CONFORMALLY INVARIANT POWERS OF THE LAPLACIAN, Q-CURVATURE, AND TRACTOR CALCULUS

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ABSTRACT. We describe an elementary algorithm for expressing, as explicit formulae in tractor calculus, the conformally invariant GJMS operators due to C.R. Graham et alia. These differential operators have leading part a power of the Laplacian. Conformal tractor calculus is the natural induced bundle calculus associated to the conformal Cartan connection. Applications discussed include standard formulae for these operators in terms of the Levi-Civita connection and its curvature and a direct definition and formula for T. Branson's so-called Q-curvature (which integrates to a global conformal invariant) as well as generalisations of the operators and the Q-curvature. Among examples, the operators of order 4, 6 and 8 and the related Q-curvatures are treated explicitly. The algorithm exploits the ambient metric construction of Fefferman and Graham and includes a procedure for converting the ambient curvature and its covariant derivatives into tractor calculus expressions. This is partly based on [12], where the relationship of the normal standard tractor bundle to the ambient construction is described.

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1. INTRODUCTION

Conformally invariant differential operators have long been known to play an important role in physics and the geometry of many structures related to and including Riemannian and conformal geometries. For example, the classical field equations describing massless particles, including the Maxwell and Dirac (neutrino) equations, depend only on conformal structure [2, 18]. More recently string theory and quantum gravity have motivated several developments in mathematics where conformally invariant operators play a key role. Many of these could be said to fall under the umbrella of geometric spectral theory where, broadly, one attempts to relate global geometry to the spectrum of some natural operators on the manifold. For example, on compact manifolds there are programmes to find extremal metrics for functional determinants of natural operators. Conformally invariant operators yield determinants with a workable formula (a so called Polyakov formula) for the conformal variation of the determinant so leading to significant progress [10, 6, 5]. In another direction there is new progress [35] in relating scattering matrices on conformally compact Einstein manifolds with conformal objects on their boundaries at infinity. This falls within the framework of the AdS/CFT correpondence of quantum gravity [43, 36, 37, 33].

In these areas it seems an especially important role is played by natural conformally invariant operators with principal part a power of the Laplacian

Δ. The earliest known of these is the conformally invariant wave operator which was first constructed for the study of massless fields on curved spacetime. More recently its Riemannian signature variation, usually called the Yamabe operator, has played a large role in the Yamabe problem on compact Riemannian manifolds. As an operator on functions it is given by the formula $\Delta-(n-2)R/(4(n-1))$, and it governs the transformation of the scalar curvature R under conformal rescaling. An operator with principal part Δ^2 is due to Paneitz [40] (see also [41, 23]), and then sixth-order analogues were constructed in [3, 44]. Graham, Jenne, Mason and Sparling (GJMS) solved a major existence problem in [32] where they used a formal geometric construction to show the existence of conformally invariant differential operators P_{2k} (to be referred to as the GJMS operators) with principal part Δ^k. In odd dimensions, k is any positive integer, while in dimension n even, k is a positive integer no more than $n/2$. The $k=1$ and $k=2$ cases recover, respectively, the Yamabe and Paneitz operators.

In dimension 2 the transformation of the scalar curvature can also be deduced from the Yamabe operator by a dimensional continuation argument, and the curvature fixing problem corresponding to the Yamabe problem is usually known as Gauss curvature prescription. In the late 1980's Branson [4, 10] observed that the Paneitz operator P is formally self-adjoint and can be expressed in the form $P^1+((n-4)/2)Q_4$, where P^1 annihilates constant functions and Q_4 is a scalar curvature invariant which could play a role parallel to the scalar curvature in higher order analogues of the Gauss curvature prescription programme. In dimension 4 the conformal transformation of Q_4 is given by the Paneitz operator, and it follows that the integral of Q_4 over compact 4-manifolds is a global conformal invariant. On conformally flat structures this is a multiple of the Euler characteristic. It has recently been established by Graham and Zworski and Fefferman and Graham [34, 35, 27] that the GJMS operators P_{2k} are formally self-adjoint, and so [5] shows that these operators yield an analogous local Riemannian invariant Q_n for each even-dimensional manifold. There has been considerable recent interest and progress in understanding Branson's Q-curvatures, especially in low dimensions and on conformally flat structures [16, 17].

In [32] the GJMS operators are derived from the Laplacian of the ambient metric of Fefferman and Graham [25, 26]. This construction is very valuable not only in itself but also because of the close links with the Poincaré metrics of the conformally compact Einstein theory. On the other hand there is another way to generate a conformally invariant operator with principal part Δ^k. The result is usually presented as a simple formulae, first due to M.G. Eastwood, as given in (15). (See [28] for a derivation and some further related developments.) Underlying this formula are two related key tools. The first is a geometric construction developed by Eastwood and others [22, 19] known as the curved translation principle. This construction is a generalised and geometric variant of the translation functor due to Zuckerman and others [45]. The second is a machinery known as tractor calculus [1, 29, 14, 13]. This calculus brings the conformally invariant Cartan connection to induced bundles and also involves other fundamental conformally invariant operators (such as the ones used in this formula). The combination is potent since on the one hand it is very easy to expand these tractor formulae in terms of the Levi-Civita connection and its curvature (which is useful for the investigation of issues such as positivity of the operators), and on the other hand the link with representation theory means one easily obtains rules for generalising the operators and how they may be composed with certain other conformally invariant operators. See for example (16). It should be pointed out that the tractor formulae are themselves complete and explicit formulae and can be readily worked with directly without using any knowledge of the representation theory aspects. That is essentially the approach below. See also [7], for example, where these tractor formulae for conformally invariant powers of the Laplacian are used to construct formally self-adjoint conformally invariant boundary problems, higher order conformally invariant Dirichlet-to-Neumann operators, and related constructions.

One problem with the tractor approach up until now has been that, on even dimension n manifolds, this had failed to yield the operators of order n except for a quotient construction in dimension 4 [28]. Here we give a similar quotient tractor construction for a sixth-order operator and show that we have in fact recovered P_4 and P_6. This brings us to one of the main purposes of this paper, which is to explicitly relate the tractor calculus approach to the GJMS construction. This is achieved in Section 4, where an algorithm is described for finding a tractor formula for any of the GJMS operators P_{2k}. Remarkably this algorithm does not require solving the Fefferman-Graham ambient construction. For low order operators it is essentially trivial and quickly recovers the simple tractor formulae for P_4 and P_6 and yields a corresponding tractor formula for P_8. See Section 4.1 and Proposition 2.3. In Proposition 2.4 we use these formulae to prove directly that these operators are formally self-adjoint (verifying directly for these cases the general results of [35, 27]). Expanding these formulae into formulae in terms of the Levi-Civita connection and its curvature simply requires repeated use of the Leibniz rule and the definitions of the tractor objects. This is easily automated and is done in Section 2.2. The nature of the formulae we use mean the calculations have a large number of built-in self-checks which ensure that the formulae used are entered and used correctly by the software. Thus overall this demonstrates an effective means to obtain explicit formulae for the GJMS operators. It should be pointed out that the formulae in Section 2.2 are not in fact the raw output from the expansion of the tractor formulae, but rather this output manipulated into the canonical form described in [21]. The authors performed these expansions and manipulations mainly by using *Mathematica* and J. Lee's Ricci programme [39]; this work was performed under the assumption of a Riemannian signature metric, but the resulting formulae are independent of the signature.

The most important outcomes of Section 4 are Proposition 4.5 and Theorem 2.5. The first of these establishes important features about the form of the tractor formulae for the GJMS operators, and the latter exploits this to provide some new invariant operators closely linked to the GJMS operators. There are several applications of these. One is a direct tractor based construction of Branson's Q-curvatures. See Proposition 2.7. In fact, this also gives a new definition for these invariants. This gives an effective way to calculate these (Q_4 and Q_6 are treated as examples), and it sheds light on their remarkable transformation properties. Another application of Theorem 2.5 is Corollary 2.6. In words this states that except for the $k = n/2$ case, the theorem yields generalisations of the GJMS operators P_{2k} that are 'strongly invariant' in the sense of [19]. That is, operators that can be composed with tractor bundle valued operators to yield further conformally invariant operators. This is one of the key ideas of the curved translation principle. Finally, Theorem 2.5 is a crucial ingredient in the general construction in [8] of an elliptic conformally invariant operator on 1-forms with close connections to the first de Rham cohomology.

There are other results presented. For example, in Section 2.3 we describe how to proliferate Riemannian invariants which are not conformally invariant but have a transformation formula similar to the Branson Q-curvatures. These can be viewed as representing terms that could be added to the Q-curvature without affecting its key properties and so play a role in generating new curvature prescription problems.

There are also many other potential applications for this work not touched upon in this article. For example, the tractor formulae for the GJMS operators could immediately be used in a construction parallel to that in [7] to produce alternative conformally invariant boundary problems and non-local operators based around the GJMS operators.

It should also be pointed out that the results and ideas in this paper should have analogues for CR structures, where one would instead be involved with CR-invariant powers of the sub-Laplacian [30] and the ambient construction of C. Fefferman [24]. The construction presented in this article is in part an application of ideas developed in the joint work of one of the authors with A. Čap. See [12] where it is described explicitly how to relate the Cartan/tractor approach to the ambient construction of Fefferman and Graham and its applications to invariant theory. The relevant aspects of this theory are summarised in Section 3.1. There is a corresponding theory for the CR case [11].

The authors are indebted to Tom Branson, Andi Čap, Mike Eastwood, and Robin Graham for several illuminating conversations. The authors would also like to thank the Mathematical Sciences Research Institute and the organisers of Spring session in 2001 for helping to make this research possible.

2. Conformal geometry and tractor calculus

We summarise here an approach to local conformal geometry that is rather useful for our applications. This is broadly based on the development presented in [13], but many of the ideas and tools had their origins in [42], [1], and [29]. The notation and conventions in general follow the last two sources.

We shall work on a real conformal n-manifold M, where $n \geq 3$. That is, we have a pair $(M, [g])$, where M is a smooth n-manifold and $[g]$ is a conformal equivalence class of metrics of signature (p, q). Two metrics g and $\widehat{g}$ are said to be *conformally equivalent* if $\widehat{g}$ is a positive scalar function multiple of g. In this case it is convenient to write $\widehat{g} = \Omega^2 g$ for some positive smooth function Ω. Although we assume that the metrics have some fixed signature, all considerations below will be signature independent. For a given conformal manifold $(M,[g])$, we shall denote by $\mathcal{Q}$ the bundle of metrics. That is, $\mathcal{Q}$ is a subbundle of S^2T^*M with fibre $\mathbb{R}^+$. The points correspond to values of metrics in the conformal class.

Let $\mathcal{E}^a$ denote the space of smooth sections of the tangent bundle TM, and similarly let $\mathcal{E}_a$ be the smooth sections of the cotangent bundle T^*M. In fact, we will generally abuse notation and also use these symbols to indicate the sheaves of germs of smooth sections and even the bundles themselves. These conventions will be carried through to all bundles that we discuss. We write $\mathcal{E}$ to denote the trivial bundle over M. Penrose's abstract index notation is embraced throughout, so tensor products of these bundles will be indicated by adorning the symbol $\mathcal{E}$ with appropriate abstract indices. For example, in this notation $\otimes^2T^*M$ is written $\mathcal{E}_{ab}$. An index which appears twice, once raised and once lowered, indicates a contraction. These conventions will be extended in an obvious way to the tractor bundles described below. In all settings indices may also be 'suppressed' (omitted) if superfluous by context.

