these are non-trivial for $0 \le k \le n+2$. For k outside this range we take $\mathbb{T}^k$ to be the zero bundle.) From the relationship between $\mathcal{E}[w]$ and $\tilde{\mathcal{E}}(w)$ and the definition of $\mathcal{T}$ in Section 3.1, it follows that the sections in $\mathcal{T}^k[w]$ are equivalent to ambient manifold form fields, along $\mathcal{Q}$, which lie in $\boldsymbol{\mathcal{T}}^k(w)|_{\mathcal{Q}}$. Clearly the operators of Propositions 3.2 and 3.3 determine operators between twisted form tractor bundles. By construction these tractor operators are conformally invariant but *ab initio* might depend on the choices in the ambient metric. Hence there is a need for specific results on when such operators are natural conformal objects.

A similar comment applies to natural tensors on the ambient manifold. The ambient curvature $\boldsymbol{R}$ is in $(\otimes^4\boldsymbol{\mathcal{T}})(-2)$ and so corresponds to a section of $(\otimes^4\mathcal{T})[-2]$. In dimensions other than 4 this tractor field depends only on the conformal structure [14, 32] and is $1/(n-4)$ times the tractor W of [29]. (In terms of a metric from the conformal class and the notation from Section 3.1 W is given by

$$\begin{aligned}(n-4)\left\{\tfrac{1}{4}\mathbb{Z}^{ab}_{AB}\mathbb{Z}^{ce}_{CE}C_{abce} - \mathbb{Z}^{ab}_{AB}\mathbb{X}^{e}_{CE}\nabla_{[a}\mathsf{P}_{b]e}\right. \\ \left. -\mathbb{X}^{b}_{AB}\mathbb{Z}^{ce}_{CE}\nabla_{[c}\mathsf{P}_{e]b}\right\} + \frac{1}{n-3}\mathbb{X}^{b}_{AB}\mathbb{X}^{e}_{CE}B_{eb},\end{aligned} \tag{26}$$

where C_{abce} is the Weyl tensor and $B_{eb} := \nabla^q\nabla^p W_{peqb} + (n-3)\mathsf{P}^{qp}W_{peqb}$ is the Bach tensor.

Closely related to $\varepsilon(\boldsymbol{D\!\!\!/})$ and $\iota(\boldsymbol{D\!\!\!/})$ is the operator $\boldsymbol{D} := \boldsymbol{\nabla}(n+2\boldsymbol{\nabla}_{\boldsymbol{X}}-2) + \boldsymbol{X}\boldsymbol{\Delta}$ of [14, 32] (and see e.g. [2] where this was used in the setting of the standard flat model). Using $2\boldsymbol{h}(\boldsymbol{X},\cdot) = dQ$, (25) and (23) it is easily verified that $\boldsymbol{D}$ is a robust tangential operator on ambient tensor fields of any rank or symmetry. For the class of ambient metrics that we consider (in particular we need (13)) the operator $\boldsymbol{D}$, restricted to homogeneous tensors along $\mathcal{Q}$, is equivalent [14, 32] to the well-known tractor-D operator D of [45, 1]. D is natural and so depends only on the conformal structure. Explicitly, for a metric from the conformal class, D is given by

$$D_AV := (n+2w-2)wY_AV + (n+2w-2)Z_A{}^a\nabla_aV + X_A\Box V, \tag{27}$$

where V is a section of any twisted tractor bundle of weight w, and writing J for the trace (by $\boldsymbol{g}^{-1}$) of the Schouten tensor, we have

$$\Box V := -(\nabla_p \nabla^p V + w \mathsf{J} V).$$

In these formulae ∇ means the coupled tractor–Levi-Civita connection.

Let us write $\sharp$ (*hash*) for the natural tensorial action of sections A of $\mathrm{End}(T\tilde{M})$ on tensors. If A is skew for $\boldsymbol{h}$ then this commutes with the raising and lowering of indices. As a section of the tensor square of the $\boldsymbol{h}$-skew bundle endomorphisms of $T\tilde{M}$, the ambient curvature has a double hash action on tensors, and, on forms, this is exactly the difference between the ambient Bochner and form Laplacians. That is we have

$$\not{\boldsymbol{\Delta}} - \boldsymbol{\Delta} = -\boldsymbol{R}\sharp\sharp. \tag{28}$$

It follows immediately that, as operators on k-forms,

$$\varepsilon(\not{\boldsymbol{D}}) = \varepsilon(\boldsymbol{D}) - \varepsilon(\boldsymbol{X})\boldsymbol{R}\sharp\sharp, \quad \iota(\not{\boldsymbol{D}}) = \iota(\boldsymbol{D}) - \iota(\boldsymbol{X})\boldsymbol{R}\sharp\sharp.$$

Summarising with some additional results we have the following.

Proposition 3.5. *The operators of Proposition 3.2 descend to natural conformally invariant differential operators*

$$\varepsilon(\not{D}) : \mathcal{T}^k[w] \to \mathcal{T}^{k+1}[w-1], \quad \iota(\not{D}) : \mathcal{T}^k[w] \to \mathcal{T}^{k-1}[w-1],$$

and $\iota(\not{D}) : \mathcal{T}^{k+1}[1-n-w] \to \mathcal{T}^k[-n-w]$ *is the formal adjoint of* $\varepsilon(\not{D}) : \mathcal{T}^k[w] \to \mathcal{T}^{k+1}[w-1]$. *These satisfy*

$$\varepsilon(\not{D}) = \varepsilon(D) - \varepsilon(X)\Omega\sharp\sharp, \quad \iota(\not{D}) = \iota(D) - \iota(X)\Omega\sharp\sharp,$$

where $\Omega\sharp\sharp$ *is a curvature action (and so has order 0 as a differential operator).*

Proof: It is immediate from Proposition 3.2 that the operators there descend to conformally invariant operators.

From the discussion above, in dimensions other than 4, $(n-4)\boldsymbol{R}|_{\mathcal{Q}}$ is equivalent to the tractor field W, where W is given explicitly in (26). Viewing W as a section of $\wedge^2\mathbb{T} \otimes \mathrm{End}(\mathbb{T})$, we note that from (26) $\varepsilon(X)W = (n-4)\varepsilon(X)\Omega$, where (we also view Ω as a section of $\wedge^2\mathbb{T} \otimes \mathrm{End}(\mathbb{T})$ and) Ω is given by

$$\tfrac{1}{4}\mathbb{Z}_{AB}^{ab}\mathbb{Z}_{CE}^{ce}C_{abce} - \mathbb{Z}_{AB}^{ab}\mathbb{X}_{CE}^{e}\nabla_{[a}\mathsf{P}_{b]e} - \mathbb{X}_{AB}^{b}\mathbb{Z}_{CE}^{ce}\nabla_{[c}\mathsf{P}_{e]b}.$$

It is easily verified that Ω has Weyl tensor type symmetries. In dimension 4, $\boldsymbol{R}$ is not equivalent to a natural tractor but, as explained in Section 3.2 of [14], $\varepsilon(X)\boldsymbol{R}|_{\mathcal{Q}}$ is determined by the conformal structure and it follows easily from the discussion there that it is equivalent to the tractor field $\varepsilon(X)\Omega$. Next, from the above, $\boldsymbol{D}$ along $\mathcal{Q}$ is equivalent to the operator D of (27) on tractors. Thus in all dimensions we have

$$\varepsilon(\not{D}) = \varepsilon(D) - \varepsilon(X)\Omega\sharp\sharp, \quad \iota(\not{D}) = \iota(D) - \iota(X)\Omega\sharp\sharp,$$

where the interpretation of the action $\Omega\sharp\sharp$ on form tractors is obvious from the corresponding action $\boldsymbol{R}\sharp\sharp$ on ambient form fields.

From the explicit formulae for Ω and D it is immediately clear that the operators $\varepsilon(\not{D})$ and $\iota(\not{D})$ are natural and differential. That $\iota(D) : \mathcal{T}^{k+1}[1-n-w] \to \mathcal{T}^k[-n-w]$ is the formal adjoint of $\varepsilon(D) : \mathcal{T}^k[w] \to \mathcal{T}^{k+1}[w-1]$ is a special case of a more general result in Sec. 7 of [10]. It follows easily from this, the Weyl-tensor-type symmetries of Ω, and the fact that Ω is clearly annihilated by contraction with X, that $\varepsilon(\not{D})$ and $\iota(\not{D})$ are mutual formal adjoints. $\square$

Remark: From Proposition 3.4 we immediately have the identities

$$\iota(\not{D})\iota(\not{D}) = 0, \quad \varepsilon(\not{D})\varepsilon(\not{D}) = 0, \quad \iota(\not{D})\varepsilon(\not{D}) + \varepsilon(\not{D})\iota(\not{D}) = 0.$$

Observe that if $k = 0$ the $\boldsymbol{R}\sharp\sharp$ action is trivial by definition while if $k = 1$ it amounts to an action of $\mathrm{Ric}(\boldsymbol{h})$ and so vanishes along $\mathcal{Q}$. In either case we have $\varepsilon(\not{\boldsymbol{D}}) = \varepsilon(\boldsymbol{D})$, and $\iota(\not{\boldsymbol{D}}) = \iota(\boldsymbol{D})$ along $\mathcal{Q}$. Thus acting on form tractors of rank $k \leq 1$ we have $\varepsilon(\not{D}) = \varepsilon(D)$ and $\iota(\not{D}) = \iota(D)$. ▮

Before we continue with the main theme let us digress briefly, to give a direct formula for the scale dependent tractor Y in terms of the invariant natural operator $\varepsilon(\not{D})$ and the choice of scale. Let $\sigma \in \mathcal{E}[1]$ be a choice of conformal scale. Then by the definition of the Levi-Civita connection on densities we have $\nabla\sigma = 0$, and, from (27) we have $\sigma^{-1}D\sigma = \sigma^{-1}\varepsilon(\not{D})\sigma = nY - X\mathsf{J}$. Let us write I_σ for $\frac{1}{n}\sigma^{-1}\varepsilon(\not{D})\sigma$. Thus we have $I_\sigma \bullet I_\sigma = -2\mathsf{J}/n$, $X \bullet I_\sigma = 1$ and so

$$Y = I_\sigma - \tfrac{1}{2}(I_\sigma \bullet I_\sigma)X. \tag{29}$$

This formula plays an important role in later calculations.

Returning to our programme of constructing natural invariant operators, we note an extension of our observation above, to the effect that when $n \neq 4$, the ambient curvature is a natural conformal invariant.

Lemma 3.6. *The ambient tensors $\boldsymbol{\nabla}^s\boldsymbol{\Delta}^t\boldsymbol{R}|_{\mathcal{Q}}$ with $s, t \in \{0, 1, 2, \ldots\}$ are equivalent to conformally invariant tractor fields. In odd dimensions these tractor fields are natural. In even dimensions $n \neq 4$, the same is true with the restriction $s + t \leq n/2 - 3$.*

Proof: The first claim is clear since the tensors are homogeneous and the ambient metric is conformally invariant. Lemma 4.4 of [32] establishes that in odd dimensions, or when $s + t \leq n/2 - 3$, $\boldsymbol{\nabla}^s\boldsymbol{\Delta}^t\boldsymbol{R}$ can be expressed as a partial contraction polynomial in $\boldsymbol{D}$, $\boldsymbol{R}$, $\boldsymbol{X}$, $\boldsymbol{h}$, and $\boldsymbol{h}^{-1}$. It follows that the corresponding conformally invariant tractor sections are given by same formal expression with the respective replacements D, $W/(n-4)$, X, h, and h^{-1}. The claims concerning naturality are now immediate as each of these is natural. $\square$

Remarks: Some related results are in Theorem 3.4 of [14]. It should also be pointed out that $\boldsymbol{\Delta}^{n/2-2}\boldsymbol{R}$ corresponds to a natural conformal invariant related to the ambient obstruction of [25] – see [33]. In even dimensions the remaining ambient tensors $\boldsymbol{\nabla}^s\boldsymbol{\Delta}^t\boldsymbol{R}|_{\mathcal{Q}}$ with $s + t > n/2 - 3$ correspond to tractor fields which depend on the choices involved in extending the ambient metric to, and beyond, order $n/2$. ▮

Proposition 3.7. *The operators of Proposition 3.2 descend to conformally invariant differential operators*

$$\not{\Delta}_m : \mathcal{T}^k[m - n/2] \to \mathcal{T}^k[-m - n/2] \quad m = 0, 1, \cdots,$$

where by convention $\not{\boldsymbol{\Delta}}^0$ and $\not{\Delta}_0$ are identity operators. In odd dimensions these are natural operators. In even dimensions the same is true with the *restrictions that either $m \leq n/2-2$; or $m \leq n/2-1$ and $k=1$; or $m \leq n/2$ and $k=0$. In the conformally flat case the operators are natural with no restrictions on $m \in \{0,1,2,\ldots\}$. In every case $\not\Delta_\ell$ has the same principal part as Δ^ℓ.*

Note that the $k=0$ cases of the above theorem are by construction exactly the GJMS operators of [36].

Proof: From the conformal invariance of the ambient construction, and the relationship between $\boldsymbol{\mathcal{T}}^k(w)$ and $\mathcal{T}^k[w]$ it is clear that the operators of Proposition 3.2 determine conformally invariant operators $\not\Delta_m : \mathcal{T}^k[m-n/2] \to \mathcal{T}^k[-m-n/2]$ for all $m \in \{0,1,2,\ldots\}$.

It remains to establish that these are differential, natural and with leading term as claimed. For odd dimensions and even dimensions, up to order n, the $k=0$ cases are dealt with in [36] and [32]. The arguments of the latter adapt easily to the more general setting here. The first observation in [32] is that for a function on $\tilde{M}$ homogeneous of weight $m-n/2$, $\boldsymbol{\Delta}^m f$ (or rather $(-1)^{m-1}\boldsymbol{X}^{m-1}\boldsymbol{\Delta}^m f$) is the leading term of $\boldsymbol{\Delta}\boldsymbol{D}^{m-1}f$. The difference between these terms involves $\boldsymbol{\nabla}$-derivatives of f and $\boldsymbol{R}$ and the main part of the argument is that these can be re-expressed as $\boldsymbol{D}$-derivatives of f and $\boldsymbol{R}$. (See in particular Section 4 and the proof of Theorem 2.5). It is easily verified that if we instead just re-express the $\boldsymbol{\nabla}$-derivatives of f in this way but do not re-express the $\boldsymbol{\nabla}^\ell\boldsymbol{R}$, then in all dimensions the argument works for all $m \in \mathbb{Z}^+$. With this variant the proof goes through, and is essentially unchanged, if we replace f with an ambient tensor field homogeneous of weight $m-n/2$. (The only difference is that more curvature terms turn up.) A similar argument can be applied to powers of $\not\boldsymbol{\Delta}$ by using (28) to first re-express $\not\boldsymbol{\Delta}^m$ in terms of $\boldsymbol{\Delta}$, $\boldsymbol{\nabla}$ and $\boldsymbol{R}$. Thus for $U \in \boldsymbol{\mathcal{T}}^k(m-n/2)$, it follows that $\not\boldsymbol{\Delta}^m U$ has an expression which is polynomial in $\boldsymbol{h}$, $\boldsymbol{h}^{-1}$, $\boldsymbol{\nabla}$-derivatives of $\boldsymbol{R}$, and is linear in $\boldsymbol{D}$-derivatives of U. It follows immediately that the operators $\not\boldsymbol{\Delta}^m$ here are differential, since $\boldsymbol{D}$ descends to the natural differential operator D. The leading term is Δ^m, since from (27) it follows easily this is the leading term of ΔD^{m-1}. Counting the powers of $\boldsymbol{\nabla}$ and $\boldsymbol{\Delta}$ that can act on $\boldsymbol{R}$ in the expansion discussed, the statements on naturality in odd dimensions and in even dimensions for $m \leq n/2-2$ are now immediate from Lemma 3.6.

The improved result when $k=1$ in even dimensions is a consequence of the observation already made that the $\boldsymbol{R}\sharp\sharp$ action reduces to a $\mathrm{Ric}(\boldsymbol{h})$ action on 1-forms while by property (iii) of the ambient metric, (12) and (25) it follows that $\boldsymbol{\nabla}^s\boldsymbol{\Delta}^t\,\mathrm{Ric}(\boldsymbol{h})|_{\mathcal{Q}} = 0$ if $s+t \leq n/2-2$. On conformally flat manifolds we take the flat ambient metric and so the claim for this setting is clear. $\square$

Next we will construct natural conformally invariant operators

$$\begin{aligned} &\tilde{d} : \mathcal{G}^k[w] \to \mathcal{G}^{k+1}[w] && \text{and} \\ &\tilde{\delta} : \mathcal{G}_{k+1}[w] \to \mathcal{G}_k[w] && \text{for } k = 0,1,\cdots,n+1, \end{aligned}$$

which in an appropriate sense generalise d and δ. The operators act between section spaces that we define as follows. First denote by $\mathbb{V}^k[w]$ the subbundle of $\mathbb{T}^k[w]$ consisting (pointwise) of k-form tractors annihilated by $\varepsilon(X)$.