The bundle $\mathcal{Q}$ is a principal bundle with group $\mathbb{R}_+$, so there are natural line bundles on $(M,[g])$ induced from the irreducible representations of $\mathbb{R}_+$. We write $\mathcal{E}[w]$ for the line bundle induced from the representation of weight $-w/2$ on $\mathbb{R}$ (that is $\mathbb{R}_+ \ni x \mapsto x^{-w/2} \in \mathrm{End}(\mathbb{R})$). Thus a section of $\mathcal{E}[w]$ corresponds to a real-valued function f on $\mathcal{Q}$ with the homogeneity property $f(x,\Omega^2 g) = \Omega^w f(x,g)$, where Ω is a positive function on M, $x \in M$, and g is a metric from the conformal class $[g]$. We use the notation $\mathcal{E}_a[w]$ for $\mathcal{E}_a \otimes \mathcal{E}[w]$ and so on. Note that for consistency with [1], this convention differs in sign from the one of [14, Section 4.15].

Let $\mathcal{E}_+[w]$ be the fibre subbundle of $\mathcal{E}[w]$ corresponding to $\mathbb{R}_+ \subset \mathbb{R}$. Choosing a metric g from the conformal class defines a function $f : \mathcal{Q} \to \mathbb{R}$ by $f(\hat{g},x) = \Omega^{-2}$, where $\hat{g} = \Omega^2 g$, and this clearly defines a smooth section of $\mathcal{E}_+[-2]$. Conversely, if f is such a section, then $f(g,x)g$ is constant up the fibres of $\mathcal{Q}$ and so defines a metric in the conformal class. Thus $\mathcal{E}_+[-2]$ is canonically isomorphic to $\mathcal{Q}$, and the *conformal metric* $\boldsymbol{g}_{ab}$ is the tautological section of $\mathcal{E}_{ab}[2]$ that represents the map $\mathcal{E}_+[-2] \cong \mathcal{Q} \to \mathcal{E}_{(ab)}$. From this there is a canonical section $\boldsymbol{g}^{ab}$ of $\mathcal{E}^{ab}[-2]$ such that $\boldsymbol{g}_{ab}\boldsymbol{g}^{bc} = \delta_a{}^c$ (where $\delta_a{}^c$ is the section of $\mathcal{E}_a{}^c$ corresponding to the identity endomorphism of the the tangent bundle). The conformal metric (and its inverse $\boldsymbol{g}^{ab}$) will be used to raise and lower indices without further mention. Given a choice of metric g from the conformal class, we write ∇_a for the corresponding Levi-Civita connection. With these conventions the Laplacian Δ is given by $\Delta = \boldsymbol{g}^{ab}\nabla_a\nabla_b = \nabla^b\nabla_b$. In view of the isomorphism $\mathcal{E}_+[-2] \cong \mathcal{Q}$, a choice of metric also trivialises the bundles $\mathcal{E}[w]$. In particular we will write ξ^g for the canonical section of $\mathcal{E}[1]$ satisfying $g = (\xi^g)^{-2}\boldsymbol{g}$. Conversely a section of $\mathcal{E}_+[1]$ clearly determines a metric by this relation, so such a ξ^g is termed a choice of conformal scale. This determines a connection on $\mathcal{E}[w]$ via the corresponding trivialisation of $\mathcal{E}[w]$ and the exterior derivative on functions. We shall also denote such a connection by ∇_a and refer to it as the Levi-Civita connection. Note in particular then that, by definition, $\nabla_a \xi^g = 0$, so ∇_a also preserves the conformal metric. The curvature $R_{ab}{}^c{}_d$ of the Levi-Civita connection is known as the Riemannian curvature, and is defined by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)v^c = R_{ab}{}^c{}_d v^d.$$

This can be decomposed into the totally trace-free Weyl curvature C_{abcd} and a remaining part described by the symmetric *Rho-tensor* P_{ab}, according to

$$R_{abcd} = C_{abcd} + 2\boldsymbol{g}_{c[a}\mathsf{P}_{b]d} + 2\boldsymbol{g}_{d[b}\mathsf{P}_{a]c},$$

where $[\cdots]$ indicates the antisymmetrization over the enclosed indices. The Rho-tensor is a trace modification of the Ricci tensor R_{ab}. We write J for the trace $\mathsf{P}^a{}_a$ of P.

Under a *conformal transformation* we replace our choice of metric g by the metric $\hat{g} = \Omega^2 g$, where Ω is a positive smooth function. The Levi-Civita connection then transforms as follows:

$$\widehat{\nabla_a u_b} = \nabla_a u_b - \Upsilon_a u_b - \Upsilon_b u_a + \boldsymbol{g}_{ab}\Upsilon^c u_c \quad \widehat{\nabla_a\sigma} = \nabla_a\sigma + w\Upsilon_a\sigma. \tag{1}$$

Here $u_b \in \mathcal{E}_b$, $\sigma \in \mathcal{E}[w]$, and $\Upsilon_a := \Omega^{-1}\nabla_a\Omega$. The Weyl curvature is conformally invariant, that is $\widehat{C}_{abcd} = C_{abcd}$, and the Rho-tensor transforms by

$$\widehat{\mathsf{P}}_{ab} = \mathsf{P}_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \tfrac{1}{2}\Upsilon^c\Upsilon_c\boldsymbol{g}_{ab}. \tag{2}$$

For the the density bundle $\mathcal{E}[1]$, we have the jet exact sequence at 2-jets,

$$0 \to \mathcal{E}_{(ab)}[1] \to J^2(\mathcal{E}[1]) \to J^1(\mathcal{E}[1]) \to 0,$$

where $(\cdots)$ indicates symmetrization over the enclosed indices. Note we have a bundle homomorphism $\mathcal{E}_{(ab)}[1] \to \mathcal{E}[-1]$ given by complete contraction with $\boldsymbol{g}^{ab}$. This is split via $\rho \mapsto \frac{1}{n}\rho\boldsymbol{g}_{ab}$ and so the conformal structure decomposes $\mathcal{E}_{(ab)}[1]$ into the direct sum $\mathcal{E}_{(ab)_0}[1] \oplus \mathcal{E}[-1]$. Clearly then $\mathcal{E}_{(ab)_0}[1]$ is a smooth subbundle of $J^2(\mathcal{E}[1])$, and we define $\mathcal{E}^A$ to be the quotient bundle. That is, the *standard tractor bundle* $\mathcal{E}^A$ is defined by the exact sequence

$$0 \to \mathcal{E}_{(ab)_0}[1] \to J^2(\mathcal{E}[1]) \to \mathcal{E}^A \to 0. \tag{3}$$

The jet exact sequence at 2-jets and the corresponding sequence at 1-jets, viz $0 \to \mathcal{E}_a[1] \to J^1(\mathcal{E}[1]) \to \mathcal{E}[1] \to 0$, determine a composition series for $\mathcal{E}^A$ which we can summarise via a self-explanatory semi-direct sum notation $\mathcal{E}^A = \mathcal{E}[1] \oplus\!\!\!\!+ \mathcal{E}_a[1] \oplus\!\!\!\!+ \mathcal{E}[-1]$. We denote by X^A the canonical section of $\mathcal{E}^A[1] := \mathcal{E}^A \otimes \mathcal{E}[1]$ corresponding to the mapping $\mathcal{E}[-1] \to \mathcal{E}^A$.

Composing the canonical projection $J^2(\mathcal{E}[1]) \to \mathcal{E}^A$ with the 2-jet operator j^2 yields an invariant differential operator $\frac{1}{n}D^A : \mathcal{E}[1] \to \mathcal{E}^A$. On the other hand, if we choose a metric g from the conformal class, then the map

$$j^2_x\sigma \mapsto [\tfrac{1}{n}D^A\sigma(x)]_g := (\sigma(x), \nabla_a\sigma(x), -\tfrac{1}{n}(\Delta + \mathsf{J})\sigma(x)),$$

induces an isomorphism $\mathcal{E}^A \to \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1] =: [\mathcal{E}^A]_g$ of vector bundles. Tautologically the displayed formula for $\frac{1}{n}[D^A\sigma(x)]_g$ gives the operator D^A in terms of this decomposition. If the image of $V^A \in \mathcal{E}^A$ is $[V^A]_g = (\sigma, \mu_a, \tau)$, then from the change in the Levi-Civita connection (1) we get

$$[V^A]_{\hat{g}} = \widehat{(\sigma, \mu_a, \tau)} = (\sigma, \mu_a + \sigma\Upsilon_a, \tau - \Upsilon_b\mu^b - \tfrac{1}{2}\sigma\Upsilon_b\Upsilon^b).$$

This transformation formula characterises sections of $\mathcal{E}^A$ in terms of triples in $\mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$. In this notation $[X^A]_g = (0, 0, 1)$. It is convenient

	Y^A	Z^{Ac}	X^A
Y_A	0	0	1
Z_{Ab}	0	$\delta_b{}^c$	0
X_A	1	0	0

FIGURE 1. Tractor inner product

to introduce scale-dependent sections $Z^{Ab} \in \mathcal{E}^{Ab}[-1]$ and $Y^A \in \mathcal{E}^A[-1]$ mapping into the other slots of these triples so that $[V^A]_g = (\sigma, \mu_a, \tau)$ is equivalent to

$$V^A = Y^A\sigma + Z^{Ab}\mu_b + X^A\tau.$$

If $\hat{Y}^A$ and $\hat{Z}^A{}_b$ are the corresponding quantities in terms of the metric $\hat{g} = \Omega^2 g$ then we have

$$\hat{Z}^{Ab} = Z^{Ab} + \Upsilon^b X^A, \quad \hat{Y}^A = Y^A - \Upsilon_b Z^{Ab} - \tfrac{1}{2}\Upsilon_b\Upsilon^b X^A. \tag{4}$$

The standard tractor bundle has an invariant metric h_{AB} of signature $(p+1, q+1)$ and an invariant connection, which we shall also denote by ∇_a, preserving h_{AB}. If V^A is as above and $\underline{V}^B \in \mathcal{E}^B$ is given by $[\underline{V}^B]_g = (\underline{\sigma}, \underline{\mu}_b, \underline{\tau})$, then

$$h_{AB}V^A\underline{V}^B = \mu^b\underline{\mu}_b + \sigma\underline{\tau} + \tau\underline{\sigma}.$$

Using h_{AB} and its inverse to raise and lower indices, we immediately see that

$$Y_A X^A = 1, \quad Z_{Ab}Z^A{}_c = \boldsymbol{g}_{bc}$$

and that all other quadratic combinations that contract the tractor index vanish. This is summarised in figure 1. Thus we also have $Y_A V^A = \tau$, $X_A V^A = \sigma$, $Z_{Ab}V^A = \mu_b$ and the metric may be decomposed into a sum of projections, $h_{AB} = Z_A{}^c Z_{Bc} + X_A Y_B + Y_A X_B$.

If for a metric g from the conformal class $V^A \in \mathcal{E}^A$ is given by $[V^A]_g = (\sigma, \mu_a, \tau)$, then the invariant connection is given by

$$[\nabla_a V^B]_g = \begin{pmatrix} \nabla_a\sigma - \mu_a \\ \nabla_a\mu_b + \boldsymbol{g}_{ab}\tau + \mathsf{P}_{ab}\sigma \\ \nabla_a\tau - \mathsf{P}_{ab}\mu^b \end{pmatrix}. \tag{5}$$

The tractor metric will be used to raise and lower indices without further comment. We shall use either "horizontal" (as in $[V^B]_g = (\sigma, \mu_b, \tau)$) or "vertical" (as in (5)) notation, depending on which is clearer in each given situation.