Denote by $\mathbb{F}^k[w]$ the subbundle of $\mathbb{T}^k[w]$ consisting of k-form tractors annihilated by $\iota(X)$. (Equivalently $\mathbb{V}^k[w]$ is the subbundle of form tractors of the form $\varepsilon(X)S$ for some $(k-1)$ form tractor S and similarly $\mathbb{F}^k[w]$ is the subbundle of tractors of the form $\iota(X)F$ for some $(k+1)$ form tractor F. Also note that the notation for the bundles defined here is consistent with Section 3.1 in the sense that $\mathbb{V}^1 = \mathbb{V}$ and $\mathbb{F}^1 = \mathbb{F}$.) Then for $k = 0, 1, \cdots, n+1$ we have the definitions:

$$\begin{aligned} &\mathbb{G}^k[w] := \mathbb{T}^k[w-k]/\mathbb{V}^k[w-k] && E^{k-1}[w] \oplus E^k[w] \\ &\mathbb{G}_k[w] := \mathbb{F}^k[w+k-n] && E_k[w] \oplus E_{k-1}[w]. \end{aligned} \tag{30}$$

The second column here gives the composition series of the bundle defined (which follow at once from (16) and the definitions here). We use the following notations for the section spaces:

$$\begin{aligned} \mathcal{G}^k[w] &:= \Gamma(\mathbb{G}^k[w]), & \mathcal{G}_k[w] &:= \Gamma(\mathbb{G}_k[w]), \\ \mathcal{F}^k[w] &:= \Gamma(\mathbb{F}^k[w]), & \mathcal{V}^k[w] &:= \Gamma(\mathbb{V}^k[w]). \end{aligned}$$

We note that via the form tractor metric $\mathbb{G}_k[n-w]$ is identified with the bundle dual to $\mathbb{G}^k[w]$, and so there is an integral pairing between $\mathcal{G}^k[w]$ and $\mathcal{G}_k[-w]$. The latter is part of the reason for using $\mathbb{G}_k[w]$ as an alternative notation for $\mathbb{F}^k[w+k-n]$. Along the lines of conventions for other spaces we write $\mathbb{G}^k := \mathbb{G}^k[0]$ and $\mathbb{G}_k := \mathbb{G}_k[0]$. Note that the wedge product $\triangle$ of (17) induces a wedge product carrying $\mathbb{G}^k[w] \times \mathbb{G}^{k'}[w']$ to $\mathbb{G}^{k+k'}[w+w']$, and similarly for the weighted $\mathbb{G}_k$ bundles.

Since X and Y are null tractor fields, we have the well-defined operations

$$\begin{aligned} \iota(X) &: \mathcal{G}^{k+1}[w] \to \mathcal{G}^k[w], & \varepsilon(Y) &: \mathcal{G}^k[w] \to \mathcal{G}^{k+1}[w] \\ \iota(Y) &: \mathcal{G}_{k+1}[w] \to \mathcal{G}_k[w], & \varepsilon(X) &: \mathcal{G}_k[w] \to \mathcal{G}_{k+1}[w], \end{aligned}$$

arising from interior and exterior multiplication on $\mathcal{T}^k[w]$. Thus we have that $\{\iota(X), \varepsilon(Y)\}$ acts as the identity on $\mathcal{G}^k[w]$ and $\{\iota(Y), \varepsilon(X)\}$ acts as the identity on $\mathcal{G}_k[w]$. Also $\varepsilon(X) : \mathcal{G}^k[w] \to \mathcal{V}^{k+1}[w-k+1]$ is clearly well defined (and in fact injective since $\{\iota(Y), \varepsilon(X)\}$ is the identity on $\mathcal{T}^k[w-k]$) while $\iota(X)$ acts as 0 on $\mathcal{G}_k[w]$.

Remark: There are other "subtractor" bundles similar to those defined in (30). These are used for the constructions outlined in Section 2.1. We outline briefly their relationship to $\mathbb{G}^k[w]$ and $\mathbb{G}_k[w]$.

Suppose M is orientable, with an orientation given by the conformal volume form $\boldsymbol{\epsilon}$. Recall that $\boldsymbol{\epsilon}$ is a section of $\mathcal{E}^n[n]$ with the property that, for each choice of conformal scale σ, $\sigma^{-n}\boldsymbol{\epsilon}$ is the unique volume form compatible with the orientation and the metric $g = \sigma^{-2}\boldsymbol{g}$. Then

$$\mathbb{W}^{n+2} \cdot \boldsymbol{\epsilon}$$

is a conformally invariant canonical section of $\mathcal{T}^{n+2}$. In an obvious way one can use this and the tractor metric to define a Hodge star operator, which we denote $\star$, for form tractor bundles.

The usual Hodge relations apply to $\star$ and in particular we have

$$\begin{aligned} &\star\varepsilon(X) = (-1)^k \iota(X)\star, \qquad \varepsilon(X)\star = (-1)^{k-1}\star\iota(X), \\ &\text{and } \star\star = (-1)^{k(n+2-k)+q+1}. \end{aligned}$$

It follows that on oriented manifolds we have an isomorphisms

$$\star : \mathbb{G}^k[w] \to \mathbb{G}^\star_{n-k}[w], \quad \star : \mathbb{G}_k[w] \to \mathbb{G}_\star^{n-k}[w],$$

where $\mathbb{G}^\star_{n-k}[w]$ is the quotient of $\mathbb{T}^{n+2-k}[w-k]$ with composition series $E_{n-k+1}[w] \oplus E_{n-k}[w]$ and $\mathbb{G}_\star^{n-k}[w]$ is subbundle of $\mathbb{T}^{n+2-k}[w+k-n]$ with composition series $E^{n-k}[w] \oplus E^{n-k+1}[w]$. Put another way, $\mathbb{G}^\star_{n-k}[w]$ is the pointwise quotient of $\mathbb{T}^{n+2-k}[w-k]$ by form tractors of the form $\iota(X)F$, and $\mathbb{G}_\star^{n-k}[w]$ is the subbundle of $\mathbb{T}^{n+2-k}[w+k-n]$ consisting of forms annihilated by $\varepsilon(X)$. We write $\mathcal{G}^\star_{n-k}[w]$ and $\mathcal{G}_\star^{n-k}[w]$ for the section spaces of, respectively, $\mathbb{G}^\star_{n-k}[w]$ and $\mathbb{G}_\star^{n-k}[w]$. ‖

The operators $\tilde{d}$ and $\tilde{\delta}$, to be constructed, are simply restrictions of the exterior derivative and its formal adjoint on $\mathcal{Q}$. For our current purposes, it is useful to see how they arise from tangential operators in the ambient picture. First observe that it is clear that, when acting on homogeneous tensors, $\boldsymbol{d}$ preserves homogeneous degree while increasing rank by 1, and so lowers the homogeneity weight by 1. Since the ambient metric is homogeneous of weight 0 it follows that $\boldsymbol{\delta}$ also lowers weight by 1. In summary

$$\boldsymbol{d} : \boldsymbol{\mathcal{T}}^k(w) \to \boldsymbol{\mathcal{T}}^{k+1}(w-1), \qquad \boldsymbol{\delta} : \boldsymbol{\mathcal{T}}^{k+1}(w) \to \boldsymbol{\mathcal{T}}^k(w-1).$$

Next observe that for any form field F, $\boldsymbol{d}\varepsilon(\boldsymbol{X})F = -\varepsilon(\boldsymbol{X})\boldsymbol{d}F$ so $\boldsymbol{d}$ preserves the space of forms of the form $\varepsilon(\boldsymbol{X})F$. Once again using the Tables 1 and 2, we also note that $\boldsymbol{d}QF = 2\varepsilon(\boldsymbol{X})F + Q\boldsymbol{d}F$. Let us use the informal notation $\Lambda^k T^*\tilde{M}/\varepsilon(\boldsymbol{X})$ for the quotient bundle which is the pointwise quotient of $\Lambda^k T^*\tilde{M}$ by forms of the form $\varepsilon(\boldsymbol{X})F$. The space of smooth sections of this quotient bundle is identified with the space $\Gamma(\Lambda^k T^*\tilde{M})$ modulo the subspace of smooth sections of the form $\varepsilon(\boldsymbol{X})F$. Via this and our observations just above it is clear that $\boldsymbol{d}$ determines a robust tangential operator (also to be denoted $\boldsymbol{d}$) on $\Lambda^k T^*\tilde{M}/\varepsilon(\boldsymbol{X})$. For each $w \in \mathbb{R}$ we have an inclusion $\boldsymbol{\mathcal{T}}^k(w) \hookrightarrow \Gamma(\Lambda^k T^*\tilde{M})$ and so $\boldsymbol{\mathcal{T}}^k(w)$ has an image in this quotient of section spaces. Let us use $\boldsymbol{\mathcal{G}}^k(w+k)$ to denote this. Once again using the notation $\boldsymbol{d}$ for the restriction to this subspace, it follows that

$$\boldsymbol{d} : \boldsymbol{\mathcal{G}}^k(w) \to \boldsymbol{\mathcal{G}}^{k+1}(w)$$

is a tangential operator.

On $\tilde{M}$ there is a natural integral pairing between the sections of $\Lambda^k T^*\tilde{M}/\varepsilon(\boldsymbol{X})$ and the space of sections of $\Lambda^k T^*\tilde{M}$ which are annihilated by $\iota(\boldsymbol{X})$. Since $\boldsymbol{\delta}$ is the formal adjoint of the exterior derivative $\boldsymbol{d}$, it preserves this subspace, and its restriction to this subspace (denoted $\boldsymbol{\delta}$) may be viewed as a formal adjoint of $\boldsymbol{d}$ on the quotient $\Lambda^{k-1}T^*\tilde{M}/\varepsilon(\boldsymbol{X})$. Since the latter commutes with Q it follows at once that, as an operator on the space of sections of $\Lambda^k T^*\tilde{M}$ which are annihilated by $\iota(\boldsymbol{X})$, $\boldsymbol{\delta}$ also commutes with Q. We write $\boldsymbol{\mathcal{G}}_k(w-k+n)$ (or alternatively $\boldsymbol{\mathcal{F}}^k(w)$) for the intersection of this last space with $\boldsymbol{\mathcal{T}}^k(w)$.

From the definition of the tractor bundle $\mathbb{T}$ in terms of the ambient manifold in Section 3.1 and the relationship between $X \in \mathcal{T}[1]$ and $\boldsymbol{X} \in \boldsymbol{\mathcal{T}}(1)$ it follows that sections of $\mathcal{G}^k[w]$ are equivalent to sections from the space $\boldsymbol{\mathcal{G}}^k(w)|_{\mathcal{Q}}$. Similarly sections of $\mathcal{F}^k[w]$ are equivalent to sections of $\boldsymbol{\mathcal{F}}^k(w)$. We now have the following.

Theorem 3.8. *The operators*

$$\boldsymbol{d} : \boldsymbol{\mathcal{G}}^k(w) \to \boldsymbol{\mathcal{G}}^{k+1}(w) \quad \text{and} \quad \boldsymbol{\delta} : \boldsymbol{\mathcal{G}}_k(w) \to \boldsymbol{\mathcal{G}}_{k-1}(w)$$

are tangential and satisfy $\boldsymbol{d}^2 = 0 = \boldsymbol{\delta}^2$. *These operators determine first order conformally invariant differential operators*

$$\tilde{d} : \mathcal{G}^k[w] \to \mathcal{G}^{k+1}[w] \quad \text{and} \quad \tilde{\delta} : \mathcal{G}_k[w] \to \mathcal{G}_{k-1}[w]$$

on M *which satisfy* $\tilde{d}^2 = 0 = \tilde{\delta}^2$. *The operator* $\tilde{d}$ *satisfies the anti-derivation rule* $\tilde{d}(\varepsilon(U)V) = \varepsilon(\tilde{d}U)V + (-1)^k \varepsilon(U)\tilde{d}V$ *for* U *in* $\mathcal{G}^k[w]$ *and* V *in any* $\mathcal{G}^{k'}[w']$.

Proof: We have already shown the operators are tangential. For $V \in \mathcal{G}^k[w]$, let $\tilde{V}$ be any homogeneous smooth extension to a section of $\boldsymbol{\mathcal{G}}^k(w)$ of the equivalent section from the space $\boldsymbol{\mathcal{G}}^k(w)|_{\mathcal{Q}}$. Then $\boldsymbol{d}\tilde{V}|_{\mathcal{Q}}$ is a section of $\boldsymbol{\mathcal{G}}^{k+1}(w)|_{\mathcal{Q}}$ dependent only on $\tilde{V}|_{\mathcal{Q}}$ and we write $\tilde{d}V$ for the equivalent section of $\mathcal{G}^{k+1}[w]$. This defines the operator $\tilde{d}$. By taking coordinates on the ambient manifold $\tilde{M}$, it is easily verified that $\tilde{d}$ is differential and first order. (See formulae (32,33) below.) The result $\boldsymbol{d}^2 = 0$ (resp. $\tilde{d}^2 = 0$) on $\boldsymbol{\mathcal{G}}^k(w)$ (resp. $\mathcal{G}^k[w]$) follows from the same result for the exterior derivative on $\Lambda^k\tilde{M}$, since if $\tilde{V}'$ is a homogeneous section of $\Lambda^k\tilde{M}$ representing $\tilde{V} \in \boldsymbol{\mathcal{G}}^k(w)$ (resp. $V \in \mathcal{G}^k(w)$), then $\boldsymbol{d}\tilde{V}'$ is a section of $\Lambda^{k+1}\tilde{M}$ representing $\boldsymbol{d}\tilde{V} \in \boldsymbol{\mathcal{G}}^{k+1}(w)$ (resp. $\tilde{d}V \in \mathcal{G}^{k+1}[w]$).

The corresponding results for $\boldsymbol{\delta}$ and $\tilde{\delta}$ follow by an analogous argument. The anti-derivation rule for $\tilde{d}$ is immediate from its definition. $\square$

It is useful to understand the geometric origins of the results above. Recall that we write $i : \mathcal{Q} \to \tilde{M}$ for the embedding of $\mathcal{Q}$ in $\tilde{M}$. The identification of $\mathcal{Q}$ with $i(\mathcal{Q})$ induces an identification of $T^*\mathcal{Q} = i^*(T^*\tilde{M})$ with $(i_*T\mathcal{Q})^*$. This in turn is canonically identified with the quotient of $T^*\tilde{M}|_{\mathcal{Q}}$ by the conormal bundle. Since (along $\mathcal{Q}$) $2\boldsymbol{X} = \boldsymbol{d}Q$ is a section of the latter it follows immediately that this quotient is, in terms of the notation above, precisely $\Big(T^*\tilde{M}/\varepsilon(\boldsymbol{X})\Big)|_{\mathcal{Q}}$. Taking exterior powers we have $\Lambda^k T^*\mathcal{Q}$ identified with $\Big(\Lambda^k T^*\tilde{M}/\varepsilon(\boldsymbol{X})\Big)|_{\mathcal{Q}}$. Now for $\varphi \in \Gamma(\Lambda^k T^*\mathcal{Q})$ let us say that $\tilde{\varphi} \in \Gamma(\Lambda^k\tilde{M})$ is an *extension* of φ if $i^*\tilde{\varphi} = \varphi$. Writing $\boldsymbol{d}$ also for the exterior derivative on $\mathcal{Q}$ we have $\boldsymbol{d}i^*\tilde{\varphi} = i^*\boldsymbol{d}\tilde{\varphi}$. So $\boldsymbol{d}\tilde{\varphi}$ is an extension of $\boldsymbol{d}\varphi$ and hence $\boldsymbol{d}$ is tangential as an operator on $\Lambda^k T^*M/\varepsilon(\boldsymbol{X})$. Thus $\boldsymbol{d} : \boldsymbol{\mathcal{G}}^k(w)|_{\mathcal{Q}} \to \boldsymbol{\mathcal{G}}^{k+1}(w)|_{\mathcal{Q}}$ (is again shown to be well defined and) is really just a restriction of the exterior derivative on $\mathcal{Q}$ to homogeneous sections.

Similarly $\boldsymbol{\delta} : \boldsymbol{\mathcal{G}}_k(w) \to \boldsymbol{\mathcal{G}}_{k-1}(w)$ may be viewed as a restriction of the formal adjoint of the exterior derivative on $\mathcal{Q}$. Since $\mathcal{Q}$ has no nondegenerate metric, or even conformal structure, we should view the latter as an operator on weighted exterior powers of the tangent bundle $T\mathcal{Q}$. From this viewpoint the properties of $\boldsymbol{\delta}$ on $\boldsymbol{\mathcal{G}}_k(w)$ follow easily. Rather than introduce new notation to explain this explicitly, essentially the same argument may be phrased in terms of the ambient metric as follows. Pick (just locally if necessary) a volume form for the ambient metric. Then we have an ambient Hodge star $\bigstar$. Note that $\bigstar\varepsilon(\boldsymbol{X})\boldsymbol{d}$ and $\boldsymbol{\delta}\bigstar\varepsilon(\boldsymbol{X})$ agree up to sign. Similarly, $\bigstar\varepsilon(\boldsymbol{X})$ and $\iota(\boldsymbol{X})\star$ agree up to sign, while $\bigstar\varepsilon(\boldsymbol{X})$ gives an isomorphism between $\Gamma(\Lambda^k T^*\tilde{M}/\varepsilon(\boldsymbol{X}))$ and the subspace of $\Gamma(\Lambda^{n+1-k}T^*\tilde{M})$ consisting of sections annihilated by $\iota(\boldsymbol{X})$. It follows now that the results in the theorem for $\boldsymbol{\delta}$ are equivalent to the corresponding results for $\boldsymbol{d}$.

Remark: From these observations it is clear that $\tilde{d}$ and $\tilde{\delta}$ do not depend on the ambient construction. In fact, in a sense that we will presently describe, they do not depend on the conformal structure either. On any n-manifold M we may view the total space $\tilde{\mathcal{Q}}$ of $\wedge^n T^*M$, with zero section removed, as a principal $\mathbb{R}_\times$-bundle. Densities of weight w on the underlying manifold correspond to functions f on $\tilde{\mathcal{Q}}$ which are homogeneous in the sense that $\rho^s_* f = |s|^{w/n} f$, where ρ denotes the $\mathbb{R}_\times$ action. Let $\sim$ denote equivalence on the total space of $\wedge^k T^*\tilde{\mathcal{Q}}$ given by $U \sim V$ if $U = \rho^s_* V$ for some $s \in \mathbb{R}_\times$. Then the quotient $\wedge^k T^*\tilde{\mathcal{Q}}/\sim$ may be viewed as a vector bundle $\tilde{\mathbb{G}}^k$ on M and this has a composition series $E^{k-1} \oplus\!\!\!\!\!\!- E^k$. A section U of $\wedge^k T^*\tilde{\mathcal{Q}}$ which is homogeneous of degree w (i.e. $\rho^s_* U = |s|^w U$ at each point of $\tilde{\mathcal{Q}}$) is equivalent to a section of the bundle $\tilde{\mathbb{G}}^k[w] = \tilde{\mathbb{G}}^k \otimes E[w]$, where $E[w]$ indicates the bundle of densities of weight w on M. The exterior derivative on $\tilde{\mathcal{Q}}$ preserves homogeneity and so determines an operator $\tilde{d} : \tilde{\mathbb{G}}^k[w] \to \tilde{\mathbb{G}}^{k+1}[w]$. If we write $\tilde{\mathbb{G}}_k[w]$ for the bundle $(\tilde{\mathbb{G}}^k)^* \otimes E[w-n]$ then we may define $\tilde{\delta} : \tilde{\mathbb{G}}_{k+1}[-w] \to \tilde{\mathbb{G}}_k[-w]$ as the formal adjoint of $\tilde{d}$. (Alternatively $\tilde{\delta}$ may be obtained from the formal adjoint of the exterior derivative on $\tilde{\mathcal{Q}}$.) It is not difficult to show that when M has a conformal structure these operators agree with the operators $\tilde{d}$ and $\tilde{\delta}$ defined above.