Tensor products of the standard tractor bundle, skew or symmetric parts of these and so forth are all termed tractor bundles. The bundle tensor product of such a bundle with $\mathcal{E}[w]$, for some real number weight w, is termed a weighted tractor bundle. For example $\mathcal{E}_{A_1A_2\cdots A_\ell}[w] = \mathcal{E}_{A_1} \otimes \cdots \otimes \mathcal{E}_{A_\ell} \otimes \mathcal{E}[w]$ is a weighted tractor bundle. Given a choice of conformal scale we have the corresponding Levi-Civita connection on tensor and density bundles. In this setting we can use the coupled Levi-Civita tractor connection to act on sections of the tensor product of a tensor bundle with a tractor bundle. This is defined by the Leibniz rule in the usual way. For example if $u^b V^C \sigma \in \mathcal{E}^b \otimes \mathcal{E}^C \otimes \mathcal{E}[w] =: \mathcal{E}^{bC}[w]$, then $\nabla_a u^b V^C \sigma = (\nabla_a u^b)V^C\sigma + u^b(\nabla_a V^C)\sigma +$

$u^bV^C\nabla_a\sigma$. Here ∇ means the Levi-Civita connection on $u^b \in \mathcal{E}^b$ and $\sigma \in \mathcal{E}[w]$, while it denotes the tractor connection on $V^C \in \mathcal{E}^C$. In particular with this convention we have

$$\nabla_a X_A = Z_{Aa}\,, \quad \nabla_a Z_{Ab} = -\mathsf{P}_{ab}X_A - Y_A\boldsymbol{g}_{ab}\,, \quad \nabla_a Y_A = \mathsf{P}_{ab}Z_A{}^b, \tag{6}$$

which for the purposes of automating calculations is a very useful description of the tractor connection.

Note that if V is a section of $\mathcal{E}^\Phi[w]$, which means simply some tractor bundle of weight w, then the coupled Levi-Civita tractor connection is not confomally invariant but transforms just as the Levi-Civita connection transforms on densities of the same weight. That is

$$\widehat{\nabla}_a V = \nabla_a V + w\Upsilon_a V$$

under the conformal rescaling $g \mapsto \hat{g} = \Omega^2 g$ (cf. (1)). It is an elementary exercise using the last transformation formulae and (4) to show that, for $V \in \mathcal{E}^\Phi[w]$, the formula

$$D^{AP}V := 2wX^{[P}Y^{A]}V + 2X^{[P}Z^{A]b}\nabla_b V \tag{7}$$

determines an invariant operator $D^{AP} : \mathcal{E}^\Phi[w] \to \mathcal{E}^{[AP]} \otimes \mathcal{E}^\Phi[w]$. (This was first developed in early versions of [29] and is closely related to the 'fundamental D' operator developed in [14].) Since we can vary the weight and the tractor bundle $\mathcal{E}^\Phi$, D^{AP} is really an entire family of operators. The point is that with the way we have defined ∇, the same formula works for the entire family, and so it is reasonable to let the single symbol D^{AP} denote all of these operators. We abuse terminology and describe it as an operator. (The Levi-Civita connection is usually used this way.) If we have a single formula Op that gives a family of conformally invariant operators

$$\mathrm{Op} : \mathcal{E}^\Phi \otimes \mathcal{V} \to \mathcal{E}^\Phi \otimes \mathcal{W}$$

as we range over all tractor bundles $\mathcal{E}^\Phi$ then, following [19], we describe Op as a *strongly invariant* operator. For example D^{AP} is strongly invariant. As already pointed out D^{AP} is rather more universal since we can vary the weight w as well. Thus we can form compositions of this operator with itself, and in particular we consider $h^{AB}D_{A(Q}D_{|B|P)_0}V$ for $V \in \mathcal{E}^\Phi[w]$ some weighted tractor. Expanding this out using (6), (7), and the Leibniz rule for ∇, it is easily verified that it may be re-expressed in the form

$$h^{AB}D_{A(Q}D_{|B|P)_0}V = -X_{(Q}D_{P)_0}V,$$

where D is some operator determined explicitly in the calculation. Since the map $\mathcal{E}_P[w-1] \to \mathcal{E}_{(PQ)_0}[w]$ given by $S_P \mapsto X_{(Q}S_{P)_0}$ is injective, this establishes $D_A : \mathcal{E}^\Phi[w] \to \mathcal{E}_A \otimes \mathcal{E}^\Phi[w-1]$ as a conformally invariant differential operator on weighted tractor bundles. For $V \in \mathcal{E}^\Phi[w]$, this is given by

$$D^AV := (n+2w-2)wY^AV + (n+2w-2)Z^{Aa}\nabla_a V - X^A\Box V, \tag{8}$$

where

$$\Box V := \nabla_p\nabla^p V + w\mathsf{J}V. \tag{9}$$

So D_A is in fact precisely the tractor D-operator in [1]. Note the identity

$$D_A X^A V = (n+2w+2)(n+w)V, \tag{10}$$

which we will use later.

The curvature Ω of the tractor connection is defined by

$$[\nabla_a, \nabla_b]V^C = \Omega_{ab}{}^C{}_E V^E$$

for $V^C \in \mathcal{E}^C$ and is precisely the local obstruction to conformal flatness. (That is locally there is a flat metric in the conformal class if and only if this curvature vanishes.) Using (6) and the usual formulae for the curvature of the Levi-Civita connection we calculate (cf. [1])

$$\Omega_{abCE} = Z_C{}^c Z_E{}^e C_{abce} - 4X_{[C}Z_{E]}{}^e \nabla_{[a}\mathsf{P}_{b]e}. \tag{11}$$

It is straightforward to use this and (6) to show that if $V \in \mathcal{E}_{CE\cdots F}[w]$, then

$$\begin{aligned} &[D_A, D_B]V_{CE\cdots F} = \\ &\quad (n+2w-2)[W_{ABC}{}^Q V_{QE\cdots F} + 2w\Omega_{ABC}{}^Q V_{QE\cdots F} \\ &\quad +4X_{[A}\Omega_{B]}{}^s{}_C{}^Q \nabla_s V_{QE\cdots F} + \cdots + W_{ABF}{}^Q V_{CE\cdots Q} \\ &\quad +2w\Omega_{ABF}{}^Q V_{CE\cdots Q} + 4X_{[A}\Omega_{B]}{}^s{}_F{}^Q \nabla_s V_{CE\cdots Q}]. \end{aligned} \tag{12}$$

Here $\Omega_{ABCE} = Z_A{}^a Z_B{}^b \Omega_{abCE}$, $\Omega_{BsCE} = Z_B{}^b \Omega_{bsCE}$, and

$$W_{ABCE} = (n-4)\Omega_{ABCE} - 2X_{[A}Z_{B]}^b \nabla^p \Omega_{pbCE}. \tag{13}$$

It follows that on conformally flat structures $[D_B, D_C]V_{CE\cdots F} = 0$. Similarly it is easily verified that $[D_B, D_C]$ annihilates densities.

We should point out some features of W_{ABCE}. Firstly, it is conformally invariant. One can already see this from (12) by setting $w = 0$ and then considering sections V_C of $\mathcal{E}_C$ such that $\nabla_a V_C$ vanishes at a given point. This is also immediately clear from the formula $W_{AB}{}^K{}_L := \frac{3}{n-2} D^P X_{[P}\Omega_{AB]}{}^K{}_L$ (see [29]), which is readily verified. From this several things are immediately clear. Firstly, W_{ABCE} vanishes on conformally flat structures. Next, we have that $W_{ABCE} = W_{[AB][CE]}$ and that it is trace-free (since Ω_{ABCE} is annihilated by contraction with X^P on any index). Furthermore expanding (13) reveals that $W_{[ABC]E} = 0$. Thus W_{ABCE} has 'Weyl tensor symmetries'. Whence it is immediately clear that W_{ABCE} is also annihilated upon contraction with X^P.

Finally we should comment on the uniqueness of this tractor calculus. In sections 2.6 and 2.7 of [13] it is shown that the transformation properties (4) and the form of the connection (6) identify $\mathcal{E}^A$ and its tractor connection ∇_a as above as a normal tractor bundle and connection corresponding to the defining representation of $\mathrm{SO}(p+1, q+1)$. Let $\mathbb{V}$ be $\mathbb{R}^{n+2}$ as the representation space for the standard (or defining) representation of $\mathrm{SO}(p+1, q+1)$. We can construct [14] from the pair $(\mathcal{E}^A, \nabla_a)$ a principal bundle $\mathcal{G}$ which is the frame bundle for $\mathcal{E}^A$ corresponding to the metric and filtration. This has fibre P, a certain parabolic subgroup of $\mathrm{SO}(p+1, q+1)$. A Cartan connection ω on $\mathcal{G}$ is determined by ∇. This is the normal Cartan connection on $\mathcal{G}$ such that ∇_a is the vector bundle connection induced from ω. That is the normality condition on the pair $(\mathcal{E}^A, \nabla_a)$ is equivalent to the pair $(\mathcal{G}, \omega)$ being a normal Cartan bundle and connection in the sense of [15].

2.1. **Conformally invariant powers of the Laplacian.** Since the tractor-D operator constructed above is well defined on any weighted tractor bundle, we can compose the tractor-D operators. It is clear from the formula for the tractor-D operator that any such composition will yield a *natural* operator, that is an operator which can be written as a polynomial formula in terms of a representative metric, its inverse, the metric connection and its curvature. On densities of the appropriate weight and with some minor adjustment a composition of this form will lead to conformally invariant operators with principal part a power of the Laplacian.

First let us observe how the conformal Laplacian arises from the tractor machinery. Let $f \in \mathcal{E}[1-n/2]$. Then observe that immediately from (8) we have $D_A f = -X_A \Box f$. Since D_A is conformally invariant we have immediately that for $f \in \mathcal{E}[1-n/2]$, $\Box f$ is conformally invariant. From (9) this is $(\nabla^a \nabla_a + \frac{2-n}{2}\mathsf{J})f$ – the usual conformal Laplacian.

Now suppose that instead we have $f \in \mathcal{E}^\Phi[1-n/2]$, where here and below $\mathcal{E}^\Phi[w]$ will be used to indicate any tractor bundle of weight w. We still have

$$D_A f = -X_A \Box f, \tag{14}$$

but now in $\Box f = (\nabla^a \nabla_a + \frac{2-n}{2}\mathsf{J})f$, ∇ means the Levi-Civita tractor coupled connection. In particular this establishes a strongly invariant generalisation of the Yamabe operator on tractor sections of the said weight.

It is clear from our observations that that there is a conformally invariant operator

$$\Box D_{A_1} D_{A_2} \cdots D_{A_{k-1}} : \mathcal{E}[k-n/2] \to \mathcal{E}_{A_1 A_2 \cdots A_{k-1}}[-1-n/2].$$

In the conformally flat case this already yields an operator between densities (cf. [28]).

Proposition 2.1. *On conformally flat structures, if $f \in \mathcal{E}[k-n/2]$, then*

$$\Box D_{A_{k-1}} \cdots D_{A_1} f = (-1)^{k-1} X_{A_1} \cdots X_{A_{k-1}} \Box_{2k} f,$$

where $\Box_{2k} : \mathcal{E}[k-n/2] \to \mathcal{E}[-k-n/2]$ is a conformally invariant operator. Locally we can choose a flat metric from the conformal class. This determines a connection in terms of which we have $\Box_{2k} f = \Delta^k f$.