Given the construction described here, it is clear that there are analogues of $\tilde{d}$ of $\tilde{\delta}$ for many other geometries and situations. This will be taken up elsewhere. ∎

Next we compare $\tilde{d}$ with the exterior derivative on M. More precisely writing q_k for the natural injection $\mathcal{E}^k[w] \to \mathcal{G}^k[w]$, we want to compare the compositions $q_{k+1}d$ and $\tilde{d}q_k$ as operators on $\mathcal{E}^k$. We write $\boldsymbol{\mathcal{F}} := \boldsymbol{\mathcal{F}}^1$. Recall that $2\boldsymbol{h}(\boldsymbol{X},\cdot) = dQ$, so $\boldsymbol{\mathcal{F}}|_{\mathcal{Q}}$ is the subspace of $\boldsymbol{\mathcal{T}}$ consisting of sections that take values in $T\mathcal{Q} \subset T\tilde{M}|_{\mathcal{Q}}$. The elements of $\boldsymbol{\mathcal{F}}$ are equivalent to sections of the tractor subbundle $\mathbb{F}$. $T^*\tilde{M}/\varepsilon(\boldsymbol{X})$ is the bundle dual to the subbundle of $T\tilde{M}$ consisting of vectors annihilated by $\iota(\boldsymbol{X})$. (We have already observed above that $T^*\tilde{M}/\varepsilon(\boldsymbol{X})|_{\mathcal{Q}}$ is the bundle dual to $T\mathcal{Q} \subset T\tilde{M}_{\mathcal{Q}}$, which is what this statement amounts to along $\mathcal{Q}$.) Via the ambient metric $T^*\tilde{M}/\varepsilon(\boldsymbol{X})|_{\mathcal{Q}} \cong T\tilde{M}/\varepsilon(\boldsymbol{X})|_{\mathcal{Q}}$ (where, as usual, in the index free notation we are not distinguishing forms from their contravariant equivalents obtained by the ambient metric). Also from above the homogeneous sections of $T\tilde{M}/\varepsilon(\boldsymbol{X})$ of weight w are denoted $\boldsymbol{\mathcal{G}}^1(w)$ (with $\boldsymbol{\mathcal{G}}^1 := \boldsymbol{\mathcal{G}}^1(0)$) and so $\boldsymbol{\mathcal{G}}^1(w)$ is a natural dual space to $\boldsymbol{\mathcal{F}}(-w)$. Now dualising the discussion of Section 3.1 it is clear that $\pi^*(\mathcal{E}^1[1])$ is the subspace of $\boldsymbol{\mathcal{G}}^1|_{\mathcal{Q}}$ consisting of sections which annihilate vertical fields in $\boldsymbol{\mathcal{F}}|_{\mathcal{Q}}$. Since the vertical vector fields are generated by $\boldsymbol{X}$, this is the subspace of $\boldsymbol{\mathcal{G}}^1(1)|_{\mathcal{Q}}$ annihilated by (the form field) $\boldsymbol{X}$. This is well defined since $\boldsymbol{X}$ is null along $\mathcal{Q}$. So if $\omega \in \mathcal{E}^1[1]$ then the section of $\boldsymbol{\mathcal{G}}^1(1)|_{\mathcal{Q}}$ equivalent to $q_1\omega \in \mathcal{G}^1[1]$ is exactly $\pi^*\omega$. Taking exterior powers and tensoring with an appropriate density bundle we conclude that similarly the section of $\boldsymbol{\mathcal{G}}^k|_{\mathcal{Q}}$ equivalent to $q_k\varphi$, for $\varphi \in \mathcal{E}^k$, is $\pi^*\varphi$. Now since, as forms on $\mathcal{Q}$, we have $\boldsymbol{d}\pi^*\varphi = \pi^* d\varphi$ we have the first result of the following proposition. The second result here follows immediately from Proposition 3.10 below.

Proposition 3.9. *As operators $\mathcal{E}^k \to \mathcal{G}^{k+1}$ we have*

$$\tilde{d} q_k = q_{k+1} d.$$

Similarly

$$q^k \tilde{\delta} = \delta q^{k+1}$$

as operators $\mathcal{G}_{k+1} \to \mathcal{E}_k$. Here $q^k : \mathbb{G}_k \to E_k$ is the bundle morphism algebraically dual to q_k.

The operators $\tilde{d}$ and $\tilde{\delta}$ are readily described in terms of metrics from the conformal class using the machinery from Section 3.1. For $\sigma \in \mathcal{E}[1]$ a choice of conformal scale let $\tilde{\sigma} \in \tilde{\mathcal{E}}(1)$ be any section such that $\tilde{\sigma}|_{\mathcal{Q}}$ is the homogeneous function equivalent to σ. Define $\boldsymbol{Y} := \boldsymbol{I}_\sigma - \frac{1}{2}(\boldsymbol{I}_\sigma \bullet \boldsymbol{I}_\sigma)\boldsymbol{X}$ where $\boldsymbol{I}_\sigma := \frac{1}{n}\tilde{\sigma}^{-1}\varepsilon(\boldsymbol{D\!\!\!/})\tilde{\sigma}$. Then $\boldsymbol{Y}$ is a section of $\boldsymbol{\mathcal{T}}(-1)$ such that $\boldsymbol{h}(\boldsymbol{X}, \boldsymbol{Y}) = 1$ and by construction $\boldsymbol{Y}|_{\mathcal{Q}}$ is the homogeneous field along $\mathcal{Q}$ equivalent to the Y corresponding to $\sigma \in \mathcal{E}[1]$ (via (29)). We make the definition $\tilde{\boldsymbol{D}} := \boldsymbol{\nabla} - \boldsymbol{X} \otimes \boldsymbol{\nabla}_{\boldsymbol{Y}}$, where of course $\boldsymbol{X}$ means the 1-form $dQ/2$. Note that $\tilde{\boldsymbol{D}}_{\boldsymbol{Y}} = 0$, and that $\tilde{\boldsymbol{D}} Q = 0$ and so $\tilde{\boldsymbol{D}}$ is a robust tangential operator. Also if, along $\mathcal{Q}$, $V \in \mathfrak{X}(\tilde{M})$ takes values in $T\mathcal{Q}$ then we have (along $\mathcal{Q}$) $\tilde{\boldsymbol{D}}_V = \boldsymbol{\nabla}_V$. Furthermore if $\tilde{F}$ is an ambient tensor field homogeneous of weight w then we have $\tilde{\boldsymbol{D}}_{\boldsymbol{X}} \tilde{F} = w\tilde{F}$. With these observations it follows that that $\tilde{\boldsymbol{D}}$ descends to the operator on weighted tractor bundles given by

$$\tilde{D} F = wY \otimes F + Z \cdot \nabla F \tag{31}$$

for F of weight w, where ∇ means the coupled tractor–Levi-Civita connection. (This operator has already played a role in conformal geometry [29, 30].) To see this observe first that if $w = 0$ then, aside from straightforward details, the result is tautological, given the definition of the tractor connection in terms of the ambient connection. For other weights observe that upon restriction to vector fields with values in $T\mathcal{Q}$ we have $\tilde{\boldsymbol{D}} \tilde{F} = \tilde{\sigma}^w \boldsymbol{\nabla} \tilde{\sigma}^{-w} \tilde{F} + w\boldsymbol{Y} \otimes \tilde{F}$. Since $\tilde{\sigma}^{-w}\tilde{F}$ has weight 0, by the result for the weight zero case this is equivalent to $\sigma^w Z \cdot \nabla \sigma^{-w} F + wY \otimes F$. But by the definition of the coupled connection this is exactly $\tilde{D} F$ as given. By construction $\tilde{D}$ depends on σ.

Since the ambient connection is torsion free we have, for ambient k-forms F, $\boldsymbol{d}F = \varepsilon(\boldsymbol{\nabla})F$. Considering tangential components of this it follows that for $V \in \mathcal{G}^k[w]$ we have

$$\tilde{d} V = \varepsilon(\tilde{D}) V.$$

The right-hand side here means that we start with any section $V' \in \mathcal{T}^k[w-k]$ representing V, and then take the class of $\varepsilon(\tilde{D})V'$ in $\mathcal{G}^{k+1}[w]$. The equality with the left-hand-side guarantees that this is a well defined operation, with a result that is independent of the choice of conformal scale σ. This may also be verified directly by first noting that $\varepsilon(X)\varepsilon(\tilde{D})$ is independent of Y, and then, via (15), that $Z_{[A}{}^a \nabla_a X_{B]} = 0$, and thus $\{\varepsilon(\tilde{D}), \varepsilon(X)\} = \varepsilon(Y)\varepsilon(X)$.

The sections $\mathbb{Y}^k$ and $\mathbb{Z}^k$ may be used in the obvious way to give scale-dependent decompositions of the $\mathcal{G}^k[w]$ bundles. For example, $\mathbb{Y}^k \cdot \alpha + \mathbb{Z}^k \cdot \mu$ denotes the image in $\mathcal{G}^k[w]$ of $\mathbb{Y}^k \cdot \alpha + \mathbb{Z}^k \cdot \mu + \mathbb{W}^k \cdot \varphi + \mathbb{X}^k \cdot \rho \in \mathcal{T}^k[w-k]$ (for any φ and ρ), and $q_k : \mathcal{E}^k[w] \to \mathcal{G}^k[w]$ is given by $\mu \mapsto \mathbb{Z}^k{\cdot}\mu$. With these conventions and using the formulae (20) to calculate $\varepsilon(\tilde{D})V$, we obtain the very simple formula

$$\tilde{d}V = \mathbb{Y}^{k+1}{\cdot}(w\mu - \varepsilon(\nabla)\sigma) + \mathbb{Z}^{k+1}{\cdot}\varepsilon(\nabla)\mu. \tag{32}$$

Next recall that $\tilde{\delta}$ on $\mathcal{F}^{k+1}[w]$ arises from the action of $\boldsymbol{\delta} = -\iota(\boldsymbol{\nabla})$ on $\boldsymbol{\mathcal{F}}^{k+1}(w)$. Now $\iota(\boldsymbol{\nabla}) = \iota(\tilde{\boldsymbol{D}}) + \iota(\boldsymbol{X})\boldsymbol{\nabla}_{\boldsymbol{Y}}$. So using that $\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{X} = \boldsymbol{Y}$ we have, on $\boldsymbol{\mathcal{F}}^{k+1}(w)$, that

$$\iota(\boldsymbol{\nabla}) = \iota(\tilde{\boldsymbol{D}}) - \iota(\boldsymbol{Y}).$$

The operators on the right-hand side here are both tangential on $\boldsymbol{\mathcal{T}}^{k+1}(w)$. Thus $\tilde{\delta}$ on $\mathcal{F}^{k+1}[w]$ is given by $\iota(Y) - \iota(\tilde{D}) = (1-w)\iota(Y) - \iota(Z{\cdot}\nabla)$. We can once again use (27) and (20) to expand this; for $F = \mathbb{Z}^{k+1}{\cdot}\mu + \mathbb{X}^{k+1}{\cdot}\rho \in \mathcal{F}^{k+1}[w]$ we obtain $\tilde{\delta}F = \mathbb{Z}^k{\cdot}((k-n-w)\rho - \iota(\nabla)\mu) + \mathbb{X}^k{\cdot}\iota(\nabla)\rho$. Recall that $\mathcal{G}_k[w]$ is an alternative notation for $\mathcal{F}^k[w+k-n]$, so if now we suppose that $F \in \mathcal{G}_{k+1}[-w]$, we have

$$\tilde{\delta}F = \mathbb{Z}^k{\cdot}(w\rho - \iota(\nabla)\mu) + \mathbb{X}^k{\cdot}\iota(\nabla)\rho. \tag{33}$$

The naturality of $\tilde{d}$ and $\tilde{\delta}$ (and the fact that they are first-order differential operators) is immediate from these explicit formulae, and using these formulae, it is now straightforward to verify that $\tilde{\delta}$ and $\tilde{d}$ are formal adjoints with respect to the integral pairing between $\mathcal{G}_k[-w]$ and $\mathcal{G}^k[w]$. A further inspection of these formulae also reveals parts (*ii*) and (*iii*) of the next proposition.

Proposition 3.10. (*i*) *The operators $\tilde{\delta} : \mathcal{G}_{k+1}[-w] \to \mathcal{G}_k[-w]$ and $\tilde{d} : \mathcal{G}^k[w] \to \mathcal{G}^{k+1}[w]$ are natural and are mutual formal adjoints.*
(*ii*) *When $w \neq 0$, $\tilde{d}q_k$ is, up to a constant multiple, a differential splitting of the canonical surjection $\mathcal{G}^{k+1}[w] \to \mathcal{E}^k[w]$ and $q^k\tilde{\delta}$ is, up to a constant multiple, a differential splitting of the inclusion $\mathcal{E}_k[-w] \to \mathcal{G}_{k+1}[-w]$.*
(*iii*) *If $w \neq 0$ then $\mathcal{N}(\tilde{d} : \mathcal{G}^k[w] \to \mathcal{G}^{k+1}[w]) = \mathcal{R}(\tilde{d} : \mathcal{G}^{k-1}[w] \to \mathcal{G}^k[w])$ and $\mathcal{N}(\tilde{\delta} : \mathcal{G}_k[-w] \to \mathcal{G}_{k-1}[-w]) = \mathcal{R}(\tilde{\delta} : \mathcal{G}_{k+1}[-w] \to \mathcal{G}_k[-w])$.*

Before Proposition 3.9, we observed that the section of $\boldsymbol{\mathcal{G}}^k|_{\mathcal{Q}}$ equivalent to $q_k\varphi$, for $\varphi \in \mathcal{E}^k$, is exactly the lift $\pi^*\varphi$. As a section of $\boldsymbol{\mathcal{G}}^{k-1}(1)|_{\mathcal{Q}}$, $\iota(\boldsymbol{X})\pi^*\varphi$ vanishes and from the considerations above it is clear that conversely if $\psi \in \boldsymbol{\mathcal{G}}^k|_{\mathcal{Q}}$ such that $\iota(\boldsymbol{X})\psi = 0 \in \boldsymbol{\mathcal{G}}^{k-1}(1)|_{\mathcal{Q}}$ then $\psi = \pi^*\varphi$ for some $\varphi \in \mathcal{E}^k$. We will extend the use of the term 'lift' as follows. If the restriction of $\tilde{F} \in \boldsymbol{\mathcal{T}}^k(w)$ to $\mathcal{Q}$ (i.e. $\tilde{F}|_{\mathcal{Q}}$) is, up to a non-zero constant multiple, the homogeneous section equivalent to $F \in \mathcal{T}^k[w]$ then $\tilde{F}$ is termed an *ambient lift* of F while $\tilde{F}|_{\mathcal{Q}}$ will be called a *lift* of F. For $T \in \mathcal{G}^k[w]$, the term *ambient lift* will be used for $\tilde{T} \in \boldsymbol{\mathcal{G}}^k(w)$ with the property that $\tilde{T}|_{\mathcal{Q}}$ is, up to a constant non-zero scale, equivalent to T; and also for any representative of this in $\boldsymbol{\mathcal{T}}^k(w)$. As for the other cases, $\tilde{T}|_{\mathcal{Q}}$ will be called a *lift* of T. If $T = q_k t$ for $t \in \mathcal{E}^k[w+k]$ then $\tilde{T}$ or any representative of this in $\boldsymbol{\mathcal{T}}^k(w)$ will also be said to be *ambient lifts* of t.

In each case the lift of a section is unique (up to a constant multiple), whereas there is choice in an ambient lift. For most weights w (in a sense made precise in part (*ii*) of the proposition just below), there is a special ambient lift of $t \in \mathcal{E}^k[w]$ so that at least the restriction of this to $\mathcal{Q}$ is unique.

First note that it is obvious that acting with $\varepsilon(\boldsymbol{X})$ gives a well defined operator from $\Gamma(\Lambda^k T^*\tilde{M}/\varepsilon(\boldsymbol{X})|\mathcal{Q}) \to \Gamma(\Lambda^{k+1}T^*\tilde{M})$ (which is, in fact, injective) and so also $\varepsilon(\boldsymbol{X}) : \boldsymbol{\mathcal{G}}^k(w) \to \boldsymbol{\mathcal{T}}^{k+1}(w-k+1)$ is well defined. By construction the composition $\iota(\boldsymbol{D\!\!\!/})\varepsilon(\boldsymbol{X}) : \boldsymbol{\mathcal{G}}^k(w) \to \boldsymbol{\mathcal{T}}^k(w-k)$ acts tangentially along $\mathcal{Q}$.

Proposition 3.11. *Let $U \in \boldsymbol{\mathcal{T}}^k(\ell - n/2)$ be an ambient lift of $u \in \mathcal{E}^k[w]$, where $w = k + \ell - n/2$. Then*
(i) $\iota(\boldsymbol{X})\iota(\boldsymbol{D\!\!\!/})\varepsilon(\boldsymbol{X})U|_{\mathcal{Q}} = 0$;
(ii) Up to scale $\iota(\boldsymbol{D\!\!\!/})\varepsilon(\boldsymbol{X})U$ is an ambient lift of u if and only if $\ell \neq -1$ and $k \neq \ell + n/2$.