Proof. In any choice of conformal scale, expand out $\Box D_{A_{k-1}} \cdots D_{A_1} f$ via the formula (8) and move the X, Y, Z's to the left of all ∇'s via the identities (6). It is immediately clear that the highest order term is precisely $(-1)^{k-1} X_{A_1} \cdots X_{A_{k-1}} \Delta^k f$ and that any other coefficient of $X_{A_1} \cdots X_{A_{k-1}}$ involves the curvature P_{ab} or its trace.

On the other hand, on conformally flat structures, $[D_A, D_B]V = 0$ for V any weighted tractor field. Thus $D_{A_\ell} \cdots D_{A_1} f$ is completely symmetric for any $\ell \in \mathbb{Z}_+$. In particular $\Box D_{A_{k-1}} \cdots D_{A_1} f \in \mathcal{E}_{(A_1 \cdots A_{k-1})}[-1-n/2]$ and $D_{A_k} D_{A_{k-1}} \cdots D_{A_1} f \in \mathcal{E}_{(A_1 \cdots A_k)}[-n/2]$. Consequently it must be that $0 = D_{[A_k} D_{A_{k-1}]} \cdots D_{A_1} f$. But $D_{A_{k-1}} \cdots D_{A_1} f$ has weight $1-n/2$, so from (14) this implies $X_{[A_k} \Box D_{A_{k-1}]} D_{A_{k-2}} \cdots D_{A_1} f = 0$. It follows immediately that $\Box D_{A_{k-1}} D_{A_{k-2}} \cdots D_{A_1} f = X_{A_1} \cdots X_{A_{k-1}} \Box_{2k} f$ for some operator $\Box_{2k}$. With the above we are done. $\square$

If we are happy to work in the scale of a flat metric then there is an even simpler proof along the lines of the proof of Proposition 4.3. We leave this for the reader.

By the same ideas as in the proof above, it is easy to use (12) and (8) to show that, if $k \geq 3$, then $X_{[A_k}\Box D_{A_{k-1}]}D_{A_{k-2}}\cdots D_{A_1}f \neq 0$ for $f \in \mathcal{E}[k-n/2]$ on a general conformally curved manifold. Thus the proposition fails if we remove the requirement of conformal flatness. One way to generalise the $\Box_{2k}$ is as follows.

Consider

$$D^{A_1}\cdots D^{A_{k-1}}\Box D_{A_{k-1}}\cdots D_{A_1}f$$

for $f \in \mathcal{E}^\Phi[k-n/2]$. This is manifestly strongly conformally invariant in all dimensions and for all positive integers k. Furthermore by the identity (10) we have that, on conformally flat structures,

$$D^{A_1}\cdots D^{A_{k-1}}\Box D_{A_{k-1}}\cdots D_{A_1}f = \prod_{i=2}^{k}(2i-n)(i-1)\Box_{2k}f. \tag{15}$$

On the other hand for general conformally curved structures, suppose that the dimension n is odd or satisfies $2k < n$. Then we can *define* $\Box_{2k}f$ by (15), and this gives a conformally invariant operator

$$\Box_{2k} : \mathcal{E}^\Phi[k-n/2] \to \mathcal{E}^\Phi[-k-n/2] \tag{16}$$

with principal part Δ^k. Here, as usual, $\mathcal{E}^\Phi[k-n/2]$ indicates any tractor bundle of weight $k-n/2$. In these dimensions this generalises the operator $\Box_{2k}$ of the proposition. Although we do not wish to describe the curved translation principle [22, 19], it is worth pointing out that it is partly illustrated here. The tractor formula (15) manifests $\Box_{2k}$ as a 'translate' of the Yamabe operator $\Box$. In fact proceeding in smaller steps it demonstrates $\Box_{2k}$ as a translate of $\Box_{2k-2}$.

Before we move on, let us demonstrate that the operators $\Box_{2k}$ are formally self-adjoint. We summarise some results we need from Section 7 of [7] in the following proposition. These results can be verified easily using the definitions above, and it is important for our needs to note that this works in a rather formal manner. That is, we can leave the dimension and weight as unknown in the calculations.

Proposition 2.2. *On a conformal manifold M we have*
(i) If $\psi^B \in \Gamma\mathcal{E}^B[w]$ and $\varphi \in \Gamma\mathcal{E}[1-n-w]$ is compactly supported on M, then

$$\int_M \varphi D_A\psi^A\boldsymbol{\epsilon} = \int_M (D_A\varphi)\psi^A\boldsymbol{\epsilon}.$$

(ii) If $\mathcal{E}^\Phi$ is any tractor bundle, then $\mathcal{E}^\Phi$ is canonically isomorphic to its dual $\mathcal{E}_\Phi$ via the tractor metric and for any pair $\psi^\Phi, \varphi^\Phi \in \Gamma\mathcal{E}^\Phi[1-n/2]$, ($\mathcal{E}^\Phi[1-n/2] := \mathcal{E}^\Phi \otimes \mathcal{E}[1-n/2]$) we have

$$\int_M \varphi^\Phi\Box\psi_\Phi\boldsymbol{\epsilon} = \int \psi^\Phi\Box\varphi_\Phi\boldsymbol{\epsilon}.$$

Here $\boldsymbol{\epsilon}$ is the canonical conformal volume form, that is the canonical section of $\mathcal{E}_{[a_1a_2\cdots a_n]}[n]$ compatible with the conformal metric. Because this volume form has weight n, the integrals in the proposition are conformally invariant. Now part (ii) of the proposition asserts $\Box_2 = \Box$ is formally self-adjoint, while the same result for $D^{A_1} \cdots D^{A_{k-1}} \Box D_{A_{k-1}} \cdots D_{A_1}$ follows immediately from this and repeated use of (i). So the formal self-adjoint property of $\Box_{2k}$ is proved. It should be pointed out that as well as observing that $D^{A_1} \cdots D^{A_{k-1}} \Box D_{A_{k-1}} \cdots D_{A_1} f$ recovers a conformally invariant power of the Laplacian, M.G. Eastwood also observed the formal self-adjoint property. It is clear from (15) that, unfortunately, this formula does not yield a conformally invariant operator of order n on even dimensional structures, yet the existence of such an operator is guaranteed by the construction of [32].

Recall from Section 2 that if $f \in \mathcal{E}[w]$, then $[D_A, D_B]f = 0$, and so the $k = 2$ case of the proposition does hold on general conformal structures. In particular, as observed in [28], for $f \in \mathcal{E}[2 - n/2]$ we can define $P_4' f$ by the quotient formula

$$\Box D_A f = -X_A P_4' f.$$

Then P_4' has principle part Δ^2. This construction works even when $n = 4$, and in other dimensions $P_4' = \Box_4$ as defined above. It is not hard to do the next even order in a similar way. If now $f \in \mathcal{E}[3 - n/2]$, then $[D_B, D_C]f = 0$ and hence $D_B D_C f = D_{(B} D_{C)} f$. Now it is a short exercise, using (12) and the definition of W_{ABCD} once more, to show that

$$(n-4) X_{[A} \Box D_{B]} D_C f = -2 X_{[A} W_{B]}{}^S{}_C{}^T D_S D_T f.$$

Now from the fact that $D_S D_T f$ is symmetric and that W_{BSCT} has Weyl tensor type symmetries, we can deduce that in dimensions $n \neq 4$,

$$P_{BC} f := \Box D_B D_C f + \frac{2}{n-4} W_B{}^S{}_C{}^T D_S D_T f \tag{17}$$

is symmetric (i.e. $P_{BC} f \in \mathcal{E}_{(BC)}[-1 - n/2]$). On the other hand, from the previous display $X_{[A} D_{B]C} f = 0$. Thus, for $n \neq 4$,

$$P_{BC} f = X_B X_C P_6' f,$$

where P_6' is a conformally invariant operator $\mathcal{E}[3 - n/2] \to \mathcal{E}[-3 - n/2]$ generalising (for the allowed dimensions) the sixth-order operator of Proposition 2.1. In particular this works in dimension 6.

We should point out that although $\Box_{2k}$ as defined by (15) is manifestly strongly invariant, we cannot conclude this for P_6' as defined here. The operator P_{BC} defined above is clearly invariant when acting on weighted tractors, but the argument here to deduce that $P_{BC} f$ has the form $X_B X_C P_6' f$ relies on the vanishing of $[D_A, D_B]f$.

We will establish in Section 4 (see in particular Subsection 4.1) the following result.

Proposition 2.3. *The operators P_4' and P_6' defined by the tractor expressions above are precisely the fourth-order and sixth-order GJMS operators. That is $P_4' = P_4$, $P_6' = P_6$. A tractor expression for the eighth-order GJMS operator*

P_8 is as follows:

$$
\begin{aligned}
X_A X_B X_C P_8 f = {}& \\
&-\Box D_A D_B D_C f - \tfrac{2}{n-4} W_A{}^P{}_B{}^Q D_P D_Q D_C f - \tfrac{2}{n-4} W_A{}^P{}_C{}^Q D_P D_B D_Q f \\
&- \tfrac{4}{n-6} X_A U_B{}^P{}_C{}^Q D_P D_Q f + \tfrac{2}{(n-4)(n-6)} X_A D^E W_B{}^P{}_C{}^Q D_E D_P D_Q f \\
&+4 \tfrac{n-2}{(n-4)^2(n-6)} X_A W_B{}^P{}_C{}^Q W_P{}^S{}_Q{}^T D_S D_T f,
\end{aligned}
$$

where all operators act on everything to their right, in a given term, and $U_B{}^P{}_C{}^Q$ is the tractor field

$$\tfrac{2}{(n-4)^2}\left(W^{AP}{}_B{}^F W_{FAC}{}^Q + W^{AP}{}_C{}^F W_{BAF}{}^Q + W^{APQF} W_{BACF}\right).$$

Here $D^E W_B{}^P{}_C{}^Q D_E D_P D_Q f$ means $D^E(W_B{}^P{}_C{}^Q D_E D_P D_Q f)$ (and *not* $(D^E W_B{}^P{}_C{}^Q) D_E D_P D_Q f$). This and similar conventions for other operators and situations will apply throughout the paper.

We should emphasise at this point that the tractor formulae for P_4, P_6 and P_8 above, and similar ones for the higher order P_{2k} that we could easily construct via the algorithm of Section 4, are genuine formulae for the GJMS operators. No further algorithm is required. They are valid on any conformal manifold where the given GJMS operators exist. In this tractor form they are already suitable for many applications, such as establishing strong invariance or constructing related operators. The remainder of the section will demonstrate this.