Proof:
(i) First note that since $U \in \boldsymbol{\mathcal{T}}^k(\ell - n/2)$ is an ambient lift of $u \in \mathcal{E}^k[w]$ the image of $\iota(\boldsymbol{X})U|_{\mathcal{Q}}$ in $\boldsymbol{\mathcal{G}}^{k-1}(w))|_{\mathcal{Q}}$ must vanish. Thus $\iota(\boldsymbol{X})U = \varepsilon(\boldsymbol{X})V + QW$ for some ambient forms V and W. The last two form fields are not independent. Since all forms are smooth and

$$0 = \iota(\boldsymbol{X})\iota(\boldsymbol{X})U = \iota(\boldsymbol{X})(\varepsilon(\boldsymbol{X})V + QW),$$

a short calculation gives $\varepsilon(\boldsymbol{X})V + \varepsilon(X)\iota(\boldsymbol{X})W = 0$, and so $\iota(\boldsymbol{X})U = \iota(\boldsymbol{X})\varepsilon(\boldsymbol{X})W$. Thus we may assume without loss of generality that $V = -\iota(\boldsymbol{X})W$.

Now for an ambient k-form F of weight s we have

$$\begin{aligned}
&\iota(\boldsymbol{D\!\!\!/})F = -(n+2s-2)\boldsymbol{\delta}F + \iota(\boldsymbol{X})\boldsymbol{\Delta\!\!\!\!/}F, \qquad \text{and so}\\
&\iota(\boldsymbol{X})\iota(\boldsymbol{D\!\!\!/})F = -(n+2s-2)\iota(\boldsymbol{X})\boldsymbol{\delta}F = (n+2s-2)\boldsymbol{\delta}\iota(\boldsymbol{X})F.
\end{aligned}$$

Thus $\iota(\boldsymbol{X})\iota(\boldsymbol{D\!\!\!/})\varepsilon(\boldsymbol{X})U = (n + 2w - 2k)\boldsymbol{\delta}\iota(\boldsymbol{X})\varepsilon(\boldsymbol{X})U$. Since $\iota(\boldsymbol{X})\varepsilon(\boldsymbol{X}) + \varepsilon(\boldsymbol{X})\iota(\boldsymbol{X})$ is Q as a left multiplication operator, we have $(n+2w-2k)[\boldsymbol{\delta}QU - \boldsymbol{\delta}\varepsilon(\boldsymbol{X})\iota(\boldsymbol{X})U]$. Recall that $[\boldsymbol{\delta}, Q] = -2\iota(\boldsymbol{X})$. Thus

$$\begin{aligned}
&\boldsymbol{\delta}QU = -2\iota(\boldsymbol{X})U + O(Q) = -2\iota(\boldsymbol{X})\varepsilon(\boldsymbol{X})W + O(Q) \qquad \text{and}\\
&-\boldsymbol{\delta}\varepsilon(\boldsymbol{X})\iota(\boldsymbol{X})U = -\boldsymbol{\delta}\varepsilon(\boldsymbol{X})\iota(\boldsymbol{X})\varepsilon(\boldsymbol{X})W = -\boldsymbol{\delta}Q\varepsilon(\boldsymbol{X})W\\
&\qquad = 2\iota(\boldsymbol{X})\varepsilon(\boldsymbol{X})W + O(Q) = O(Q).
\end{aligned}$$

So $\iota(\boldsymbol{X})\iota(\boldsymbol{D\!\!\!/})\varepsilon(\boldsymbol{X})U = O(Q)$, as required. □

(ii) We must show that, up to a non-zero constant multiple, $\iota(\boldsymbol{D\!\!\!/})\varepsilon(\boldsymbol{X})U|_{\mathcal{Q}}$ and $U|_{\mathcal{Q}}$ represent the same section of $\boldsymbol{\mathcal{G}}^k(w)|_{\mathcal{Q}}$. Let us re-express $\iota(\boldsymbol{D\!\!\!/})\varepsilon(\boldsymbol{X})U$. Using $\{\boldsymbol{\delta}, \varepsilon(\boldsymbol{X})\} = \boldsymbol{\mathcal{L}}^*_{\boldsymbol{X}}$ and the weight of U, we have

$$\boldsymbol{\delta}\varepsilon(\boldsymbol{X})U = -\varepsilon(\boldsymbol{X})\boldsymbol{\delta}U - (n - 2k + w + 2)U.$$

Also $[\boldsymbol{\Delta\!\!\!\!/}, \varepsilon(\boldsymbol{X})]U = -2\boldsymbol{d}U$, so

$$\begin{aligned}
\iota(\boldsymbol{X})\boldsymbol{\Delta\!\!\!\!/}\varepsilon(\boldsymbol{X})U &= -2\iota(\boldsymbol{X})\boldsymbol{d}U + \iota(\boldsymbol{X})\varepsilon(\boldsymbol{X})\boldsymbol{\Delta\!\!\!\!/}U\\
&= -2\boldsymbol{\mathcal{L}}_{\boldsymbol{X}}U + 2\boldsymbol{d}\iota(\boldsymbol{X})U - \varepsilon(\boldsymbol{X})\iota(\boldsymbol{X})\boldsymbol{\Delta\!\!\!\!/}U + O(Q)\\
&= -2wU + 2\boldsymbol{d}\iota(\boldsymbol{X})U - \varepsilon(\boldsymbol{X})\iota(\boldsymbol{X})\boldsymbol{\Delta\!\!\!\!/}U + O(Q).
\end{aligned}$$

Combining these yields

$$\begin{aligned}
\iota(\boldsymbol{D\!\!\!/})\varepsilon(\boldsymbol{X})U &= (n+w-2k)(n+2w-2k+2)U + (n+2w-2k)\varepsilon(\boldsymbol{X})\boldsymbol{\delta}U\\
&\qquad -\varepsilon(\boldsymbol{X})\iota(\boldsymbol{X})\boldsymbol{\Delta\!\!\!\!/}U + 2\boldsymbol{d}\iota(\boldsymbol{X})U.
\end{aligned}$$

As observed above, since U is an ambient lift of u, $\iota(\boldsymbol{X})U = \varepsilon(\boldsymbol{X})V + QW$ where V is a $(k-2)$-form and W a $(k-1)$-form. Thus $\boldsymbol{d}\iota(\boldsymbol{X})U =$

$-\varepsilon(\boldsymbol{X})\boldsymbol{d}V+2\varepsilon(\boldsymbol{X})W+O(Q)$. As a result, upon restriction to vectors from $T\mathcal{Q}$, $\iota(\boldsymbol{D\!\!\!/})\varepsilon(\boldsymbol{X})U$ agrees precisely with $(n+w-2k)(n+2w-2k+2)U$. This completes the proof, since $(n+w-2k)(n+2w-2k+2)=2(n/2+\ell-k)(\ell+1)$. $\square$

Remark: The proof of part (i) above can be shortened somewhat if one first observes that for $u\in\mathcal{E}^k$ there is an ambient lift $\tilde{U}$ such that $\iota(\boldsymbol{X})\tilde{U}=0$. For example take $\tilde{U}=\iota(\boldsymbol{X})\varepsilon(\boldsymbol{Y})U$ where U is an ambient lift of u as above and as usual $\boldsymbol{Y}$ is a section of $\boldsymbol{\mathcal{T}}(-1)$ such that $\boldsymbol{h}(\boldsymbol{X},\boldsymbol{Y})=1$. Since $\iota(\boldsymbol{X})\varepsilon(\boldsymbol{Y})+\varepsilon(\boldsymbol{Y})\iota(\boldsymbol{X})$ is the identity on forms we have

$$\begin{aligned}\tilde{U}&=U-\varepsilon(\boldsymbol{Y})\iota(\boldsymbol{X})U\\&=U+\varepsilon(\boldsymbol{X})\varepsilon(\boldsymbol{Y})V-Q\varepsilon(\boldsymbol{Y})W\end{aligned}$$

from which is clear that $\tilde{U}\in\boldsymbol{\mathcal{T}}^k(\ell-n/2)$ and $U\in\boldsymbol{\mathcal{T}}^k(\ell-n/2)$ represent the same section of $\boldsymbol{\mathcal{G}}^k(w)|_{\mathcal{Q}}$. On the other hand from the last display and Proposition 3.3 it is also clear that $\iota(\boldsymbol{D\!\!\!/})\varepsilon(\boldsymbol{X})U|_{\mathcal{Q}}=\iota(\boldsymbol{D\!\!\!/})\varepsilon(\boldsymbol{X})\tilde{U}|_{\mathcal{Q}}$. ‖

We observed above that the tangential operator $\iota(\boldsymbol{D\!\!\!/})\varepsilon(\boldsymbol{X}):\boldsymbol{\mathcal{G}}^k(w)\to\boldsymbol{\mathcal{T}}^k(w-k)$ is well defined. By Proposition 3.5 this descends to a natural conformally invariant operator $\iota(D\!\!\!/)\varepsilon(X):\mathcal{G}^k[w]\to\mathcal{T}^k[w]$ for all weights w. Next note that from part (i) of the last proposition the composition $\iota(D\!\!\!/)\varepsilon(X)q_k$, acting on $\mathcal{E}^k[w]$, takes values in the subbundle $\mathbb{F}^k[w-k]$ of $\mathbb{T}^k[w-k]$. Part (ii) shows that for $\varphi\in\mathcal{E}^k[w]$ the image of $\iota(D\!\!\!/)\varepsilon(X)q_k\varphi$ in $\mathcal{G}^k[w]$ (under that natural quotient mapping $\mathbb{T}^k[w-k]\to\mathbb{G}^k[w]$) is in general a non-zero multiple of $q_k\varphi$. Now recall $\mathbb{G}^k[w]=E^{k-1}[w]\oplus E^k[w]$. From the definitions of $\mathbb{F}^k[w-k]$ and $\mathbb{G}^k[w]$ it follows that the composition $\mathbb{F}^k[w-k]\to\mathbb{T}^k[w-k]\to\mathbb{G}^k[w]$ takes values in the composition factor isomorphic to $E^k[w]$. So finally applying q_k^{-1} to this we obtain a non-zero map $p_k:\mathbb{F}^k[w-k]\to E^k[w]$ and this is a constant multiple of the canonical surjection $q^k:\mathbb{F}^k[w-k]\to E^k[w]$. Gathering these observations and results we have the following.

Proposition 3.12. *The composition*

$$\iota(D\!\!\!/)\varepsilon(X):\mathcal{G}^k[w]\to\mathcal{T}^k[w]$$

is a conformally invariant natural differential operator. If $w=k+\ell-n/2$, then up to a constant non-zero multiple,

$$\iota(D\!\!\!/)\varepsilon(X)q_k:\mathcal{E}^k[w]\to\mathcal{F}^k[w-k]$$

is a differential splitting of the canonical surjection $q^k:\mathcal{F}^k[w-k]\to\mathcal{E}^k[w]$ if and only if $\ell\neq-1$ and $k\neq\ell+n/2$. That is, $q^k\iota(D\!\!\!/)\varepsilon(X)q_k$ is a multiple of the identity on $\mathcal{E}^k[w]$ and the multiple is non-zero if and only if $\ell\neq-1$ and $k\neq\ell+n/2$ (or equivalently $k-1-n/2\neq w\neq 2k-n$).

4. Proofs of the main theorems and their extensions

The following set of simple but remarkable results are central to the constructions which follow.

Lemma 4.1. *If $V\in\boldsymbol{\mathcal{T}}^k(\ell-n/2+1)$ and $U\in\boldsymbol{\mathcal{T}}^k(\ell-n/2)$ then for $\ell=0,1,\cdots$ we have*

$$\boldsymbol{\Delta\!\!\!\!/}^{\ell}\varepsilon(\boldsymbol{D\!\!\!/})V=\varepsilon(\boldsymbol{X})\boldsymbol{\Delta\!\!\!\!/}^{\ell+1}V,\quad \boldsymbol{\Delta\!\!\!\!/}^{\ell}\iota(\boldsymbol{D\!\!\!/})V=\iota(\boldsymbol{X})\boldsymbol{\Delta\!\!\!\!/}^{\ell+1}V$$

and

$$\varepsilon(\not{\boldsymbol{D}})\not{\boldsymbol{\Delta}}^{\ell}U = \not{\boldsymbol{\Delta}}^{\ell+1}\varepsilon(\boldsymbol{X})U, \quad \iota(\not{\boldsymbol{D}})\not{\boldsymbol{\Delta}}^{\ell}U = \not{\boldsymbol{\Delta}}^{\ell+1}\iota(\boldsymbol{X})U.$$

Here $\not{\boldsymbol{\Delta}}^0$ *means 1.*

Proof: We will prove the first identity; the proofs of the others are similar. First observe that acting on any ambient form field, we have

$$\not{\boldsymbol{\Delta}}^{\ell}(2\ell\boldsymbol{d}+\varepsilon(\boldsymbol{X})\not{\boldsymbol{\Delta}}) = 2\ell\not{\boldsymbol{\Delta}}^{\ell}\boldsymbol{d}+\not{\boldsymbol{\Delta}}^{\ell}\varepsilon(\boldsymbol{X})\not{\boldsymbol{\Delta}} = 2\ell\not{\boldsymbol{\Delta}}^{\ell}\boldsymbol{d}+[\not{\boldsymbol{\Delta}}^{\ell},\varepsilon(\boldsymbol{X})]\not{\boldsymbol{\Delta}}+\varepsilon(\boldsymbol{X})\not{\boldsymbol{\Delta}}^{\ell+1}.$$

Now recall from the Tables 1 and 2 that $[\not{\boldsymbol{\Delta}},\varepsilon(\boldsymbol{X})] = -2\boldsymbol{d}$, and that $\not{\boldsymbol{\Delta}}$ and $\boldsymbol{d}$ commute. Thus $[\not{\boldsymbol{\Delta}}^{\ell},\varepsilon(\boldsymbol{X})]\not{\boldsymbol{\Delta}} = -2\ell\not{\boldsymbol{\Delta}}^{\ell}\boldsymbol{d}$, giving

$$\not{\boldsymbol{\Delta}}^{\ell}(2\ell\boldsymbol{d}+\varepsilon(\boldsymbol{X})\not{\boldsymbol{\Delta}}) = \varepsilon(\boldsymbol{X})\not{\boldsymbol{\Delta}}^{\ell+1}.$$

On the other hand, from the definition of $\varepsilon(\not{\boldsymbol{D}})$, we have that $\varepsilon(\not{\boldsymbol{D}})V = (2\ell\boldsymbol{d}+\varepsilon(\boldsymbol{X})\not{\boldsymbol{\Delta}})V$ for $V \in \boldsymbol{\mathcal{T}}^{k-1}(\ell-n/2+1)$. □

Recall from Proposition 3.7 that the powers $\not{\boldsymbol{\Delta}}^m$ determine conformally invariant operators $\not{\Delta}_m : \mathcal{T}^k[m-n/2] \to \mathcal{T}^k[-m-n/2]$ for $m \in \{0,1,2,\ldots\}$. Since, via the metric on $\mathcal{T}^k$, sections of $\mathcal{T}^k[m-n/2]$ and $\mathcal{T}^k[-m-n/2]$ pair to give sections of $\mathcal{E}[-n]$, it follows that the formal adjoint of $\not{\Delta}_m$ is a conformally invariant differential operator between the same spaces:

$$\not{\Delta}_m^* : \mathcal{T}^k[m-n/2] \to \mathcal{T}^k[-m-n/2].$$

For each m, let us define the operator $\not{\Box}_m$ to be the average of these, namely

(34)
$$\tfrac{1}{2}(\not{\Delta}_m+\not{\Delta}_m^*) =: \not{\Box}_m : \mathcal{T}^k[m-n/2] \to \mathcal{T}^k[-m-n/2] \quad m \in \{0,1,2,\ldots\}.$$

By construction $\not{\Box}_m$ is formally self-adjoint. We now have the following consequence of the lemma above.

Lemma 4.2. *If* $V \in \mathcal{T}^k(\ell-n/2+1)$ *and* $U \in \mathcal{T}^k(\ell-n/2)$ *then for* $\ell = 0,1,\cdots$ *we have*

$$\not{\Box}_\ell\varepsilon(\not{D})V = \varepsilon(X)\not{\Box}_{\ell+1}V, \quad \not{\Box}_\ell\iota(\not{D})V = \iota(X)\not{\Box}_{\ell+1}V$$

and

$$\varepsilon(\not{D})\not{\Box}_\ell U = \not{\Box}_{\ell+1}\varepsilon(X)U, \quad \iota(\not{D})\not{\Box}_\ell U = \not{\Box}_{\ell+1}\iota(X)U.$$

Here $\not{\Box}_0$ *means 1.*

Proof: Recall from Proposition 3.5 that $\iota(\not{D})$ and $\varepsilon(\not{D})$ are formal adjoints. Similarly $\iota(X)$ and $\varepsilon(X)$ are formal adjoints. From Lemma 4.1 we have $\not{\Delta}_\ell\varepsilon(\not{D}) = \varepsilon(X)\not{\Delta}_{\ell+1}$ on $\mathcal{T}^k(\ell-n/2+1)$ and $\iota(\not{D})\not{\Delta}_\ell = \not{\Delta}_{\ell+1}\iota(X)$ on $\mathcal{T}^{k+1}(\ell-n/2)$. The formal adjoint of the latter gives $\not{\Delta}_\ell^*\varepsilon(\not{D}) = \varepsilon(X)\not{\Delta}_{\ell+1}^*$ on $\mathcal{T}^k(\ell-n/2+1)$. Averaging this with the former gives the first result. Then $\not{\Box}_\ell\iota(\not{D})V = \iota(X)\not{\Box}_{\ell+1}V$ follows from a similar argument. Taking formal adjoints on both sides of these two results then gives the remaining identities. □

For $\ell \in \mathbb{Z}$ let us define $\mathbb{K}_k^\ell : \mathcal{T}^k[\ell-n/2] \to \mathcal{T}^k[-\ell-n/2]$ by

(35)
$$\mathbb{K}_k^\ell = \begin{cases} \not{\Box}_\ell\iota(\not{D})\varepsilon(X) & \text{if} \quad \ell \geq 0, \\ \iota(X)\varepsilon(X) & \text{if} \quad \ell = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Each $\mathbb{K}^\ell_k$ is a composition of conformally invariant operators and so is conformally invariant. Note that for $U\in\mathcal{T}^k[\ell-n/2]$, Lemma 4.2 implies that

$$\not{\Box}_\ell\iota(\not{D})\varepsilon(X)U=\iota(X)\not{\Box}_{\ell+1}\varepsilon(X)U=\iota(X)\varepsilon(\not{D})\not{\Box}_\ell U,$$

and so these are each alternative expressions for $\mathbb{K}^\ell_k$. From the form $\mathbb{K}^\ell_k=\iota(X)\not{\Box}_{\ell+1}\varepsilon(X)$, it is immediate that $\mathbb{K}^\ell_k$ is formally self-adjoint and that $\iota(X)\mathbb{K}^\ell_k=0$. Thus $\mathbb{K}^\ell_k:\mathcal{T}^k[\ell-n/2]\to\mathcal{T}^k[-\ell-n/2]$ takes values in $\mathcal{G}_k[-w]\subset\mathcal{T}^k[-\ell-n/2]$. (Recall $w:=k+\ell-n/2$.) On the other hand it is also clear that the composition $\mathbb{K}^\ell_k\varepsilon(X)$ vanishes, and so we may naturally view $\mathbb{K}^\ell_k$ as a formally self-adjoint operator between $\mathcal{G}^k[w]$ and $\mathcal{G}_k[-w]$. Except where otherwise mentioned, we will take this point of view.