We begin by using the tractor formulae directly to show that the operators P_4, P_6 and P_8 are formally self-adjoint (FSA). We treat these in order. For $f \in \mathcal{E}[2-n/2]$, we have

$$\Box D_A f = -X_A P_4 f \quad \text{which implies} \quad D^A \Box D_A f = (n-4) P_4 f. \tag{18}$$

We have already observed that $D^{A_1} \cdots D^{A_{k-1}} \Box D_{A_{k-1}} \cdots D_{A_1}$ is FSA on $\mathcal{E}[k-n/2]$. So from the second of these it is clear that P_4 is FSA in dimensions other than 4. From the expressions (8) and (9) it follows that $D^A \Box D_A f$ and $\Box D_A f$, as expressions in terms of Levi-Civita covariant derivatives of f, P_{ab} and J, are polynomial in n. So from (18) it is clear that $(4-n)$ divides this expression for $D^A \Box D_A f$ and so $P_4 f$ is also given as a formula polynomial in n and the Levi-Civita covariant derivatives of f, P_{ab} and J. Working among tensors of this form a calculation to verify the FSA property of P_4 (in dimensions greater than 4) can be carried out formally in dimension n, since Proposition 2.2 is established that way. It follows immediately that the same calculation must work when we set $n = 4$. Thus P_4 is also FSA in dimension 4. Now for P_6, let $f \in \mathcal{E}[3-n/2]$ and note that $\Box D_B D_C f$ and $W_B{}^S{}_C{}^T D_S D_T f$ are polynomial in n. Thus $P_{BC} f$ is rational in n with a singularity only at $n = 4$. From $P_{BC} f = X_B X_C P_6 f$ we have

$$
\begin{aligned}
&(n-4) D^C D^B P_{BC} f \\
&\quad = (n-4) D^C D^B \Box D_B D_C f + 2 D^C D^B W_B{}^S{}_C{}^T D_S D_T f \\
&\quad = 2(n-4)^2 (n-6) P_6 f.
\end{aligned}
$$

Now since W_{BSCT} has the Weyl tensor symmetries (in fact here we just need $W_{BSCT} = W_{CTBS} = W_{TCSB}$), it follows from Proposition 2.2 that $D^C D^B W_B{}^S{}_C{}^T D_S D_T$ is FSA on $\mathcal{E}[3-n/2]$. We know $D^C D^B \Box D_B D_C f$ is also FSA and as expressions in terms of Levi-Civita covariant derivatives of f, C_{abcd}, P_{ab}, and J, both of these and $(n-4)P_{BC}$ are polynomial in n. Thus the expression like this for $D^C D^B P_{BC} f$ is divisible by $(n-6)$, so reasoning as for P_4, we quickly conclude that P_6 is FSA in all dimensions for which it is defined. Finally, since U_{ABCD} also has Weyl tensor symmetry (as readily verified directly or since it corresponds to $\boldsymbol{\Delta R}_{ABCD}$ as in Section 3), it follows that $D^C D^B U_B{}^P{}_C{}^Q D_P D_Q f$ is FSA for $f \in \mathcal{E}[4-n/2]$. A similar comment applies to the other terms on the right-hand side of the formula,

$$\begin{aligned}
&6(n-4)^2(n-6)(n-8)P_8 f = \\
&\quad +(n-4)D^C D^B D^A \Box D_A D_B D_C f + 2D^C D^B D^A W_A{}^P{}_B{}^Q D_P D_Q D_C f \\
&\quad +2D^C D^B D^A W_A{}^P{}_C{}^Q D_P D_B D_Q f - 4\tfrac{(n-4)^2}{n-6} D^C D^B U_B{}^P{}_C{}^Q D_P D_Q f \\
&\quad +2\tfrac{n-4}{n-6} D^C D^B D^E W_B{}^P{}_C{}^Q D_E D_P D_Q f \\
&\quad +4\tfrac{(n-2)}{n-6} D^C D^B W_B{}^P{}_C{}^Q W_P{}^S{}_Q{}^T D_S D_T f,
\end{aligned}$$

which follows from the earlier display for P_8. This shows immediately that $P_8 f$ is FSA in dimensions other than 8, and then, arguing as in the previous cases, we can deduce that it is also FSA in dimension 8. We have directly proved the following.

Proposition 2.4. *The GJMS operators P_4, P_6 and P_8 are formally self-adjoint.*

In fact the result is also immediate from the formulae for these operators in Section 2.2. The formulae there are given in terms of the Levi-Civita connection and its curvature and are in a canonical form that manifests the formal self-adjoint symmetry. (It should be pointed out that in deriving those formulae formal self-adjointness was not assumed.)

Recently the entire family of operators P_{2k} have been shown to be formally self-adjoint by other means [35, 27].

In Section 4 we will show that there are similar tractor formulae for all of the GJMS operators and that these tractor formulae share some of the qualitative features of the examples above. In particular we will prove the following theorem.

Theorem 2.5. *(i) In each dimension n and for each integer $2 \leq k$, with $k \leq n/2$ if n is even, there is a conformally invariant differential operator*

$$E_{CD}{}^{AB} : \mathcal{E}_{AB}[k-2-n/2] \to \mathcal{E}_{CD}[2-k-n/2]$$

such that

$$E_{CD}{}^{AB} D_A D_B f = X_C X_D P_{2k} f$$

for $f \in \mathcal{E}[k-n/2]$, where $P_{2k} : \mathcal{E}[k-n/2] \to \mathcal{E}[-k-n/2]$ is the order $2k$ GJMS operator. The operator is given by a formula which is a partial contraction polynomial in $\Box$, D_A, W_{ABCD}, X_A, h_{AB} and its inverse h^{AB}.

(ii) With n and k as in (i) and for $\mathcal{E}^\Phi$ any tractor bundle, there is a conformally invariant differential operator

$$E_{CD}{}^{AB} : \mathcal{E}^\Phi \otimes \mathcal{E}_{AB}[k-2-n/2] \to \mathcal{E}^\Phi \otimes \mathcal{E}_{CD}[2-k-n/2]$$

which generalises the operator of part (i).
(iii) On conformally flat structures the operator E is, up to a non-zero scale, $\Box_{2k-4}$. In this case, given a choice of flat metric from the conformal class, E is, up to a non-zero scale, Δ^{k-2}.

Proof. We have already observed that for $f \in \mathcal{E}[2-n/2]$, we have (see (18)) $\Box D_A f = -X_A P_4 f$. Since then $D_A f \in \mathcal{E}_A[1-n/2]$ we have $X_B \Box D_A f = -D_B D_A f$, and so we have (i) for $k=2$. Otherwise establishing part (i) is the primary purpose of Section 4. More precisely, from the discussion there we obtain Proposition 4.5, which asserts that

$$X_{A_1} \cdots X_{A_{k-1}} P_{2k} f = (-1)^{k-1} \Box D_{A_1} \cdots D_{A_{k-1}} f + \Psi_{A_{k-1} \cdots A_1}{}^{PQ} D_P D_Q f.$$

Applying $D^{A_3} \cdots D^{A_{k-1}}$ to both sides of this and using (10), we obtain

$$X_{A_1} X_{A_2} P_{2k} f = E_{A_1 A_2}{}^{PQ} D_P D_Q f,$$

where
(19)

$$\begin{aligned} E_{A_1 A_2}{}^{PQ} &= \Box_{2(k-2)} \delta_{A_1}{}^P \delta_{A_2}{}^Q + \\ &\quad \left(\textstyle\prod_{i=2}^{k-2} (n-2i)(i-1)\right)^{-1} D^{A_3} \cdots D^{A_{k-1}} \Psi_{A_{k-1} \cdots A_3 A_2 A_1}{}^{PQ}. \end{aligned}$$

As explained in Proposition 4.5, Ψ is given explicitly by a sum of terms each of which is a monomial in D_A, W_{ABCD}, X_A, h_{AB}, and its inverse h^{AB}. Each such monomial is thus a composition of strongly conformally invariant operators. So $E_{A_1 A_2}{}^{PQ}$ is a sum of compositions of strongly conformally invariant operators. Just knowing that $E_{A_1 A_2}{}^{PQ}$ is a sum of compositions of conformally invariant operators of this form gives part (i). Then part (ii) is immediate from the fact that these are strongly invariant operators.

Next we show part (iii). From the Proposition 4.5 each term in the expression for Ψ is of degree at least 1 in W_{ABCD}. The latter vanishes on conformally flat structures. Thus, from (19), on such structures E is just $\Box_{2k-4}$. From Proposition 4.3, given a choice of flat metric from the conformal class, we have $\Box_{2k-4} = \Delta^{k-2}$. $\Box$

Remarks: In regard to part (i) of the theorem we should point out that Section 4 not only establishes the existence of a formula for E which is a partial contraction polynomial in D_A, W_{ABCD}, X_A, h_{AB}, and its inverse h^{AB}, but describes an algorithm for finding such a formula.

It seems likely that the operator E in the theorem is formally self-adjoint. Note for example, if we write E^* for the formal adjoint of E, then from Proposition 2.2, the identity (10), and the result that P_{2k} is formally self-adjoint, we have $(\prod_{i=k-1}^{k} (n-2i)(i-1)) P_{2k} = D^A D^B E^*_{AB}{}^{PQ} D_P D_Q$ on $\mathcal{E}[k-n/2]$.

Finally we should add that the terms which distinguish the P_{2k} from the $\Box_{2k}$ do not vanish in general. At least we have verified by direct calculation that for $f \in \mathcal{E}[3-n/2]$ (and $n \neq 4$) the leading term of $D^C D^B W_B{}^S{}_C{}^T D_S D_T f$

is a non-zero scalar multiple of $C^a{}_{cde}C^{bcde}\nabla_a\nabla_b f$. Thus P_6 is not simply a scalar multiple of $\Box_6$.

It is a non-trivial matter to know when conformally invariant operators have strongly invariant generalisations. Some do not. For example in dimension 4 we know there is a conformally invariant operator $P_4 : \mathcal{E} \to \mathcal{E}[-4]$ with principal part Δ^2. Suppose there were a strongly invariant generalisation of this. Then, in particular, it would give a conformally invariant operator $H_A{}^B : \mathcal{E}_B \to \mathcal{E}_A[-4]$ with principal part Δ^2. (Here we mean the principal part as an operator between the reducible bundles indicated.) Then, in the case of Riemannian signature conformal 4-manifolds, using the ellipticity of this, (10), Proposition 2.2 and the differential operator existence results in the conformally flat setting (cf. [23]) we can conclude that we would have a conformally invariant operator $D^A H_A{}^B D_B : \mathcal{E}[1] \to \mathcal{E}[-5]$ with principal part Δ^3 (on arbitrary conformal 4-manifolds). This contradicts C.R. Graham's non-existence result [31], and so we can conclude the operator $H_A{}^B$ does not exist. (See also [22].) However P_4 does have a strongly invariant generalisation in all other dimensions. This is just $\Box_4$ as a special case of (16). More generally, a consequence of the part (ii) of the theorem is that the GJMS operators P_{2k} admit strongly invariant generalisations except in the critical dimension $n = 2k$. That is, we have the following proposition on n-dimensional conformal manifolds:

Corollary 2.6. *For each integer $k \geq 1$, with $2k < n$ if n is even, there is a (tractor) formula that gives, for each tractor bundle $\mathcal{E}^\Phi$, a formally self-adjoint differential operator*

$$P_{2k}^\Phi : \mathcal{E}^\Phi[k - n/2] \to \mathcal{E}^\Phi[-k - n/2],$$

where $\mathcal{E}^\Phi[w] := \mathcal{E}^\Phi \otimes \mathcal{E}[w]$. The operator has principal part Δ^k and can be expressed as a sum of $\Box_{2k}$ and a contraction polynomial in D_A, W_{ABCD}, X_A, h_{AB}, and its inverse h^{AB}. In the conformally flat case the operator is $\Box_{2k}$. In the case that $\mathcal{E}^\Phi = \mathcal{E}$ then $P_{2k}^\Phi = P_{2k}$.

Proof. Since D_A is strongly invariant and since also, from (ii) of the theorem, $E_{AB}{}^{PQ}$ is strongly invariant, it follows that there is a conformally invariant operator $(F := (\prod_{i=k-1}^{k}(n-2i)(i-1))^{-1} D^A D^B E_{AB}{}^{PQ} D_P D_Q) : \mathcal{E}^\Phi[k - n/2] \to \mathcal{E}^\Phi[-k - n/2]$ for any tractor bundle. By (10) this precisely recovers the GJMS operator P_{2k} if $\mathcal{E}^\Phi[k - n/2]$ is simply the density bundle $\mathcal{E}[k - n/2]$. Now consider the formal adjoint F^* of F. This is another conformally invariant operator $\mathcal{E}^\Phi[k - n/2] \to \mathcal{E}^\Phi[-k - n/2]$ (where as usual we identify $\mathcal{E}^\Phi$ with its dual via the tractor metric). Since P_{2k} is formally self-adjoint, it is clear that, when applied to $\mathcal{E}[k - n/2]$, F^* also recovers the GJMS operator. Thus $(F + F^*)/2$ is the required formally self-adjoint operator.