Proposition 4.3. *The expressions of* (35) *define formally self-adjoint conformally invariant differential operators*

$$\mathbb{K}^\ell_k:\mathcal{G}^k[w]\to\mathcal{G}_k[-w].$$

These are natural when $\ell=-1$, or when ℓ is in the range of m given in Proposition 3.7. For $V\in\mathcal{G}^{k-1}[w]$, $U\in\mathcal{G}^k[w]$ where $n/2+w-k=\ell\geq-1$ we have

$$\varepsilon(X)\mathbb{K}^{\ell+1}_{k-1}V=2(\ell+2)\mathbb{K}^\ell_k\tilde{d}V\quad\textit{and}\quad\mathbb{K}^{\ell+1}_{k-1}\iota(X)U=2(\ell+2)\tilde{\delta}\mathbb{K}^\ell_kU.\tag{36}$$

Proof: The first statement is established above. The naturality assertion is immediate from Propositions 3.7 and 3.12 since $\iota(\not{D})$, $\iota(X)$ and $\varepsilon(X)$ are natural.

Next observe that $\mathbb{K}^{\ell+1}_{k-1}\iota(X)U$ is given by the expression

$$\iota(X)\not{\Box}_{\ell+2}\varepsilon(X)\iota(X)U=-\iota(X)\not{\Box}_{\ell+2}\iota(X)\varepsilon(X)U.$$

From Lemma 4.2, $\iota(X)\not{\Box}_{\ell+2}\iota(X)\varepsilon(X)U=\iota(X)\iota(\not{D})\not{\Box}_{\ell+1}\varepsilon(X)U$. Now from Proposition 3.5 is is clear that $\iota(X)\iota(\not{D})=\iota(X)\iota(D)$. On the other hand, given a choice of scale, we have $\iota(X)\iota(D)F=-(n+2w-2)\iota(X)\iota(\tilde{D})F$, for F any form tractor of weight w. Here we have used (27) and (31). Thus noting that $\not{\Box}_{\ell+1}\varepsilon(X)U$ has weight $-(n/2+\ell+1)$ we have $\iota(X)\iota(\not{D})\not{\Box}_{\ell+1}\varepsilon(X)U=2(\ell+2)\iota(X)\iota(\tilde{D})\not{\Box}_{\ell+1}\varepsilon(X)U$. Next, using the formulae (20) for the tractor connection it is straightforward to verify that $[\tilde{D},X]=h-X\otimes Y$. Thus we have the identity $\iota(X)\iota(\tilde{D})=(\iota(Y)-\iota(\tilde{D}))\iota(X)=\tilde{\delta}\iota(X)$, and so

$$\iota(X)\not{\Box}_{\ell+2}\varepsilon(X)\iota(X)U|_{\mathcal{Q}}=2(\ell+2)\tilde{\delta}\iota(X)\not{\Box}_{\ell+1}\varepsilon(X)U|_{\mathcal{Q}},$$

which is the second identity of (36). (Note that none of the operators composed on either side depend on a choice of scale.) The first identity of (36) follows immediately by taking formal adjoints. $\square$

Remark: One can establish the identity $\iota(X)\iota(\tilde{D})=\tilde{\delta}\iota(X)$ without a choice of scale via the ambient metric. Simply observe that since $-\boldsymbol{\delta}=\iota(\boldsymbol{\nabla})=\iota(\tilde{\boldsymbol{D}})+\iota(\boldsymbol{X})\boldsymbol{\nabla}_{\boldsymbol{Y}}$, we have $\iota(\boldsymbol{X})\iota(\tilde{\boldsymbol{D}})=-\iota(\boldsymbol{X})\boldsymbol{\delta}=\boldsymbol{\delta}\iota(\boldsymbol{X})$. ‖

Remark: We may also define the conformally invariant operators

$$\mathbb{K}^{n-k}_{\ell,\star}=\begin{cases}\not{\Box}_\ell\varepsilon(\not{D})\iota(X) & \text{if}\quad \ell\geq0,\\ \varepsilon(X)\iota(X) & \text{if}\quad \ell=-1,\\ 0 & \text{otherwise.}\end{cases}\tag{37}$$

on $\mathcal{T}^{n-k+2}[\ell-n/2]$. By arguments similar to those above, we find that each $\mathbb{K}^{n-k}_{\ell,\star}$ descends to a well-defined formally self-adjoint conformally invariant operator

$$\mathbb{K}^{n-k}_{\ell,\star} : \mathcal{G}^{\star}_{n-k}[w] \to \mathcal{G}^{n-k}_{\star}[-w],$$

where $\mathcal{G}^{\star}_{n-k}[w]$ and $\mathcal{G}^{n-k}_{\star}[-w]$ are defined in the remark on page 30. In this context we obtain

$$\iota(X)\mathbb{K}^{n-k+1}_{\ell+1,\star} = -2(\ell+2)\mathbb{K}^{n-k}_{\ell,\star}\tilde{\delta}_{\star} \quad \text{and} \quad \mathbb{K}^{n-k+1}_{\ell+1,\star}\varepsilon(X) = -2(\ell+2)\tilde{d}^{\star}\mathbb{K}^{n-k}_{\ell,\star}.$$

Here $\tilde{\delta}_{\star} : \mathcal{G}^{\star}_{n-k+1}[w] \to \mathcal{G}^{\star}_{n-k}[w]$ and $\tilde{d}^{\star} : \mathcal{G}^{n-k}_{\star}[-w] \to \mathcal{G}^{n-k+1}_{\star}[-w]$ are first order conformally invariant operators which arise, respectively, from the ambient $\boldsymbol{\delta}$ and $\boldsymbol{d}$ by constructions parallelling the constructions of $\tilde{\delta}$ and $\tilde{d}$.

Suppose now that M is oriented. It is easily verified that on weighted k-form tractors we have

$$\iota(\not{D})\star = (-1)^k\star\varepsilon(\not{D}) \quad \text{and} \quad \varepsilon(\not{D})\star = (-1)^{k-1}\star\iota(\not{D}).$$

On the other hand, from the relation between $\star$ and the ambient volume form one obtains that on $\mathcal{T}^k[\ell-n/2]$, we have $\star\not{\Box}_\ell\star = (-1)^{k(n+2-k)+q+1}\not{\Box}_\ell$. It follows that up to a sign, $\mathbb{K}^{n-k}_{\ell,\star}$ is exactly $\star\mathbb{K}^k_\ell\star$. So on general manifolds these operators are, in a suitable sense, locally equivalent, and we may even (ignoring the issue of a possible overall sign difference) use this to give an alternative definition of $\mathbb{K}^{n-k}_{\ell,\star}$ on oriented neighbourhoods as $\star\mathbb{K}^k_\ell\star$. ∎

Recall that $\mathcal{G}^k[w]$ has the composition series $\mathcal{E}^{k-1}[w] \oplus \mathcal{E}^k[w]$ and $q_k : \mathcal{E}^k[w] \to \mathcal{G}^k[w]$ is the canonical inclusion. Dually $\mathcal{G}_k[-w]$ has the composition series $\mathcal{E}_k[-w] \oplus \mathcal{E}_{k-1}[-w]$ with canonical surjection $q^k : \mathcal{G}_k[-w] \to \mathcal{E}_k[-w]$. Thus we can define conformally invariant differential operators by compositions as follows:

$$\begin{array}{rl} (\mathbb{K}^\ell_k q_k =: \mathbb{L}^\ell_k) : & \mathcal{E}^k[w] \to \mathcal{G}_k[-w], \\ (q^k\mathbb{K}^\ell_k =: \overline{\mathbb{L}}^\ell_k) : & \mathcal{G}^k[w] \to \mathcal{E}_k[-w], \\ (q^k\mathbb{K}^\ell_k q_k =: L^\ell_k) : & \mathcal{E}^k[w] \to \mathcal{E}_k[-w]. \end{array} \tag{38}$$

Note that clearly $L^\ell_k = q^k\mathbb{L}^\ell_k = \overline{\mathbb{L}}^\ell_k q_k$. By construction and from Theorem 4.3 we have that $\mathbb{L}^\ell_k$ and $\overline{\mathbb{L}}^\ell_k$ are formal adjoints and that L^ℓ_k is formally self-adjoint. To simplify the notation, when the source bundles are true (unweighted) forms (the case $w=0$ above), and when $\ell = n/2-k$, we shall often omit the ℓ superscript.

A first concern is to verify that these operators are non-trivial. It suffices to establish this for the family L^ℓ_k.

Proposition 4.4. *For $k \in \{0,1,\cdots,n\}$ and $\ell \in \mathbb{N}$ such that $\ell + n/2 \neq k$, the operator $L^\ell_k : \mathcal{E}^k[w] \to \mathcal{E}_k[-w]$ is conformally invariant, formally self-adjoint, non-trivial and of order 2ℓ. It is quasi-Laplacian if and only if $k+\ell-n/2 =: w \neq 0$. In the Riemannian signature case the operator L^ℓ_k is elliptic if and only if $w \neq 0$, and it is positively elliptic if and only if $k \notin [n/2-\ell, n/2+\ell]$. For each k the differential operator sequence (4) is an elliptic complex.*

Proof: The claims of conformal invariance and symmetry under taking adjoints are established above.

Suppose we are in the Riemannian signature setting. Since $\not\Box_\ell$ is elliptic it has a finite-dimensional null space on compact manifolds. On the other hand, from Proposition 3.12, the range of $\iota(\not{\mathbb{D}})\varepsilon(X)q_k$ is infinite dimensional. Thus the composition $\not\Box_\ell\iota(\not{\mathbb{D}})\varepsilon(X)q_k = \mathbb{K}_k^\ell q_k =: \mathbb{L}_k^\ell$ is non-trivial in general. From Proposition 4.3, this composition takes values in the subbundle $\mathbb{G}_k[-w]$ of $\mathbb{T}^k(-n+k-w)$.

Let us consider $\mathbb{L}_k^\ell : \mathcal{E}^k[w] \to \mathcal{G}_k[-w]$ on the flat model S^n with its standard conformal structure. Recall that $\mathcal{G}_k[-w]$ has the composition series $\mathcal{E}_k[-w] \oplus \mathcal{E}_{k-1}[-w]$ and q^k is the map onto the quotient $q^k : \mathcal{G}_k[-w] \to \mathcal{E}_k[-w]$. If $q^k\mathbb{L}_k^\ell =: L_k^w : \mathcal{E}^k[w] \to \mathcal{E}_k[-w]$ is trivial then $\mathbb{L}_k^\ell$ determines a non-trivial conformally invariant operator $\mathcal{E}^k[w] \to \mathcal{E}_{k-1}[-w]$. There is no such operator [24] and so L_k^ℓ is non-trivial. (All the invariant operators between forms preserve k except d and δ, or restrictions or projections of d or δ to $n/2$-forms of one duality. δ maps $\mathcal{E}_k = \mathcal{E}^k[2k-n]$ to $\mathcal{E}_{k-1}$, so the only possibility for a true form operator is when $k = n/2$. but this is an $\ell = 0$ case, and so fails the assumption $k \neq \ell + n/2$.) By construction, Proposition 3.5 and Proposition 4.3, L_k^ℓ is natural in the conformally flat case.

Next observe that it is clear from the formulae for $\mathbb{K}^\ell$, q^k and q_k and the proof of Theorem 3.7 that L_k^ℓ has formally the same leading symbol on all structures (where we range over both signature and curvature). Thus the L_k^ℓ are non-trivial.

Specialising once again to the conformally flat Riemannian case the differential operators L_k^ℓ must be the unique (up to constant multiples) operators between the bundles concerned. For the remainder of the proof let us fix some choice of scale. Up to a non-zero constant multiple, the operator L_k^ℓ has the form

$$-u(\delta d)^\ell - w(d\delta)^\ell + \mathrm{LOT},$$

with w, u as in (1). This follows from the formulae for the operators on the sphere ([9], Remark 3.30). In particular $u \neq 0$ and $\mathcal{E}^k[u] = \mathcal{E}_k[-w]$ is the target bundle of L_k^ℓ. Thus in all cases the operators are of order 2ℓ. Next note that if $w \neq 0$ then

$$(-u^{-1}\delta d - w^{-1} d\delta + \mathrm{LOT})L_k^\ell = \not\Delta^{\ell+1} + \mathrm{LOT}.$$

The leading symbol of d is $\mathrm{i}\varepsilon(\xi)$. Thus if, on the other hand, $w = 0$, the leading symbol of L_k^ℓ annihilates the range of $\varepsilon(\xi)$, and so cannot be a right factor of the leading symbol of a power of the Laplacian. We conclude that L_k^ℓ is quasi-Laplacian if and only if $w \neq 0$.

Specialising to the Riemannian setting, it follows that L_k^ℓ is elliptic if and only if $w \neq 0$. Using the fact that the leading symbol of δ is $-\mathrm{i}\iota(\xi)$ we have that, up to a non-zero constant multiple, the leading symbol of L_k^ℓ is $|\xi|^{2\ell}(-u\iota(\xi')\varepsilon(\xi') - w\varepsilon(\xi')\iota(\xi'))$, where $\xi' = \xi/|\xi|$. But $\iota(\xi')\varepsilon(\xi')$ and $\varepsilon(\xi')\iota(\xi')$ are complementary projections on the fibre E_x^k. Thus the real linear combination $-u\iota(\xi')\varepsilon(\xi') - w\varepsilon(\xi')\iota(\xi')$ is definite if and only if w and u have the same sign. On the other hand if $w = 0$ then the leading symbol of L_k is, up to a non-vanishing scalar factor, just $\iota(\xi')\varepsilon(\xi')$, and so once again using the fact that $\iota(\xi')\varepsilon(\xi')$ and $\varepsilon(\xi')\iota(\xi')$ are complementary projections on the fibre E_x^k (and so also $E_k|_x$), it follows that the symbol sequence is exact at $\mathcal{E}^k$ and $\mathcal{E}_k$. Since the adjoint de Rham sequences are also elliptic, this shows that the sequence (4) is an elliptic complex. □

We are now ready for one of the main results.

Theorem 4.5. *(i) For $n/2+w-k=\ell\geq 0$, the operators $\mathbb{L}^\ell_k$ and $\overline{\mathbb{L}}^\ell_k$ have the factorisations*

$$\mathbb{L}^\ell_k=\tilde{\delta}\mathbb{N}^\ell_k \qquad \text{and} \qquad \overline{\mathbb{L}}^\ell_k=\overline{\mathbb{N}}^\ell_k\tilde{d}$$

where, in a choice of scale, $\mathbb{N}^\ell_k=2(\ell+1)\mathbb{K}^{\ell-1}_{k+1}\varepsilon(Y)q_k$ and $\overline{\mathbb{N}}^\ell_k=2(\ell+1)q^k\iota(Y)\mathbb{K}^{\ell-1}_{k+1}$. For $n/2-k=\ell\geq 1$, the operator L_k has the factorisation

$$L_k=\delta M_k d,$$

where

$$M^\sigma_k=-4\ell(\ell+1)q^{k+1}\iota(Y)\mathbb{K}^{\ell-2}_{k+2}\varepsilon(Y)q_{k+1}. \tag{39}$$

(ii) The operators $\mathbb{L}^\ell_k$, $\overline{\mathbb{L}}^\ell_k$ and L^ℓ_k are natural as follows: in odd dimensions for integers $-1\leq\ell$; in even dimensions for integers $-1\leq\ell\leq n/2-1$ and for $\ell=n/2$ if $k=0$.

(iii) The differential operator $G^\sigma_k:\mathcal{E}^k\to\mathcal{E}_{k-1}$, defined on even dimensional manifolds for $k\leq n/2+1$ by $G^\sigma_k:=q^{k-1}\iota(Y)\mathbb{L}_k$, for each choice of conformal scale σ, is natural. Upon restriction to the null space of L_k, G^σ_k is conformally invariant (and so we omit the argument σ). The composition $G_kd:\mathcal{E}^{k-1}\to\mathcal{E}_{k-1}$ is (up to a non-zero scale factor) L_{k-1}.

(iv) For $k=1,\cdots,n/2$, $G_k=\delta\tilde{M}_{k-1}$ where $\tilde{M}_{k-1}:\mathcal{N}(L_k)\to\mathcal{E}_k/\mathcal{N}(\delta)$ is conformally invariant. $G^\sigma_0=0$ and, up to a non-vanishing constant multiple, $G^\sigma_{n/2}$ is δ (and so is conformally invariant on $\mathcal{E}^{n/2}$).