It is clear from the Proposition 4.5 that we can express F by a formula which is a sum of $\Box_{2k}$ and a contraction polynomial in D_A, W_{ABCD}, X_A, h_{AB}, and its inverse h^{AB}. From that proposition we also have that each term in the latter polynomial expression is of degree at least 1 in W_{ABCD}. Using Proposition 2.2 and the formal self-adjoint property of $\Box_{2k}$, we see that there is an expression for F^* as a sum of $\Box_{2k}$ and a contraction polynomial in D_A, W_{ABCD}, X_A, h_{AB}, and its inverse h^{AB}. Again each term in the latter polynomial is of degree at least 1 in W_{ABCD}. So the final part of the corollary follows from these observations and Proposition 2.1. $\square$

2.2. **Conventional formulae.** There are circumstances where it is useful to have explicit formulae for the GJMS-related operators and invariants in terms of the Levi-Civita connection and its curvature. These formulae are generally cumbersome. But the various curvature terms are closely related to the spectrum of the operator, so it is important to be able to extract these explicitly. In particular, for example, issues of positivity can be investigated directly in this setting. Moreover such formulae are ready to be mechanically rewritten in local coordinates should this be required.

Here we will describe how to re-express tractor formulae for $P_{2k}f$ into formulae which are polynomial in $\boldsymbol{g}$, its inverse, ∇ (meaning the Levi-Civita connection), C, P and J, and of course linear in f.

For the most part, the process is simply an expansion of the tractor formulae using the definitions above. Consider the Paneitz operator P_4 first. We observed in Proposition 2.3 that for $f \in \mathcal{E}[2-n/2]$, $-X_A P_4 f = \Box D_A f$. So $P_4 f = -Y^A \Box D_A f$, and we could simply calculate this scalar quantity $Y^A \Box D_A f$. In fact we prefer to expand the entire tractor valued quantity $\Box D_A f$ using (8) and (9). According to its definition, D_A lowers weight by 1. Thus $\Box D_A f$ is given by

$$(\nabla_b \nabla^b + (1-\frac{n}{2})\mathsf{J})((4-n)Y_A f + 2Z_A{}^a \nabla_a f - X_A(\nabla_c \nabla^c + (2-\frac{n}{2})\mathsf{J})f).$$

Now we simply move the X_A, Y_A and Z_A^a to the left of all other operators by repeated use of (6). This is easily done by hand and simplified via the Bianchi identity to yield

$$\begin{aligned} -\Box D_A f \;&=\; X_A(\Delta^2 f - (n-2)\mathsf{J}\Delta f + 4\,\mathsf{P}^{ij}\nabla_i\nabla_j f - (n-6)(\nabla^i \mathsf{J})\nabla_i f \\ &\qquad - \tfrac{n-4}{2}(\Delta \mathsf{J})f + \tfrac{n(n-4)}{4}\mathsf{J}^2 f - (n-4)\mathsf{P}_{ij}\mathsf{P}^{ij} f). \end{aligned}$$

The coefficient of X_A on the right-hand side is a formula for the Paneitz operator. Note that the coefficient of Y_A and the coefficient of $Z_A{}^a$ both turned out to be zero. Of course this is exactly as predicted by our formula $-X_A P_4 f = \Box D_A f$, but it provides a very useful check of the formulae to verify this. So this is all there is to producing the required formula for P_4 from the tractor formula. Before we continue with the general case let us just reorganise the result.

For any linear differential operator on densities of the appropriate weight there is a canonical form for the formula which, among other features, manifests the symmetry in the formally self-adjoint and formally anti-self-adjoint parts [21]. As already observed, the Paneitz operator is formally self-adjoint. Applying this idea to the formula above yields

$$P_4 f = \nabla_i \nabla_j \mathsf{S}_4^{ijkl} \nabla_k \nabla_l f + \nabla_i \mathsf{S}_2^{ij} \nabla_j f + \tfrac{n-4}{2} Q^g_{4,n} f. \tag{20}$$

Here S_4^{ijkl} and S_2^{ij} are the tensors $(1/3)(\boldsymbol{g}^{il}\boldsymbol{g}^{jk} + \boldsymbol{g}^{ik}\boldsymbol{g}^{jl} + \boldsymbol{g}^{ij}\boldsymbol{g}^{kl})$ and

$$\tfrac{4-3n}{3}\boldsymbol{g}^{ij}\mathsf{J} - \tfrac{2(n-8)}{3}\mathsf{P}^{ij},$$

respectively, and $Q^g_{4,n}$ denotes the scalar

$$\frac{n}{2}\mathsf{J}^2 - 2\mathsf{P}_{ij}\mathsf{P}^{ij} - \Delta\mathsf{J}. \tag{21}$$

In (20), the ∇'s act on all tensors to their right within the given term.

We now discuss the general case. Explicit tractor formulae are readily produced by the algorithm described in Section 4, and so we shall suppose we are beginning with a formula for P_{2k} as described in Proposition 4.5. The formulae for P_6 and P_8 above (see Proposition 2.3) give explicit examples that can be kept in mind. These formulae are polynomial in $\Box$, D_A, W_{ABCD}, X_A, h_{AB}, and its inverse h^{AB}. We replace each of these with its formula in terms of the coupled tractor-Levi-Civita connection ∇, X_A and so forth according to the formulae (8), (9), (11), and (13). In doing so we note that W has weight -2 and that D lowers the weight of a tractor by 1. Next we move all occurrences of $X_A, Z_A{}^a$, and Y_A to the left of the ∇ via repeated use of (6). At the end of this process all tractor valued objects are to the left of the remaining ∇'s, and so at this point these ∇'s are simply Levi-Civita covariant derivative operators. Next we use the inner product rules of Figure 1 to simplify the resulting expression. The formula for $P_{2k}f$ is then simply the overall coefficient of $X_{A_1}X_{A_2}\cdots X_{A_{k-1}}$. >From Proposition 4.5 all other slots of the tractor expression vanish. That is, the sum of the terms that do *not* contain $X_{A_1}X_{A_2}\cdots X_{A_{k-1}}$ is zero. Verifying this or even partly verifying this provides a very serious check of all formulae and any software that are used in the calculation. For example, one can verify that the sum of the terms containing $Z_{A_1}{}^a X_{A_2}\cdots X_{A_{k-1}}$ vanishes.

This procedure is very simple. But there are many terms involved, as the next examples will illustrate. Thus it becomes very useful to be able calculate via a suitable computer algebra system. For the examples below the authors used *Mathematica* and J. Lee's *Ricci* program [39], which proved to be very effective. Certainly the P_8 case is beyond a reasonable hand calculation. The use of software and the self-checking nature of the formulae as discussed above mean that one can be confident of the final result.

As a technical point for these calculations, we describe a simple technique which can considerably reduce the computing time they require. One can implement this technique by developing a short computer programme. We begin by noting that certain steps in the computation may produce tractor inner products of the form $\Psi_{B_1B_2\ldots B_\ell}\Phi^{B_1B_2\ldots B_\ell}$, where the indices B_1, B_2,…, and B_ℓ appear as subscripts or superscripts attached to the tractors Y, Z, and X. Suppose that ℓ is large and that the tractors $\Psi_{B_1B_2\ldots B_\ell}$ and $\Phi^{B_1B_2\ldots B_\ell}$ are the sums of many terms. Suppose also that no derivatives of Y, Z, or X occur. The tractor $\Psi_{B_1B_2\ldots B_\ell}$ is a linear combination of the following 3^ℓ terms:

$$\begin{array}{l}
Y_{B_1}Y_{B_2}\cdots Y_{B_{\ell-1}}Y_{B_\ell}\\
Y_{B_1}Y_{B_2}\cdots Y_{B_{\ell-1}}Z_{B_\ell}{}^{b_\ell}\\
Y_{B_1}Y_{B_2}\cdots Y_{B_{\ell-1}}X_{B_\ell}\\
Y_{B_1}Y_{B_2}\cdots Z_{B_{\ell-1}}{}^{b_{\ell-1}}Y_{B_\ell}\\
Y_{B_1}Y_{B_2}\cdots Z_{B_{\ell-1}}{}^{b_{\ell-1}}Z_{B_\ell}{}^{b_\ell}\\
Y_{B_1}Y_{B_2}\cdots Z_{B_{\ell-1}}{}^{b_{\ell-1}}X_{B_\ell}
\end{array}$$

$$\vdots$$
$$X_{B_1}X_{B_2}\cdots X_{B_{\ell-1}}X_{B_\ell}$$

The coefficients of these terms may, of course, be very complicated. By raising indices we may write $\Phi^{B_1B_2\ldots B_\ell}$ as a similar linear combination. Each term of each linear combination may be paired off with at most one term in the other linear combination so as to give a nonzero inner product. We compute the 3^ℓ possible inner products and add the results.

We conclude this section with the calculation of P_6 and P_8 via these methods, beginning with the tractor formulae indicated in Proposition 2.3. As a check, the authors verified the vanishing of the overall coefficient of the $Z_B{}^iX_C$ term in the expansion of (17). In a similar fashion, they also verified the vanishing of the overall coefficients of the $X_BZ_C{}^i$ and $Z_A{}^iX_BX_C$ terms in the expansions for P_6 and P_8, respectively. This involved the use of the Bianchi identities, tensor symmetries, and changes in the order in which covariant derivatives are taken. The authors also manipulated the resulting formulae for the GJMS operators into the canonical form suggested in [21]. Here are the results:

$$\begin{aligned} P_6 &= \nabla_i\nabla_j\nabla_k\mathsf{T}_6^{ijklmp}\nabla_l\nabla_m\nabla_p f+\nabla_i\nabla_j\mathsf{T}_4^{ijkl}\nabla_k\nabla_l f+\nabla_i\mathsf{T}_2^{ij}\nabla_j f\\ &\quad+\tfrac{n-6}{2}Q^g_{6,n}f. \end{aligned}$$

Here T_6^{ijklmp} and T_4^{ijkl} are the symmetrizations of the tensors $\boldsymbol{g}^{ij}\boldsymbol{g}^{kl}\boldsymbol{g}^{mp}$ and

$$\frac{2-3n}{2}\mathsf{J}\boldsymbol{g}^{ij}\boldsymbol{g}^{kl}+(20-2n)P^{ij}\boldsymbol{g}^{kl},$$

respectively, and T_2^{ij} is the tensor

$$\begin{aligned} &-\tfrac{(88-86\,n+n^2)}{15\,(n-4)}\mathsf{P}^{ij}{}_{|k}{}^{k}-\tfrac{2\,(2176-768\,n+82\,n^2+n^3)}{15\,(n-4)}\mathsf{P}^{i}{}_{k}\mathsf{P}^{jk}\\ &-\tfrac{2\,(320-218\,n+27\,n^2)}{15\,(n-4)}\boldsymbol{g}^{ij}\mathsf{P}_{kl}\mathsf{P}^{kl}+\tfrac{(3\,n-2)\,(5\,n-54)}{15}\mathsf{P}^{ij}\mathsf{J}\\ &+\tfrac{-164-120\,n+45\,n^2}{60}\boldsymbol{g}^{ij}\mathsf{J}^2-\tfrac{2\,(5\,n-22)}{15}\boldsymbol{g}^{ij}\mathsf{J}_{|k}{}^{k}\;-\tfrac{744-250\,n+31\,n^2}{15\,(n-4)}\mathsf{J}_{|}{}^{ij}\\ &+\tfrac{2\,(296-26\,n+3\,n^2)}{15\,(n-4)}\mathsf{P}_{kl}C^{ikjl}-\tfrac{4}{15}C^{i}{}_{klm}C^{jklm}. \end{aligned}\tag{22}$$

Here and below, for typesetting convenience, we write $\mathsf{P}^{ij}{}_{|k}{}^{k}$ as an alternative notation for $\nabla^k\nabla_k\mathsf{P}^{ij}$ and so forth. Finally, $Q^g_{6,n}$ denotes the scalar

$$\begin{aligned} &-8\mathsf{P}_{ij|k}\mathsf{P}^{ij}{}_{|}{}^{k}-\tfrac{8\,(n-2)}{n-4}\mathsf{P}_{ij}\mathsf{P}^{ij}{}_{|k}{}^{k}+\tfrac{64}{n-4}\mathsf{P}_{ij}\mathsf{P}^{i}{}_{k}\mathsf{P}^{jk}\\ &+\tfrac{4\,(-4-4\,n+n^2)}{n-4}\mathsf{P}_{ij}\mathsf{P}^{ij}\mathsf{J}-\tfrac{(n-2)\,(n+2)}{4}\mathsf{J}^3+(n-6)\,\mathsf{J}_{|i}\mathsf{J}_{|}{}^{i}\\ &+\tfrac{3\,n-2}{2}\mathsf{J}\mathsf{J}_{|i}{}^{i}-\tfrac{8\,(n-6)}{n-4}\mathsf{P}_{ij}\mathsf{J}_{|}{}^{ij}-\mathsf{J}_{|i}{}^{i}{}_{j}{}^{j}-\tfrac{32}{n-4}\mathsf{P}_{ij}\mathsf{P}_{kl}C^{ikjl}. \end{aligned}\tag{23}$$

We find that P_8f is given by

$$\begin{aligned} &\nabla_i\nabla_j\nabla_k\nabla_l\mathsf{U}_8^{ijklmpqr}\nabla_m\nabla_p\nabla_q\nabla_r f+\nabla_i\nabla_j\nabla_k\mathsf{U}_6^{ijklmp}\nabla_l\nabla_m\nabla_p f\\ &+\nabla_i\nabla_j\mathsf{U}_4^{ijkl}\nabla_k\nabla_l f+\nabla_i\mathsf{U}_2^{ij}\nabla_j f+\tfrac{n-8}{2}Q^g_{8,n}f. \end{aligned}$$

$$\frac{-4\left(-19200+8468\,n-980\,n^2+27\,n^3\right)}{45\,(-6+n)\,(-4+n)}\boldsymbol{g}^{ij}\mathsf{P}_{kl|m}\mathsf{P}^{kl}{}_{|}{}^{m}$$
$$-\frac{4\left(2592-5262\,n+625\,n^2+14\,n^3\right)}{315\,(-4+n)}\boldsymbol{g}^{ij}\mathsf{P}_{kl}\mathsf{P}^{kl}{}_{|m}{}^{m}$$
$$-\frac{4\left(74928-21908\,n-968\,n^2+283\,n^3\right)}{315\,(-6+n)\,(-4+n)}\boldsymbol{g}^{ij}\mathsf{P}_{kl|m}\mathsf{P}^{km}{}_{|}{}^{l}$$
$$+\frac{8\left(311616-146460\,n+27484\,n^2-2167\,n^3+5\,n^4+7\,n^5\right)}{315\,(-6+n)\,(-4+n)}\boldsymbol{g}^{ij}\mathsf{P}_{kl}\mathsf{P}^{k}{}_{m}\mathsf{P}^{lm}$$
$$+\frac{2\left(-44928-10224\,n+14400\,n^2-3206\,n^3+203\,n^4\right)}{45\,(-6+n)\,(-4+n)}\boldsymbol{g}^{ij}\mathsf{P}_{kl}\mathsf{P}^{kl}\mathsf{J}$$
$$+\frac{-2560+1568\,n+420\,n^2-105\,n^3}{210}\boldsymbol{g}^{ij}\mathsf{J}^3+\frac{14820-3650\,n+231\,n^2}{315}\boldsymbol{g}^{ij}\mathsf{J}_{|k}\mathsf{J}_{|}{}^{k}$$
$$+\frac{4\left(-6-31\,n+5\,n^2\right)}{15}\boldsymbol{g}^{ij}\mathsf{J}\mathsf{J}_{|k}{}^{k}$$
$$+\frac{2\left(480384-238464\,n+44040\,n^2-3346\,n^3+91\,n^4\right)}{315\,(-6+n)\,(-4+n)}\boldsymbol{g}^{ij}\mathsf{P}_{kl}\mathsf{J}_{|}{}^{kl}-\frac{3\,(-48+7\,n)}{35}\boldsymbol{g}^{ij}\mathsf{J}_{|k}{}^{k}{}_{l}{}^{l}$$
$$+\frac{8\,(188+21\,n)}{315}\boldsymbol{g}^{ij}\mathsf{P}_{kl|mp}C^{kmlp}$$
$$-\frac{8\left(79488-20160\,n+1020\,n^2+50\,n^3+7\,n^4\right)}{315\,(-6+n)\,(-4+n)}\boldsymbol{g}^{ij}\mathsf{P}_{kl}\mathsf{P}_{mp}C^{kmlp}+\frac{16}{45}\boldsymbol{g}^{ij}\mathsf{J}C_{klmp}C^{kmlp}$$
$$-\frac{16}{45}\boldsymbol{g}^{ij}C_{klmp|q}C^{kmpq}{}_{|}{}^{l}+\frac{16\,(-4+n)}{45}\boldsymbol{g}^{ij}\mathsf{P}_{kl}C^{k}{}_{mpq}C^{lpmq}$$
$$-\frac{16}{45}\boldsymbol{g}^{ij}C_{klmp}C^{k}{}_{q}{}^{m}{}_{r}C^{lqpr}-\frac{16}{45}\boldsymbol{g}^{ij}C_{klmp}C^{km}{}_{qr}C^{lqpr}$$

FIGURE 2. The tensor E^{ij}

In this formula, $\mathsf{U}_8^{ijklmpqr}$ and U_6^{ijklmp} denote the symmetrizations of the tensors $\boldsymbol{g}^{ij}\boldsymbol{g}^{kl}\boldsymbol{g}^{mp}\boldsymbol{g}^{qr}$ and

$$-2n\mathsf{J}\boldsymbol{g}^{ij}\boldsymbol{g}^{kl}\boldsymbol{g}^{mp}-4(n-12)\boldsymbol{g}^{ij}\boldsymbol{g}^{kl}\mathsf{P}^{mp},$$

respectively, and U_4^{ijkl} denotes the symmetrization of the tensor

$$\frac{-8\,(-12+n)}{5}\mathsf{P}^{ij}{}_{|}{}^{kl}+\frac{4\,(-12+n)\,(-64+5\,n)}{15}\mathsf{P}^{ij}\mathsf{P}^{kl}+\frac{24\,n}{-4+n}\boldsymbol{g}^{ij}\mathsf{P}^{kl}{}_{|m}{}^{m}$$
$$-\frac{16\left(1536-530\,n+59\,n^2\right)}{15\,(-4+n)}\boldsymbol{g}^{ij}\mathsf{P}^{k}{}_{m}\mathsf{P}^{lm}-\frac{2\left(480-568\,n+67\,n^2\right)}{15\,(-4+n)}\boldsymbol{g}^{ij}\boldsymbol{g}^{kl}\mathsf{P}_{mp}\mathsf{P}^{mp}$$
$$+\frac{4\,n\,(-64+5\,n)}{5}\boldsymbol{g}^{ij}\mathsf{P}^{kl}\mathsf{J}+\frac{-96-20\,n+15\,n^2}{10}\boldsymbol{g}^{ij}\boldsymbol{g}^{kl}\mathsf{J}^2+\frac{24-5\,n}{5}\boldsymbol{g}^{ij}\boldsymbol{g}^{kl}\mathsf{J}_{|m}{}^{m}$$
$$-\frac{8\left(120-31\,n+4\,n^2\right)}{5\,(-4+n)}\boldsymbol{g}^{ij}\mathsf{J}_{|}{}^{kl}+\frac{16\left(192-7\,n+n^2\right)}{15\,(-4+n)}\boldsymbol{g}^{ij}\mathsf{P}_{mp}C^{kmlp}-\frac{16}{15}C^{i}{}_{m}{}^{j}{}_{p}C^{kmlp}$$
$$-\frac{32}{15}\boldsymbol{g}^{ij}C^{k}{}_{mpq}C^{lpmq}.$$

We let U_2^{ij} denote the symmetrization of the tensor $\mathsf{E}^{ij}+\mathsf{F}^{ij}+\mathsf{G}^{ij}$, where E^{ij}, F^{ij}, and G^{ij} are as given in Figures 2, 3, and 4. Finally, $Q^g_{8,n}$ denotes the scalar given in Figure 5.

2.3. **Branson's Q-curvature.** We have used P_{2k} to indicate a conformally invariant operator between densities, $P_{2k}:\mathcal{E}[k-n/2]\to\mathcal{E}[-k-n/2]$. Suppose we choose a metric from the conformal class. Then we can trivialise these density bundles, and so P_{2k} gives an operator P^g_{2k} between functions