(v) The operators $L^\ell_0:\mathcal{E}[\ell-n/2]\to\mathcal{E}[-\ell-n/2]$ are (up to a non-zero constant multiple) the GJMS operators. The operators $\mathbb{L}^{-1}_k$, $\overline{\mathbb{L}}^{-1}_k$, L^{-1}_k, and $G^\sigma_{n/2+1}$ all vanish. L^0_k is a multiple of the identity, and $L_{n/2}=0$.

Proof: Part (*i*). For $u\in\mathcal{E}^k[w]$ we have $\mathbb{L}^\ell_k u:=\mathbb{K}^\ell_k q_k u$. Now from the definition of q_k and (17) it follows that the composition $\iota(X)q_k$ vanishes on $\mathcal{E}^k[w']$ (for any weight w'). So, making an arbitrary choice of scale, we have $\mathbb{K}^\ell_k q_k u=\mathbb{K}^\ell_k\iota(X)\varepsilon(Y)q_k u$. From Proposition 4.3 we have immediately that $\mathbb{L}^\ell_k=\tilde{\delta}\mathbb{N}^\ell_k$ with $\mathbb{N}^\ell_k=2(\ell+1)\mathbb{K}^{\ell-1}_{k+1}\varepsilon(Y)q_k$. From this we obtain $\overline{\mathbb{L}}^\ell_k=\overline{\mathbb{N}}^\ell_k\tilde{d}$, with $\overline{\mathbb{N}}^\ell_k$ as given, by taking formal adjoints.

Next we recall that for $u\in\mathcal{E}^k$ we have $L_k u=q^k\mathbb{L}^\ell_k u$ where $\ell=n/2-k$. So from our results just above we have $L_k u=2(\ell+1)q^k\tilde{\delta}\mathbb{K}^{\ell-1}_{k+1}\varepsilon(Y)q_k u$. Now on $\mathcal{G}_{k+1}$ we have $q^k\tilde{\delta}=\delta q^{k+1}$. Using that $q^{k+1}\varepsilon(X)$ vanishes we obtain $L_k u=2(\ell+1)\delta q^{k+1}\iota(Y)\varepsilon(X)\mathbb{K}^{\ell-1}_{k+1}\varepsilon(Y)q_k u$. Calling on Proposition 4.3 then brings us to $4\ell(\ell+1)\delta q^{k+1}\iota(Y)\mathbb{K}^{\ell-2}_{k+2}\tilde{d}\varepsilon(Y)q_k u$. Now from the definition of $\tilde{d}$ in terms of the ambient exterior derivative and the relationship of Y in (29) to $\boldsymbol{Y}$ (or alternatively from (29), Proposition 3.5, and (27)) it is straightforward to verify that as an operator on $\mathcal{G}^k[w']$ (for any weight w') we have $\varepsilon(Y)=\varepsilon(\tilde{Y})$ where $\tilde{Y}:=\sigma^{-1}\tilde{d}\sigma$. (Here σ is the conformal scale determining Y.) It follows immediately that on $\mathcal{G}^k[w']$ we have $\{\tilde{d},\varepsilon(Y)\}=\{\tilde{d},\varepsilon(\tilde{Y})\}=0$.

From this and using that as operators on $\mathcal{E}^k$ we have $\tilde{d}q_k = q_{k+1}d$ brings us to $L_k u = \delta M_k du$ where $M_k^\sigma = -4\ell(\ell+1)q^{k+1}\iota(Y)\mathbb{K}_{k+2}^{\ell-2}\varepsilon(Y)q_{k+1}$, as claimed.
Part (*ii*). Since $\mathbb{L}_k^\ell = \mathbb{K}_k^\ell q_k$, $\overline{\mathbb{L}}_k^\ell = q^k\mathbb{K}_k^\ell$ and $L_k^\ell = q^k\mathbb{K}_k^\ell q_k$, from Proposition 4.3 it is immediate that these operators are natural for $\ell = -1$ and for ℓ in the range of m as in Proposition 3.7. On the other hand from part (i) above we also have $\mathbb{L}_k^\ell = 2(\ell+1)\tilde{\delta}\mathbb{K}_{k+1}^{\ell-1}\varepsilon(Y)q_k$, $\overline{\mathbb{L}}_k^\ell = 2(\ell+1)q^k\iota(Y)\mathbb{K}_{k+1}^{\ell-1}\tilde{d}$ and $L_k^\ell = q^k\mathbb{L}_k^\ell$. This shows these operators are natural for $\ell = m+1$ except for the cases $k=0$ and $k=1$. This exactly yields the claimed result. (Note that by their definitions above each of these is conformally invariant).
Part (*iii*). Since $k \le n/2+1$ we have $n/2 - k = \ell \ge -1$. Thus from part (*ii*), and by construction, the operator G_k^σ is differential, natural and takes values in $\mathcal{E}_{k-1}$. Consider $\mathbb{L}_k\varphi$ for $\varphi \in \mathcal{N}(L_k)$. Note that $q^k\mathbb{L}_k\varphi = L_k\varphi = 0$. Thus $\varepsilon(X)\mathbb{L}_k\varphi = 0$. Using that $\{\varepsilon(X), \iota(Y)\}$ is conformally invariant and the identity on $\mathcal{G}_k$ we have that the conformally invariant section $\mathbb{L}_k\varphi \in \mathcal{G}_k$ is equal to $\varepsilon(X)\iota(Y)\mathbb{L}_k\varphi$. It follows immediately that any conformal variation of $\iota(Y)\mathbb{L}_k\varphi$ has the form $\varepsilon(X)F$ and so is annihilated by q^{k-1}. Thus $G_k\varphi = q^{k-1}\iota(Y)\mathbb{L}_k\varphi$ is conformally invariant.

Recall that $\mathbb{L}_k = \mathbb{K}_k^\ell q_k$ (with $\ell = n/2-k$). So acting on $\mathcal{E}^{k-1}$ we have $2(\ell+2)G_k d = 2(\ell+2)q^{k-1}\iota(Y)\mathbb{K}_k^\ell q_k d$. Now since, for $v \in \mathcal{E}^{k-1}$, we have $q_k dv = \tilde{d}q_{k-1}v$, Proposition 4.3 gives $2(\ell+2)G_k dv = q^{k-1}\iota(Y)\varepsilon(X)\mathbb{K}_{k-1}^{\ell+1}q_{k-1}v$. Recall that the composition $q^{k-1}\varepsilon(X)$ vanishes on $\mathcal{G}_k[w']$ for any weight w', so we have

$$2(\ell+2)G_k dv = q^{k-1}\mathbb{K}_{k-1}^{\ell+1}q_{k-1}v = L_{k-1}v.$$

Part (*iv*). From part (*iii*) we have that $G_k^\sigma := q^{k-1}\iota(Y)\mathbb{L}_k$. Since $q^{k-1}\iota(Y)$ exactly recovers the coefficient of $\mathbb{X}^k$ in $\mathcal{G}_k$, the result is immediate from part (*i*) and the expression (33) for $\tilde{\delta}$.
Part (*v*). Fix $\ell \in \mathbb{N}$. Since q^0, q_0 are both identity maps we have

$$L_0^\ell f = \mathbb{K}_0^\ell f$$

for $f \in \mathcal{E}[\ell - n/2]$. Now via the relationship of $\iota(\not{D})$ and $\varepsilon(X)$ with the ambient operators $\iota(\boldsymbol{\not{D}})$ and $\varepsilon(\boldsymbol{X})$, or via (27) with (20), it is easily shown that $\iota(\not{D})\varepsilon(X)f = (\ell+1)(n+2\ell)f$. Now $\mathbb{K}_0^\ell f = \not{\Box}_\ell\iota(\not{D})\varepsilon(X)f$, and so

$$L_0^\ell f = (\ell+1)(n+2\ell)\not{\Box}_\ell f.$$

But from [37] (see also [26]) the operator $\not{\Delta}_\ell$ (of Proposition 3.7) is formally self-adjoint on $\mathcal{E}[\ell - n/2]$ and so $\not{\Box}_\ell f = \not{\Delta}_\ell f$ is the GJMS operator of order 2ℓ.

Next recall that by definition $\mathbb{L}_k^{-1} = \mathbb{K}_k^{-1}q_k$, while $\overline{\mathbb{L}}_k^{-1} = q^k\mathbb{K}_k^{-1}$ and $\mathbb{K}_k^{-1} = \iota(X)\varepsilon(X)$. But $\{\iota(X), \varepsilon(X)\} = 0$ and $q^k\varepsilon(X) = 0 = \iota(X)q_k$. So $\mathbb{L}_k^{-1}$ vanishes, and thus its formal adjoint $\overline{\mathbb{L}}_k^{-1}$ must also vanish. As a result, $L_k^{-1} = q^k\mathbb{L}_k^{-1} = 0$, and finally $G_{n/2+1}^\sigma = q^{n/2}\mathbb{L}_{n/2+1} = 0$ as $\mathbb{L}_{n/2+1} = q^{n/2}\mathbb{L}_{n/2+1}^{-1}$.

That L_k^0 is a multiple of the identity, and that this multiple is zero when $k = n/2$, is shown in Proposition 3.12, since $L_k^0 = q^k\iota(\not{D})\varepsilon(X)q_k$. $\square$

Proposition 4.6. *For $\ell \in \mathbb{N}$ and $k \in \{0, 1, \cdots, n\}$, $k \neq \ell + n/2$, the operator $\mathbb{L}_k^\ell : \mathcal{E}^k[w] \to \mathcal{G}_k[-w]$ is quasi-Laplacian. In particular in Riemannian signatures it is injectively elliptic.*

Proof: Recall that $q^k \mathbb{L}_k^\ell = L_k^\ell$. On one hand, this implies that for $w \neq 0$ the result is immediate from Proposition 4.4. On the other hand, for the cases $w = 0$, using $q^k \mathbb{L}_k^\ell = L_k^\ell$ with $G_k^\sigma = q^{k-1}\iota(Y)\mathbb{L}_k$, we have that

$$[\mathbb{L}_k u]_\sigma = \begin{pmatrix} L_k u \\ G_k^\sigma u \end{pmatrix},$$

in the splitting $[\mathcal{G}_k[-w]]_\sigma = \mathcal{E}_k[-w] \oplus \mathcal{E}_{k-1}[-w]$ of $\mathcal{G}_k[-w]$ determined by a choice of scale σ. Next we have already observed in the proof of Proposition 4.4 that, in a choice of scale, L_k is of the form $(\delta d)^\ell + \mathrm{LOT}$ up to a non-zero constant multiple, while from Theorem 4.5 part (*iii*) we have that $G_k d$ is L_{k-1} up to a non-zero constant multiple. Using this and considering possible leading symbols for G_k it follows that $G_k^\sigma = a\delta(d\delta)^\ell + b(d\delta)^\ell d + \mathrm{LOT}$ where $a, b \in \mathbb{R}$ with $a \neq 0$. Thus there is a pair $a', b' \in \mathbb{R}$ giving $(a'\delta d\ ,\ b'd)[\mathbb{L}_k u]_\sigma = \Delta\!\!\!\!/^{\,\ell+1} + \mathrm{LOT}$, showing that $\mathbb{L}_k$ is quasi-Laplacian. □

Remark: In the proof we have observed that at leading order, G_k^σ has the form $a\delta(d\delta)^\ell + b(d\delta)^\ell d$ with $a \neq 0$. From Theorem 4.5 we have $\mathbb{L}_k^\ell = \tilde{\delta}\mathbb{N}_k^\ell$. Considering also the explicit formula (33) for $\tilde{\delta}$ in a scale it follows that δ is a left factor of G_k^σ, and so $b = 0$ and G_k^σ has the form

$$G_k^\sigma = a\delta((d\delta)^\ell + \mathrm{LOT}). \qquad \blacksquare$$

4.1. **Operators generalising Q-curvature.** Let us write Y_σ for the section of $\mathcal{T}[-1]$ given by $Y = I_\sigma - \frac{1}{2}(I_\sigma \bullet I_\sigma)X$ where $I_\sigma := \frac{1}{n}\sigma^{-1}\varepsilon(D\!\!\!\!/)\sigma$ and $\sigma \in \mathcal{E}[1]$. Thus Y_σ is null and we have $X \bullet Y_\sigma = 1$. By (29), if σ is a choice of conformal scale then Y_σ is just Y as above but we want allow the possibility that σ is not (necessarily) a choice of conformal scale. Note that the canonical surjection $\mathcal{T}^1[-1] \to \mathcal{G}^1$ maps Y_σ to a section of $\mathcal{G}^1$. Explicitly this image is $\tilde{Y}_\sigma = \sigma^{-1}\tilde{d}\sigma$ (cf. the similar observation for $\tilde{Y}$ in the proof of part (*i*) of Theorem 4.5). Note that as operators on $\mathcal{G}^k[w']$ we have $\varepsilon(\tilde{Y}_\sigma) = \varepsilon(Y_\sigma)$, and on $\mathcal{G}_k[w']$ we have $\iota(\tilde{Y}_\sigma) = \iota(Y_\sigma)$. Thus we shall normally omit the *tilde* and write simply Y_σ for the section in $\mathcal{G}^1$ given by $\sigma^{-1}\tilde{d}\sigma$. We now consider the differential operator $q^k\iota(Y_\sigma)\mathbb{K}_{k+1}^{\ell-1}$ for $\ell \geq 1$. Apparently this depends on σ. For any weight $w \in \mathbb{R}$, let us denote by $\mathcal{K}^k[w]$ the subspace of $\mathcal{G}^k[w]$ consisting of $U \in \mathcal{G}^k[w]$ such that $\tilde{d}U = 0$.

Lemma 4.7. *For each $\sigma \in \mathcal{E}[1]$, the composition*

$$q^{k-1}\iota(Y_\sigma)\mathbb{K}_k^\ell : \mathcal{G}^k[w] \to \mathcal{E}_{k-1}[-w] \quad w = k + \ell - n/2$$

is a conformally invariant differential operator (natural for the range of ℓ as in Theorem 4.5 part (ii)). Restricted to $\mathcal{N}(\overline{\mathbb{L}}_k^\ell : \mathcal{G}^k[w] \to \mathcal{E}_k[w])$, it is independent of σ. Thus in particular restricted to $\mathcal{K}^k[w] \subset \mathcal{N}(\overline{\mathbb{L}}_k^\ell)$, $q^{k-1}\iota(Y_\sigma)\mathbb{K}_k^\ell$ is independent of σ.

Note that in the first statement here we mean that the operator is conformally invariant with the choice of $\sigma \in \mathcal{E}[1]$ fixed; that is, we are not linking

σ to conformal scale. This point of view will be continued below. In addition, by 'natural' here we mean, natural as an operator on $E[1]\otimes\mathbb{G}^k[w]$ (i.e. viewing $q^{k-1}\iota(Y_\sigma)\mathbb{K}^\ell_k$ as an operator on σ as well as the section of $\mathcal{G}^k[w]$.)

Proof: The first statement is clear by construction and the results above. Next, recall that the conformally invariant operator $\mathbb{K}^\ell_k$ takes values in $\mathcal{G}_k[-w]$, which has the composition series $\mathcal{E}_k[-w] \oplus \mathcal{E}_{k-1}[-w]$. The operator q_k is the natural surjection $\mathcal{G}_k[-w]\to\mathcal{E}_k[-w]$, while $q^{k-1}\iota(Y_\sigma)$ is a splitting of the natural injection $\mathcal{E}_{k-1}[-w]\to\mathcal{G}_k[-w]$. Thus since $\overline{\mathbb{L}}^\ell_k := q^k\mathbb{K}^\ell_k$, it is immediate that if $U\in\mathcal{N}(\overline{\mathbb{L}}^\ell_k:\mathcal{G}^k[w]\to\mathcal{E}_k[w])$, then $q^{k-1}\iota(Y_\sigma)\mathbb{K}^\ell_k U$ is independent of the choice of splitting, i.e. independent of σ.

For the final statement observe that when $\ell\geq 0$ we have $\overline{\mathbb{L}}^\ell_k=\overline{\mathbb{N}}^\ell_k\tilde{d}$ (see Theorem 4.5 part (*i*)), and so $\mathcal{K}^k[w]\subseteq\mathcal{N}(\overline{\mathbb{L}}^\ell_k)$. On the other hand $\overline{\mathbb{L}}^{-1}_k=0$ (Theorem 4.5, part (*v*)) so the final statement follows trivially in this case.
□

Remark: Note that the operator $q^{k-1}\iota(Y_\sigma)\mathbb{K}^\ell_k:\mathcal{G}^k[w]\to\mathcal{E}_{k-1}[-w]$ is essentially a generalisation of G^σ_k and has many properties which reflect this. In particular note that on $\mathcal{G}^{k-1}[w]$ we have $2(\ell+2)q^{k-1}\iota(Y_\sigma)\mathbb{K}^\ell_k\tilde{d}=q^{k-1}\mathbb{K}^{\ell+1}_{k-1}=\overline{\mathbb{L}}^{\ell+1}_{k-1}$ (cf. part (*iii*) of Theorem 4.5). This uses the result $2(\ell+2)\mathbb{K}^\ell_k\tilde{d}=\varepsilon(X)\mathbb{K}^{\ell+1}_{k-1}$ of Theorem 4.3. From the latter it is clear that $\varepsilon(X)$, as well as $\iota(X)$, annihilates $\mathbb{K}^\ell_k\tilde{d}$. This in turn implies that $q^k\mathbb{K}^\ell_k\tilde{d}=0$ and so

$$2(\ell+2)\mathbb{K}^\ell_k\tilde{d}=\mathbb{X}^k\cdot\overline{\mathbb{L}}^{\ell+1}_{k-1}. \tag{40}$$

This is useful in the next section. ▮

The next proposition constructs a family of operators with an interesting conformal transformation property. In this, as in the lemma above, 'natural' means natural as an operator on σ as well as the section in $\mathcal{G}^k[w]$.

Proposition 4.8. *For each choice of $\sigma\in\mathcal{E}[1]$ the differential operator*

$$(\mathbb{Q}^{\ell,\sigma}_k := -2(\ell+1)q^k\iota(Y_\sigma)\mathbb{K}^{\ell-1}_{k+1}\varepsilon(Y_\sigma)):\mathcal{G}^k[w]\to\mathcal{E}_k[-w],\quad w=k+\ell-n/2$$

is conformally invariant (and natural for $\ell-1$ as in the range of ℓ in Theorem 4.5 part (ii)). Acting on $U\in\mathcal{K}^k[w]=\mathcal{N}(\tilde{d}:\mathcal{G}^k[w]\to\mathcal{G}^{k+1}[w])$, $\mathbb{Q}^{\ell,\sigma}_k$ has the transformation law

$$\mathbb{Q}^{\ell,\hat{\sigma}}_k U=\mathbb{Q}^{\ell,\sigma}_k U+\overline{\mathbb{L}}^\ell_k(\Upsilon U)$$

where $\hat{\sigma}=e^{-\Upsilon}\sigma$.

Proof: The first statement is clear from the definition of $\mathbb{Q}^{\ell,\sigma}_k$.

Let us pick sections $\sigma_1,\sigma_2\in\mathcal{E}[1]$. Viewing Y_{σ_2} as a section of $\mathcal{G}^1$, we have $Y_{\sigma_2}=\sigma_2^{-1}\tilde{d}\sigma_2$ and so it is clear that $\tilde{d}Y_{\sigma_2}=0$. Thus if $U\in\mathcal{K}^k[w]$ then $\varepsilon(Y_\sigma)U\in\mathcal{K}^{k+1}[w]$ and it follows at once from Lemma 4.7 that $q^k\iota(Y_{\sigma_1})\mathbb{K}^{\ell-1}_{k+1}\varepsilon(Y_{\sigma_2})U$ is independent of σ_1.

Now let $\hat{\sigma}_2=e^{-\Upsilon}\sigma_2$ for some smooth function Υ. Viewing Y_{σ_2} and $Y_{\hat{\sigma}_2}$ as sections of $\mathcal{G}^1$ we have $Y_{\hat{\sigma}_2}=Y_{\sigma_2}-\tilde{d}\Upsilon$. Thus

$$q^k\iota(Y_{\sigma_1})\mathbb{K}^{\ell-1}_{k+1}\varepsilon(Y_{\hat{\sigma}_2})U-q^k\iota(Y_{\sigma_1})\mathbb{K}^{\ell-1}_{k+1}\varepsilon(Y_{\sigma_2})U=-q^k\iota(Y_{\sigma_1})\mathbb{K}^{\ell-1}_{k+1}\varepsilon(\tilde{d}\Upsilon)U.$$

Since by assumption $\tilde{d}U = 0$ we have $\varepsilon(\tilde{d}\Upsilon)U = \tilde{d}(\Upsilon U)$ and so by Proposition 4.3,

$$\begin{aligned} 2(\ell+1)q^k\iota(Y_{\sigma_1})\mathbb{K}^{\ell-1}_{k+1}\varepsilon(\tilde{d}\Upsilon)U &= q^k\iota(Y_{\sigma_1})\varepsilon(X)\mathbb{K}^{\ell}_{k}(\Upsilon U)\\ &= q^k\mathbb{K}^{\ell}_{k}(\Upsilon U) = \overline{\mathbb{L}}^{\ell}_{k}(\Upsilon U). \end{aligned}$$

Here we have used the operator equality $\{\iota(Y_\sigma), \varepsilon(X)\} = X\cdot Y_\sigma = 1$. □

We now return to the convention that σ denotes a conformal scale.

Definition: For each choice of *conformal scale* $\sigma \in \mathcal{E}[1]$ on even dimensional conformal manifolds and for each $k \leq n/2$, we let $(Q^\sigma_k := \mathbb{Q}^{\ell,\sigma}_k q_k) : \mathcal{E}^k \to \mathcal{E}_k$ be given by

$$Q^\sigma_k = -2(\ell+1)q^k\iota(Y)\mathbb{K}^{\ell-1}_{k+1}\varepsilon(Y)q_k. \tag{41}$$

The operator Q^σ_k has the properties described in Theorem 2.8, and in particular is a generalisation of Branson's Q-curvature.

Proof of Theorem 2.8: Note that by construction $2(\ell+2)Q^\sigma_k = M^\sigma_{k-1}$, where by the right hand side we mean the operator given in (39) above (viewed as an operator $\mathcal{E}^k \to \mathcal{E}_k$). Thus Part (*iii*) is already contained in Theorem 4.5.

Part (*i*) is immediate from formula (41), since $\mathbb{K}^{\ell-1}_{k+1}$ is formally self-adjoint by Theorem 4.3.

Part (*ii*). We have

$$\delta Q^\sigma_k = -2(\ell+1)\delta q^k\iota(Y)\mathbb{K}^{\ell-1}_{k+1}\varepsilon(Y)q_k.$$

Once again recall that on $\mathcal{G}^k$, $\delta q^k = q^{k-1}\tilde{\delta}$ while as operators on $\mathcal{G}^{k+1}[w']$ for any weight w', we have $\{\tilde{\delta}, \iota(Y)\} = 0$. Thus using Proposition 4.3 and $\iota(X)q_k = 0$, we get

$$\delta Q^\sigma_k = q^{k-1}\iota(Y)\mathbb{K}^\ell_k q_k = G^\sigma_k,$$

where G^σ_k is as defined in Theorem 4.5 (and its restriction to $\mathcal{N}(L_k)$ is denoted G_k).

Part (*iv*). If $u \in \mathcal{C}^k$ then $q_k u \in \mathcal{K}^k[0]$, since $\tilde{d}q_k = q_{k-1}d$. Thus the transformation law is immediate from the definition of Q^σ_k and Proposition 4.8, since $L_k = \overline{\mathbb{L}}_k q_k$, and q_k commutes with the multiplication operator $\varepsilon(\Upsilon)$, for any function Υ.

Part (*v*). We have $L^\ell_k = q^k\mathbb{K}^\ell_k q_k = \overline{\mathbb{L}}^\ell_k q_k$. Specialising to the case of densities $\mathcal{E}[w]$, observe that q^0 and q_0 are both simply identity maps. So for $f \in \mathcal{E}[w]$ we have $L^\ell := L^\ell_0 = \overline{\mathbb{L}}^\ell_0$, where $w = \ell - n/2$. From Theorem 4.5 part (*v*), this is a GJMS operator. From part (*i*) of that theorem we also have $L^\ell = 2(\ell+1)\iota(Y)\mathbb{K}^{\ell-1}_1\tilde{d}f$. Let us choose a conformal scale σ. Then on densities of weight w we have $\tilde{d} = Z\cdot\varepsilon(\nabla) + w\varepsilon(Y)$, from (32). So for a function $f \in \mathcal{E}[0]$ we have the operator

$$\begin{aligned} \sigma^{-w}L^\ell\sigma^w f &= 2(\ell+1)\sigma^{-w}\iota(Y)\mathbb{K}^{\ell-1}_1 Z\cdot\varepsilon(\nabla)\sigma^w f\\ &\quad + w2(\ell+1)\sigma^{-w}\iota(Y)\mathbb{K}^{\ell-1}_1\varepsilon(Y)\sigma^w f. \end{aligned}$$

Note that the first term on the right-hand side annihilates constant functions and so setting $f = 1$, taking the coefficient of w and setting in this $\ell = n/2$ (i.e. $w = 0$) yields (by definition) Branson's Q-curvature. Thus Branson's Q

is given, in dimension 2ℓ, by the operator $2(\ell+1)\iota(Y)\mathbb{K}_1^{\ell-1}\varepsilon(Y)1$. But from the definition (41) this is exactly $-Q_0^\sigma 1$. $\square$

4.2. **Other constructions and operators of order n.** In even dimension n, Theorem 4.5 constructs natural conformally invariant differential operators L_k^ℓ up to order $n-2$. For $k=0$ we have that $L_0^{n/2}$ is natural but Theorem 2.1 asserts the existence of curved generalisations of the conformally flat operators at order n for other k values. We obtain these by a variation on our general construction. Note that for the operators of order 4, the observation that one needs, and that there exist, such alternative constructions is detailed in [34].

Note that by the formula (32) for $\tilde{d}$ we have that, acting on $\mathcal{E}^k[w]$, $\iota(X)\tilde{d}q_k = wq_k$. (Alternatively observe that on $\tilde{M}$, if $U \in \boldsymbol{\mathcal{T}}^k(w-k)$ has the property $\iota(\boldsymbol{X})U = O(Q)$, then $\iota(\boldsymbol{X})\boldsymbol{d}U = \boldsymbol{\mathcal{L}_X}U = wU$; the result follows.) Thus with $w = k+\ell-n/2$, we have $w\mathbb{L}_k^\ell = \mathbb{K}_k^\ell\iota(X)\tilde{d}q_k$, as an operator on $\mathcal{E}^k[w]$. So by Proposition 4.3 we have $w\mathbb{L}_k^\ell = 2(\ell+1)\tilde{\delta}\mathbb{K}_{k+1}^{\ell-1}\tilde{d}q_k$. (Integrating this by parts gives an alternative formula along these lines for the formal adjoint $\overline{\mathbb{L}}_k^\ell$.) For the cases where $w \neq 0$ this provides an alternative construction of $\mathbb{L}_k^\ell$, and thus also of L_k^ℓ:

$$wL_k^\ell = 2(\ell+1)q^k\tilde{\delta}\mathbb{K}_{k+1}^{\ell-1}\tilde{d}q_k.$$

Next from (40) we have that $2(\ell+1)\mathbb{K}_{k+1}^{\ell-1}\tilde{d}q_k = \mathbb{X}^k\cdot L_k^\ell$. So, from the explicit formula (33) for $\tilde{\delta}$, it follows that when $w \neq 0$, the action of $q^k\tilde{\delta}$ here is the same as some non-zero multiple of $\mathbb{Y}^{k+1}\bullet$.

To further re-express L_k^ℓ we need the following lemma.

Lemma 4.9. *The conformally invariant differential operator*

$$\iota(\not{D})\varepsilon(X)\tilde{d}q_k : \mathcal{E}^k[k+\ell-n/2] \to \mathcal{T}^{k+1}[\ell-n/2-1]$$

is a differential splitting of the canonical conformally invariant surjection

$$\mathbb{X}^{k+1}\bullet : \mathcal{T}^{k+1}[\ell-n/2-1] \to \mathcal{E}^k[k+\ell-n/2]$$

for values of k and ℓ such that $\ell \neq \pm 1$ and $k \pm \ell \neq n/2$.

Proof: First note that it is clear from (17) that $\mathbb{X}^{k+1}\bullet\iota(\not{D})\varepsilon(X)\tilde{d}q_k$ is same as $\mathbb{Z}^k\bullet\iota(X)\iota(\not{D})\varepsilon(X)\tilde{d}q_k$. A straightforward calculation shows that on form tractors of weight $\tilde{w}$ we have $(n+2\tilde{w}-2)\iota(\not{D})\iota(X) + (n+2\tilde{w}+2)\iota(X)\iota(\not{D}) = 0$. Thus acting on $\mathcal{E}^k[w]$ we have $2(\ell+1)\iota(X)\iota(\not{D})\varepsilon(X)\tilde{d}q_k = -2(\ell-1)\iota(\not{D})\iota(X)\varepsilon(X)\tilde{d}q_k$. Now from $\varepsilon(X)\iota(X)+\iota(X)\varepsilon(X) = 0$ and our observation above that $\iota(X)\tilde{d}q_k = wq_k$ we obtain

$$2(\ell+1)\iota(X)\iota(\not{D})\varepsilon(X)\tilde{d}q_k = 2(k+\ell-n/2)(\ell-1)\iota(\not{D})\varepsilon(X)q_k$$

and the result follows immediately from Proposition 3.12 since on $\mathcal{F}^k[w-k]$, q^k is a non-zero constant multiple of $\mathbb{Z}^k\bullet$. $\square$

Let us suppose that the integers k,ℓ are as in the lemma above. From the lemma and the fact that $\mathbb{K}_{k+1}^{\ell-1}\tilde{d}q_k$ is a non-zero multiple of $\mathbb{X}^{k+1}\cdot L_k^\ell$, we see that if $u,v \in \mathcal{E}^k[w]$ then

$$(\iota(\not{D})\varepsilon(X)\tilde{d}q_k v)\bullet(\mathbb{K}_{k+1}^{\ell-1}\tilde{d}q_k u)$$

is a non-zero multiple of $v \cdot L_k^\ell u$. Integrating by parts, it follows immediately that

$$q^k \tilde{\delta} \iota(X) \varepsilon(\not{D}) \mathbb{K}_{k+1}^{\ell-1} \tilde{d} q_k = q^k \tilde{\delta} \iota(X) \varepsilon(\not{D}) \not\Box_{\ell-1} \iota(\not{D}) \varepsilon(X) \tilde{d} q_k$$

is a non-zero multiple of L_k^ℓ on $\mathcal{E}^k[w]$. From Proposition 4.3 this is natural for ℓ exactly as in Theorem 4.5 part (ii). The importance of this expression, for our current purposes, is that this formula is easily modified. In the constructions above we have used that $\not\Box_{\ell-1} \iota(\not{D}) \varepsilon(X) \tilde{d} q_k$ takes values in $\mathcal{G}_k[-w]$. Since $q^k \tilde{\delta} \iota(X) \varepsilon(\not{D})$ acts invariantly on general form tractors we can, in the right-hand-side of the last display, replace $\not\Box_{\ell-1}$ with

$$\Box_{\ell-1} = D^{A_1} \cdots D^{A_{\ell-2}} \Box D_{A_{\ell-2}} \cdots D_{A_1}.$$

For any integer $\ell \geq 2$ this is a natural, conformally invariant and formally self-adjoint differential operator on any tractor bundle of weight $\ell - 1 - n/2$. If $2 \leq \ell \leq n/2$, then $\Box_{\ell-1}$ has leading term a non-zero multiple of $\Delta^{\ell-1}$ (see e.g. [32, 29]). It follows that on $\mathcal{E}^k[w]$,

$$\tilde{L}_k^\ell := q^k \tilde{\delta} \iota(X) \varepsilon(\not{D}) \Box_{\ell-1} \iota(\not{D}) \varepsilon(X) \tilde{d} q_k$$

gives an invariant operator $\tilde{L}_k^\ell : \mathcal{E}^k[w] \to \mathcal{E}_k[-w]$ which at leading order agrees with L_k^ℓ provided $0 \neq w = k + \ell - n/2$, $\ell \leq n/2$, $\ell \neq \pm 1$, and $k \neq \ell + n/2$. In particular, since the splitting operator $\iota(\not{D}) \varepsilon(X) \tilde{d} q_k$ and its formal adjoint $q^k \tilde{\delta} \iota(X) \varepsilon(\not{D})$ are natural, we have the following result.

Theorem 4.10. *For each $0 < k < n$, the operator*

$$\tilde{L}_k^{n/2} : \mathcal{E}^k[k] \to \mathcal{E}_k[-k]$$

is natural, conformally invariant, formally self-adjoint, and of order n. It is quasi-Laplacian. In the Riemannian signature case the operator $\tilde{L}_k^{n/2}$ is elliptic.

5. Nontriviality of the cohomology maps

In Theorem 2.9 we have shown that the operators Q_k give conformally invariant maps $Q_k : \mathcal{H}^k \to H_k^L(M)$. Here we demonstrate that these maps are not trivial in general. Clearly it is sufficient to show the stronger claim that the $Q_k : \mathcal{H}^k \to H_k(M)$ of Corollary 2.10 are non-trivial. For $k = 0$ this result is already well known. It boils down to checking the same question for the Q-curvature, but from [8] this integrates to $(n-1)!\mathrm{vol}(S^n)/2$ times the Euler characteristic for conformally flat structures. Here we show non-triviality of $Q_p : \mathcal{H}^p \to H_p(M)$ for $M = S^p \times S^q$, where $p = n/2 - 1$, $q = n/2 + 1$, with the standard Riemannian conformal structure.

A straightforward expansion of (41) shows that up to a non-zero constant multiple, Q_p^σ is given by $\frac{1}{2} d\delta + \mathsf{J} - 2\mathsf{P}\sharp$. (Recall that J is the trace of the Schouten tensor P.) For our purposes here let us write $Q := \frac{1}{2} d\delta + \mathsf{J} - 2\mathsf{P}\sharp$. First we study $\mathcal{H}^p$.

Proposition 5.1. *Let $M = S^p \times S^q$, where $p = n/2 - 1$, $q = n/2 + 1$, with the standard Riemannian structure, and let $Q = \frac{1}{2} d\delta + \mathsf{J} - 2\mathsf{P}\sharp$. Then $\varphi \in \mathcal{E}^p(M)$ is in the joint null space of δd and δQ if and only if φ is harmonic.*

Proof: First note that by compactness, $\delta d\varphi = 0 \implies d\varphi = 0$.

Let φ be in the joint null space described above; then $d\varphi = 0$, $\delta Q\varphi = 0$. Let $\mathsf{U} := \mathsf{J} - 2\mathsf{P}\sharp$. Then

$$0 = 4\|\delta Q\varphi\|^2 = \|\delta d\delta\varphi\|^2 + 4\langle \delta d\delta\varphi, \delta\mathsf{U}\varphi\rangle + 4\|\delta\mathsf{U}\varphi\|^2. \tag{42}$$

Here $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are the L^2 inner product and norm. The first term on the right in (42) is $\langle\Delta\varphi, \Delta^2\varphi\rangle$, since $d\varphi = 0$. The second term may be written

$$\langle d\delta\varphi, d\delta\mathsf{U}\varphi\rangle = \langle d\delta\varphi, \Delta\mathsf{U}\varphi\rangle = \langle\Delta\varphi, \Delta\mathsf{U}\varphi\rangle.$$

The form Laplacian Δ commutes with the projections from the decomposition of $\mathcal{E}^p(M)$ as $\oplus_{r+s=p}\mathcal{E}^r(S^p) \boxtimes \mathcal{E}^s(S^q)$, where $\boxtimes$ is the external tensor product. Furthermore, U is a linear combination of these projections, since $4\mathsf{U}$ takes the eigenvalue

$$\frac{16r}{n-2} - \frac{2(n-4)}{n-1} \tag{43}$$

on (r,s)-forms. As a result, U commutes with Δ. Thus the second term in (42) may be written $(\Delta\varphi, \mathsf{U}\Delta\varphi)$. Since the third term in (42) is nonnegative, we have

$$0 = 4\|\delta Q\varphi\|^2 \geq \langle\Delta\varphi, (\Delta + 4\mathsf{U})\Delta\varphi\rangle. \tag{44}$$

If $n = 4$, (43) shows that U is a nonnegative operator. Since Δ is also nonnegative, (44) shows that both Δ and U kill $\Delta\varphi$. Since Δ is a positive operator on its range, we have $\Delta\varphi = 0$; i.e. φ is harmonic. In general even dimension $n \geq 4$, (43) shows that the eigenvalues of $4\mathsf{U}$ are > -2, so that $\Delta + 4\mathsf{U}$ can be nonpositive only on eigenspaces of the Laplacian with eigenvalue < 2. But the non-zero eigenvalues of the form Laplacian on standard S^m (see e.g. [28, 38]) are integers $\geq m$. Now the eigenvalues of the form Laplacian on M are sums of form Laplacian eigenvalues on S^p and S^q. Thus non-zero eigenvalues less than 2 of Δ can only arise for $n/2 - 1 \leq 1$. This means that either we are in the 4-dimensional case treated above, or else $\Delta\varphi$ is harmonic (and thus vanishes), so that φ is harmonic. This shows that the joint null space of δd and δQ is contained in the harmonics.

For the opposite inclusion, note that if h is a harmonic (r,s)-form, then

$$2\delta Qh = \delta(d\delta + \text{const})h = (\delta d + \text{const})\delta h = 0.$$

But the (r,s) components of a harmonic form φ are harmonic, so δQ kills harmonics. $\square$

We now use the above setup to give an example of a situation in which the cohomology map Q_p is non-trivial. The cohomology of $M = S^p \times S^q$ is 1-dimensional in the orders $0, p, q, n$, with ω, the pullback of the S^p volume form under projection onto the S^p factor, generating the harmonics H^p (as well as H_p). By the above, $\mathcal{H}^p = H^p$, so ω generates $\mathcal{H}^p$. Since $\nabla\omega = 0$, we have $Q\omega = \mathsf{U}\omega$, and (43) shows that U takes the eigenvalue $3n/2(n-1)$ on $(p,0)$-forms. Thus $Q\omega = 3n\omega/2(n-1)$. In particular, we have:

Theorem 5.2. *Let $M = S^p \times S^q$, where $p = n/2 - 1$, $q = n/2 + 1$, with the standard Riemannian conformal structure. Then $Q_p : \mathcal{H}^p \to H_p(M)$ is non-trivial.*

6. Variations on the theme of Q

Proposition 2.8 of [32] described one way to proliferate natural scalar fields with transformation properties similar to the Q-curvature in the sense that their conformal variation is by a linear conformally invariant differential operator acting on the variation function Υ. Our first objective here is to observe that this generalises. We follow this by making some connections with other recent constructions of the Q-curvature.

6.1. **Semi-invariant operators.** Recall that as an operator on $\mathcal{G}^k[w]$ we have that $\varepsilon(Y)$ is the same as $\varepsilon(Y_\sigma)$ where $Y_\sigma := \sigma^{-1}\tilde{d}\sigma$. Under a conformal transformation given by $\sigma \mapsto \hat{\sigma} = e^{-\Upsilon}\sigma$, we thus have $\varepsilon(Y_{\hat{\sigma}}) = \varepsilon(Y_\sigma) - \varepsilon(\tilde{d}\Upsilon)$. Now for each natural conformally invariant operator $S : \mathcal{T}^{k+1}[w-k-1] \to \mathcal{T}^{k'+1}[w'+k'-n+1]$ and choice of conformal scale σ there is an invariant operator

$$(\mathbb{S}^\sigma := q^{k'}\tilde{\delta}\iota(X)\varepsilon(\not{D})S\iota(\not{D})\varepsilon(X)\varepsilon(Y_\sigma)) : \mathcal{G}^k[w] \to \mathcal{E}_{k'}[w'].$$

From the transformation law for $\varepsilon(Y_\sigma)$ it follows that upon restriction to $\mathcal{K}^k[w]$, our operator has the conformal transformation (cf. Proposition 4.8)

$$\mathbb{S}^{\hat{\sigma}} = \mathbb{S}^\sigma - q^{k'}\tilde{\delta}\iota(X)\varepsilon(\not{D})S\iota(\not{D})\varepsilon(X)\tilde{d}\varepsilon(\Upsilon),$$

where (as above) $\varepsilon(\Upsilon)$ is Υ viewed as a multiplication operator. Note that $q^{k'}\tilde{\delta}\iota(X)\varepsilon(\not{D})S\iota(\not{D})\varepsilon(X)\tilde{d}$ is a composition of conformally invariant operators giving an operator $\mathcal{G}^k[w] \to \mathcal{E}_{k'}[w']$. Composing $\mathbb{S}^\sigma$ with q_k and restricting to $\mathcal{E}^k$ we obtain the following result.

Proposition 6.1. *For each natural conformally invariant operator*

$$S : \mathcal{T}^{k+1}[-k-1] \to \mathcal{T}^{k'+1}[w'+k'-n+1]$$

and choice of conformal scale σ, there is a natural invariant operator

$$\mathbb{S}^\sigma q_k : \mathcal{E}^k \to \mathcal{E}_{k'}[w'].$$

Upon restriction to $\mathcal{C}^k$ this has the conformal transformation

$$\mathbb{S}^{\hat{\sigma}}q_k = \mathbb{S}^\sigma q_k - q^{k'}\tilde{\delta}\iota(X)\varepsilon(\not{D})S\iota(\not{D})\varepsilon(X)q_k d\varepsilon(\Upsilon).$$

Acting between $\mathcal{E}^k$ and $\mathcal{E}_{k'}[w']$, the natural differential operator

$$q^{k'}\tilde{\delta}\iota(X)\varepsilon(\not{D})S\iota(\not{D})\varepsilon(X)q_k d : \mathcal{E}^k \to \mathcal{E}_{k'}[w']$$

is conformally invariant. If $k = k'$ and $w' = 0$ then we may re-express this by the formula $\delta q^k\iota(X)\varepsilon(\not{D})S\iota(\not{D})\varepsilon(X)q_k d$, and this is is formally self-adjoint if S is.

The final statement on formal self-adjointness is clear from the symmetry of the formula and our earlier observations (identifying $\iota(\not{D})$ as the formal adjoint of $\varepsilon(\not{D})$ and so forth).

Of course the operators $\mathbb{S}^\sigma$ are most interesting in the cases where $k = k'$ and $-w' = w = k + \ell - n/2$ for some $\ell \in \mathbb{N}$, since then they may be added to $\mathbb{Q}_k^{\ell,\sigma}$ without altering its properties significantly. If in addition n is even and $w = w' = 0$, then $\mathbb{S}^\sigma q_k$ operates between $\mathcal{E}^k$ and $\mathcal{E}_k$, and so similarly provides a possible modification to the operator Q_k^σ. Note that if we take such an operator, with S also formally self-adjoint, and form the new "Q-operator"

$Q_k^\sigma + \mathbb{S}^\sigma q_k$, then this satisfies parts (*i*), (*ii*) and (*iii*) of Theorem 2.8. Part (*iv*) of that theorem also holds with the qualification that the invariant operator in the conformal variation formula is a modification of L_k by the addition of a constant multiple of the operator $-\delta q^k \iota(X)\varepsilon(\not{D})S\iota(\not{D})\varepsilon(X)q_k d$. (Note that $L_k - \delta q^k \iota(X)\varepsilon(\not{D})S\iota(\not{D})\varepsilon(X)q_k d$ has the general form (3).) Finally (as pointed out in [32]) $(Q_0^\sigma + \mathbb{S}^\sigma q_0)1$ gives an alternative to Branson's Q-curvature.

It is easy to construct non-trivial examples. For example one can take S to be $|C|^2$ (where C is the Weyl curvature), viewed as multiplication operator, to obtain

$$q^{k'}\tilde{\delta}\iota(X)\varepsilon(\not{D})|C|^2\iota(\not{D})\varepsilon(X)\varepsilon(Y_\sigma)) : \mathcal{G}^k[w] \to \mathcal{E}_k[w'],$$

where $w' = w - 2k + n - 6$. Specialising to even dimensions, $w = 0$, $k = n/2-3$ and composing with q_k we obtain

$$\delta q^{n/2-3}\iota(X)\varepsilon(\not{D})|C|^2\iota(\not{D})\varepsilon(X)\varepsilon(Y_\sigma))q_{n/2-3} : \mathcal{E}^{n/2-3} \to \mathcal{E}_{n/2-3}.$$

Using the explicit formulae for $\varepsilon(\not{D})$, etc. it is easy expand this and verify that in (even) dimensions $n \geq 6$ this is non-trivial. (In dimension 6 this boils down to a case treated this way in [32].)

Finally on this point we should remark that in constructing $\mathbb{S}^\sigma$ we have have made no attempt to produce the most general object with a transformation law similar to the Q_k operators. Since, on $\mathcal{G}^k[w]$, $\varepsilon(Y_\sigma)$ has the conformal transformation $\varepsilon(Y_{\hat{\sigma}}) = \varepsilon(Y_\sigma) - \varepsilon(\tilde{d}\Upsilon)$ it follows that *any* conformally invariant operator P which acts on $\mathcal{G}^{k+1}[w]$ (for any weight w) may be composed with $\varepsilon(Y_\sigma)$ to yield an operator with a similar transformation law to the $\mathbb{Q}$ operators. In the case $w = 0$ we may form the composition $P\varepsilon(Y_\sigma)q_k$, which has a transformation law similar to that of the Q_k^σ operators. However we envisage that the main interest should be the $\mathbb{S}^\sigma q_k$ that operate between $\mathcal{E}^k$ and $\mathcal{E}_k$, since these may play a role in understanding the nature of the Q operators and the Q-curvature.

6.2. **Other recent constructions of Q-curvature.** Up to a scale our construction here gives Branson's curvature Q (as a multiplication operator) by $\iota(Y_\sigma)\mathbb{K}_{k+1}^{n/2-1}\varepsilon(Y_\sigma)1 = \iota(Y_\sigma)\not{\Box}_{n/2-1}\iota(\not{D})\varepsilon(X)\varepsilon(Y_\sigma)1$. Since $\not{\Delta}_\ell$ is formally self-adjoint on densities [37, 26] the argument above leading to this conclusion (in the proof of part (*v*) of Theorem 2.8) works equally if we start with $\not{\Delta}_\ell$ in place $\not{\Box}_\ell$ in the formulae for the L^ℓ. Thus up to a multiple, Q_0^σ is given by $\iota(Y_\sigma)\not{\Delta}_{n/2-1}\iota(\not{D})\varepsilon(X)\varepsilon(Y_\sigma)$. Applying this to the constant function 1, we have $\iota(Y_\sigma)\not{\Delta}_{n/2-1}\iota(\not{D})\varepsilon(X)\varepsilon(Y_\sigma)1 = \iota(Y_\sigma)\not{\Delta}_{n/2-1}\iota(\not{D})\varepsilon(X)Y_\sigma$. Suppose we define I^g by $nI^g := \iota(\not{D})\varepsilon(X)Y_\sigma$, where g is the metric corresponding to the conformal scale σ. Then an elementary calculation using $Y_\sigma = \sigma^{-1}\tilde{d}\sigma$ and the formulae in Section 3.1 shows that I^g is exactly the scale dependent standard tractor defined and used in Section 2.3 of [32] to give a new construction of the Q-curvature. So in this notation the Q-curvature is given, up to a non-zero constant multiple, by

$$\iota(Y_\sigma)\not{\Delta}_{n/2-1}I^g. \tag{45}$$

This is essentially the formula for Q given by Gover and Peterson in [32] (see Proposition 2.7). In fact, in that paper, Q is given by $\iota(Y_\sigma)FI^g$, where

$F : \mathcal{T}[-1] \to \mathcal{T}[1]$ is an invariant operator derived from the ambient powers of the Laplacian in a similar, but not identical way to $\not\Delta_{n/2-1}$. It is possible that F and $\not\Delta_{n/2-1}$ differ as operators on $\mathcal{T}[-1]$, but up to scale they agree on I^g.

Next we observe that re-expressing our formula for the Q-curvature recovers a formula given recently by Fefferman and Hirachi. An ambient expression naturally corresponding to $\iota(Y_\sigma)\not\Delta_{n/2-1}I^g = \iota(Y_\sigma)\not\Delta_{n/2-1}\iota(\not D)\varepsilon(X)Y_\sigma$ is $\iota(\boldsymbol{Y}_\sigma)\not\boldsymbol{\Delta}^{n/2-1}\iota(\not\boldsymbol{D})\varepsilon(\boldsymbol{X})\boldsymbol{Y}_\sigma$, where, with $\tilde{\sigma} \in \tilde{\mathcal{E}}(1)$ a homogeneous function on $\tilde{M}$ corresponding to $\sigma \in \mathcal{E}[1]$, we define $\boldsymbol{Y}_\sigma := \tilde{\sigma}^{-1}\boldsymbol{d}\tilde{\sigma}$. Now using Lemma 4.1 we can re-express this as $\iota(\boldsymbol{Y}_\sigma)\iota(\boldsymbol{X})\varepsilon(\not\boldsymbol{D})\not\boldsymbol{\Delta}^{n/2-1}\boldsymbol{Y}_\sigma$. Next observe that $\boldsymbol{Y}_\sigma = \boldsymbol{d}\log\tilde{\sigma}$ and that $[\not\boldsymbol{\Delta}^{n/2-1}, \boldsymbol{d}] = 0$. So we obtain

$$\iota(\boldsymbol{Y}_\sigma)\iota(\boldsymbol{X})\varepsilon(\not\boldsymbol{D})\boldsymbol{d}\not\boldsymbol{\Delta}^{n/2-1}\log\tilde{\sigma}.$$

But, from the formula for $\varepsilon(\not\boldsymbol{D})$ in Proposition 3.3, we have $\varepsilon(\not\boldsymbol{D})\boldsymbol{d} = \varepsilon(\boldsymbol{X})\boldsymbol{d}\not\boldsymbol{\Delta}$. Also note that $\{\iota(\boldsymbol{X}), \varepsilon(\boldsymbol{X})\}$ vanishes modulo $O(Q)$, while

$$\{\iota(\boldsymbol{Y}_\sigma), \varepsilon(\boldsymbol{X})\} = h(\boldsymbol{X}, \boldsymbol{Y}) = 1.$$

So, modulo $O(Q)$ terms, we get to $-\iota(\boldsymbol{X})\boldsymbol{d}\not\boldsymbol{\Delta}^{n/2}\log\tilde{\sigma} = -\mathcal{L}_{\boldsymbol{X}}\not\boldsymbol{\Delta}^{n/2}\log\tilde{\sigma}$. Since $\not\boldsymbol{\Delta}^{n/2}\log\tilde{\sigma}$ is homogeneous of degree $-n$, we obtain finally

$$n\boldsymbol{\Delta}^{n/2}\log\tilde{\sigma}, \tag{46}$$

modulo $O(Q)$ terms. Up to a non-zero constant multiple, this is the ambient expression for the Q-curvature given in [27]. In summary, we see that under the identification of tractor sections with appropriate homogeneous ambient quantities, and with the use of standard identities from exterior calculus, (46) and (45) are really identical formulae for the Q-curvature, and both are generalised by Q_k^σ as in Theorem 2.8.

There is a corresponding ambient expression for the cases $k \geq 1$. Recall that $\mathbb{Q}_k^{\ell,\sigma}$ is given by $-2(\ell+1)q^k\iota(Y_\sigma)\not\Box_{\ell-1}\iota(\not D)\varepsilon(X)\varepsilon(Y_\sigma)$, where $\not\Box_{\ell-1} = \frac{1}{2}(\not\Delta_{\ell-1} + (\not\Delta_{\ell-1})^*)$. Following the case above, if we replace $\not\Box_{\ell-1}$ by $\not\Delta_{\ell-1}$, then we obtain an alternative operator

$$\tilde{\mathbb{Q}}_k^{\ell,\sigma} := -2(\ell+1)q^k\iota(Y_\sigma)\not\Delta_{\ell-1}\iota(\not D)\varepsilon(X)\varepsilon(Y_\sigma)$$

that agrees with $\mathbb{Q}_k^{\ell,\sigma}$ at leading order and has a conformal transformation law very similar to that of $\mathbb{Q}_k^{\ell,\sigma}$. Up to a multiple the ambient expression naturally corresponding to $\tilde{\mathbb{Q}}_k^{\ell,\sigma}$ is $\boldsymbol{q}^k\iota(\boldsymbol{Y}_\sigma)\not\boldsymbol{\Delta}^{\ell-1}\iota(\not\boldsymbol{D})\varepsilon(\boldsymbol{X})\varepsilon(\boldsymbol{Y}_\sigma)$. Here $\boldsymbol{q}^k$ is an algebraic map on $\boldsymbol{\mathcal{G}}_k$ corresponding to q^k on $\mathcal{G}_k$ (so that along $\mathcal{Q}$ these maps are equivalent). Now viewing this as an operator on ambient forms U of degree k and such that $\varepsilon(\boldsymbol{X})\boldsymbol{d}U = O(Q)$ along $\mathcal{Q}$ we obtain, by a very similar argument to that above, a re-expression of this as

$$-\boldsymbol{q}^k\iota(\boldsymbol{X})\boldsymbol{d}\not\boldsymbol{\Delta}^{\ell}\varepsilon(\log\tilde{\sigma}).$$

Finally we should say that, in an obvious way, the constructions of Section 6.1 above may be carried out on the ambient manifold, and in that setting, the observation $\boldsymbol{Y}_\sigma = \boldsymbol{d}\log\tilde{\sigma}$ may be used to express the $\mathbb{S}^\sigma$ operators in terms of ambient operators acting on $\log\tilde{\sigma}$. It follows that the constructions there may also be viewed as a generalisation of Theorem 2.2 of [27].

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