$$
\begin{aligned}
&\frac{8\left(63648-43740\,n+8300\,n^2-405\,n^3+37\,n^4\right)}{315\,(-6+n)\,(-4+n)}\mathsf{P}_{kl}\mathsf{P}^{ij}{}_{|}{}^{kl}\\
&-\frac{2\left(-1728+1164\,n-292\,n^2+n^3\right)}{105\,(-6+n)\,(-4+n)}\mathsf{P}^{ij}{}_{|k}{}^{k}{}_{l}{}^{l}+\frac{8\left(-11108+5601\,n-447\,n^2+14\,n^3\right)}{315\,(-4+n)}\mathsf{P}_{kl|}{}^{j}\mathsf{P}^{ik}{}_{|}{}^{l}\\
&-\frac{16\left(-57312+18900\,n-1240\,n^2-5\,n^3+27\,n^4\right)}{315\,(-6+n)\,(-4+n)}\mathsf{P}_{kl}\mathsf{P}^{ik}{}_{|}{}^{jl}\\
&-\frac{4\left(-23040+98136\,n-18188\,n^2+134\,n^3+93\,n^4\right)}{315\,(-6+n)\,(-4+n)}\mathsf{P}^{i}{}_{k|l}{}^{l}\mathsf{P}^{jk}\\
&-\frac{4\left(-7754-1224\,n+29\,n^2\right)}{315}\mathsf{P}^{i}{}_{k|l}\mathsf{P}^{jk}{}_{|}{}^{l}\\
&+\frac{8\left(589824-282624\,n+54804\,n^2-4472\,n^3+17\,n^4+6\,n^5\right)}{105\,(-6+n)\,(-4+n)}\mathsf{P}_{kl}\mathsf{P}^{ik}\mathsf{P}^{jl}\\
&-\frac{4\left(-51384+36938\,n-9529\,n^2+662\,n^3+93\,n^4\right)}{315\,(-6+n)\,(-4+n)}\mathsf{P}^{i}{}_{k|l}\mathsf{P}^{jl}{}_{|}{}^{k}\\
&+\frac{4\left(47232-112776\,n+38488\,n^2-4444\,n^3+165\,n^4\right)}{105\,(-6+n)\,(-4+n)}\mathsf{P}_{kl}\mathsf{P}^{ij}\mathsf{P}^{kl}\\
&+\frac{2\left(-53720+22678\,n-3194\,n^2+63\,n^3\right)}{315\,(-4+n)}\mathsf{P}_{kl|}{}^{i}\mathsf{P}^{kl}{}_{|}{}^{j}\\
&+\frac{4\left(184896-132840\,n+30900\,n^2-2390\,n^3+59\,n^4\right)}{315\,(-6+n)\,(-4+n)}\mathsf{P}_{kl}\mathsf{P}^{kl}{}_{|}{}^{ij}\\
&+\frac{2\left(-2880-37008\,n+17564\,n^2-2432\,n^3+21\,n^4\right)}{315\,(-6+n)\,(-4+n)}\mathsf{P}^{ij}{}_{|k}{}^{k}\mathsf{J}\\
&+\frac{4\left(-33408-425736\,n+228064\,n^2-37810\,n^3+2094\,n^4+21\,n^5\right)}{315\,(-6+n)\,(-4+n)}\mathsf{P}^{i}{}_{k}\mathsf{P}^{jk}\mathsf{J}\\
&+\frac{-9216-1264\,n+1568\,n^2-105\,n^3}{105}\mathsf{P}^{ij}\mathsf{J}^2+\frac{9336-8206\,n+875\,n^2}{315}\mathsf{J}_{|}{}^{i}\mathsf{J}_{|}{}^{j}\\
&+\frac{8\left(-45432+19134\,n-2965\,n^2+138\,n^3\right)}{315\,(-6+n)}\mathsf{P}^{i}{}_{k|}{}^{j}\mathsf{J}_{|}{}^{k}\\
&-\frac{2\left(24768-1404\,n-2584\,n^2+215\,n^3\right)}{315\,(-6+n)}\mathsf{P}^{ij}{}_{|k}\mathsf{J}_{|}{}^{k}+\frac{4\left(6108-1487\,n+70\,n^2\right)}{315}\mathsf{P}^{ij}\mathsf{J}_{|k}{}^{k}\\
&+\frac{2\left(-40320-77904\,n+52144\,n^2-9826\,n^3+651\,n^4\right)}{315\,(-6+n)\,(-4+n)}\mathsf{J}\mathsf{J}_{|}{}^{ij}\\
&+\frac{4\left(934848-481176\,n+106800\,n^2-11644\,n^3+457\,n^4\right)}{315\,(-6+n)\,(-4+n)}\mathsf{P}^{i}{}_{k}\mathsf{J}_{|}{}^{jk}\\
&-\frac{4\left(-15984+8490\,n-1411\,n^2+85\,n^3\right)}{105\,(-6+n)\,(-4+n)}\mathsf{J}_{|}{}^{ij}{}_{k}{}^{k}
\end{aligned}
$$

FIGURE 3. The tensor F^{ij}

on the Riemannian (or pseudo-Riemannian) structure given by the choice of g. If we write $(\xi^g)^w$ for the operator given by multiplication by $(\xi^g)^w \in \mathcal{E}[w]$, then $P^g_{2k} f = (\xi^g)^{k+n/2} P_{2k} (\xi^g)^{k-n/2k} f$. The GJMS operators as discussed, for example, in [5] are the P^g_{2k}. In this form the operators are not invariant but rather covariant (see below), and conformally invariant operators are often discussed entirely in this setting. For many purposes the difference between P_{2k} and P^g_{2k} is rather small. In particular, since the Levi-Civita connection corresponding to g annihilates ξ^g, the formulae above for the P_{2k} also serve as formulae for the operators P^g_{2k}. From Theorem 2.5 and the formulae (8), (9),(11) and (13), there is a universal expression for P^g_{2k} which is polynomial in g, g^{-1}, C, P, ∇, and the ∇ covariant derivatives of C and P, and the coefficients in this universal expression are real rational functions of the dimension n which are regular for all odd n and all $n \geq 2k$.

$$\frac{8\left(-3984+4492\,n-296\,n^2+13\,n^3\right)}{315\,(-6+n)\,(-4+n)}\mathsf{P}_{kl|m}{}^{m}C^{ikjl}$$
$$-\frac{4\left(33984-21624\,n+4540\,n^2-346\,n^3+21\,n^4\right)}{105\,(-6+n)\,(-4+n)}\mathsf{P}_{kl}\mathsf{J}C^{ikjl}$$
$$-\frac{8\left(-19392+14302\,n-2403\,n^2+128\,n^3\right)}{315\,(-6+n)\,(-4+n)}\mathsf{J}_{|kl}C^{ikjl}$$
$$+\frac{4\left(-33408+13680\,n-1082\,n^2+95\,n^3\right)}{315\,(-6+n)\,(-4+n)}\mathsf{P}_{kl|m}C^{ikjm}{}_{|}{}^{l}-\tfrac{80}{63}C_{klmp}C^{ikjm}{}_{|}{}^{lp}$$
$$-\frac{16\left(-52992+13248\,n-472\,n^2-20\,n^3+3\,n^4\right)}{63\,(-6+n)\,(-4+n)}\mathsf{P}_{kl}\mathsf{P}^{j}{}_{m}C^{iklm}$$
$$+\frac{4\left(69168-25940\,n+1472\,n^2+15\,n^3\right)}{315\,(-6+n)\,(-4+n)}\mathsf{P}_{kl|m}C^{iklm}{}_{|}{}^{j}$$
$$+\frac{8\left(17088-16264\,n+652\,n^2+94\,n^3+5\,n^4\right)}{315\,(-6+n)\,(-4+n)}\mathsf{P}_{kl}\mathsf{P}^{k}{}_{m}C^{iljm}+\frac{128\,(31+2\,n)}{315}\mathsf{P}_{kl|}{}^{i}{}_{m}C^{jklm}$$
$$-\tfrac{8}{105}C^{i}{}_{klm|p}C^{jklp}{}_{|}{}^{m}+\frac{16\,(-7+3\,n)}{45}\mathsf{J}C^{i}{}_{klm}C^{jlkm}-\tfrac{68}{63}C^{i}{}_{klm|p}C^{jlkp}{}_{|}{}^{m}$$
$$+\frac{32\,(231+2\,n)}{315}\mathsf{P}^{i}{}_{k|lm}C^{jmkl}+\frac{8\left(8016-1576\,n-506\,n^2+21\,n^3\right)}{315\,(-6+n)\,(-4+n)}\mathsf{P}_{kl}C^{i}{}_{m}{}^{k}{}_{p}C^{jmlp}$$
$$+\tfrac{64}{35}C_{klmp}C^{ikl}{}_{q}C^{jmpq}-\frac{16\left(22200-7798\,n+215\,n^2+8\,n^3\right)}{315\,(-6+n)\,(-4+n)}\mathsf{P}_{kl}C^{i}{}_{m}{}^{k}{}_{p}C^{jplm}$$
$$+\tfrac{176}{105}C_{klmp}C^{ikm}{}_{q}C^{jplq}-\tfrac{32}{45}\mathsf{P}^{ij}C_{klmp}C^{kmlp}$$
$$-\frac{16\left(2692-143\,n+25\,n^2\right)}{315\,(-4+n)}\mathsf{P}_{kl}C^{i}{}_{m}{}^{j}{}_{p}C^{kmlp}-\frac{16\,(-33+20\,n)}{315}\mathsf{P}^{i}{}_{k}C^{j}{}_{lmp}C^{kmlp}$$
$$-\tfrac{32}{45}C_{klmp}C^{i}{}_{q}{}^{jk}C^{lmpq}$$

FIGURE 4. The tensor G^{ij}

Let $\tilde{Q}_{2k,n}$ be the local invariant $P^g_{2k}1$ on a conformal n-manifold. Since P^g_{2k} is formally self-adjoint (FSA) it is clear we can write it in the form $P^g_{2k}=P^{g,1}_{2k}+\tilde{Q}_{2k,n}$, where $P^{g,1}_{2k}$ has the form $\delta S^g_{2k}d$ with δ the formal adjoint of d and S^g_{2k} an order $2k-2$ differential operator.

By setting $w=0$ in the formulae (8) and (9) we see that, as an operator on $\mathcal{E}$, D_A factors through the exterior derivative d. At least this is true given a choice of metric g from the conformal class. Thus in dimension $n_0=2k$ it is clear from Theorem 2.5 that $P^g_{n_0}$ is also a composition with d. Thus the term $\tilde{Q}_{2k,n}$ vanishes in dimension $2k=n_0$. Using this and a careful use of classical invariant theory one can conclude that in fact $\tilde{Q}_{2k,n}=\frac{n-2k}{2}Q^g_{2k,n}$. In the previous section we gave explicit formulae for $Q^g_{4,n}, Q^g_{6,n}$ and $Q^g_{8,n}$.

Clearly $Q^g_{2k,n}$ is also given by a formula rational in n and regular at $n=n_0:=2k$. In dimension n_0, $Q^g_{n_0}:=Q^g_{n_0,n_0}$ is by definition (modulo a sign $(-1)^k$) Branson's Q-curvature, and for compact conformal n_0-manifolds, $\int_M Q^g_{n_0}$ is a global conformal invariant. To see this, observe that the conformal invariance of P_{2k} is equivalent to the covariance law

$$\Omega^{\frac{n+2k}{2}}P^{\hat{g}}_{2k}=P^g_{2k}\Omega^{\frac{n-2k}{2}},$$

where $\hat{g}=\Omega^2 g$ and we regard the powers of Ω as multiplication operators. Applying both sides to the constant function 1 we obtain $\frac{n-2k}{2}\Omega^{\frac{n+2k}{2}}Q^{\hat{g}}_{2k}=$

$$
\begin{aligned}
&\frac{-12\left(8-4\,n+n^2\right)}{(-4+n)^2}\mathsf{P}_{ij|k}{}^{k}\mathsf{P}^{ij}{}_{|l}{}^{l}-24\mathsf{P}_{ij|kl}\mathsf{P}^{ij}{}_{|}{}^{kl}-\frac{48\,(-18+7\,n)}{-6+n}\mathsf{P}_{ij}\mathsf{P}_{kl}\mathsf{P}^{ij}{}_{|}{}^{kl}\\
&-\frac{48\,(-2+n)}{-4+n}\mathsf{P}_{ij|k}\mathsf{P}^{ij}{}_{|}{}^{k}{}_{l}{}^{l}-\frac{12\,(-2+n)}{-6+n}\mathsf{P}_{ij}\mathsf{P}^{ij}{}_{|k}{}^{k}{}_{l}{}^{l}+\frac{192\,n}{-6+n}\mathsf{P}_{ij}\mathsf{P}_{kl}\mathsf{P}^{ik}{}_{|}{}^{jl}\\
&-\frac{48\left(-32+44\,n-12\,n^2+n^3\right)}{(-6+n)\,(-4+n)}\mathsf{P}_{ij}\mathsf{P}_{kl|}{}^{i}\mathsf{P}^{jk}{}_{|}{}^{l}+\frac{384\,(-16+3\,n)}{(-6+n)\,(-4+n)}\mathsf{P}_{ij}\mathsf{P}^{i}{}_{k|l}\mathsf{P}^{jk}{}_{|}{}^{l}\\
&+\frac{384\left(4-6\,n+n^2\right)}{(-6+n)\,(-4+n)^2}\mathsf{P}_{ij}\mathsf{P}^{i}{}_{k}\mathsf{P}^{jk}{}_{|l}{}^{l}+\frac{192\left(144-32\,n-4\,n^2+n^3\right)}{(-6+n)\,(-4+n)^2}\mathsf{P}_{ij}\mathsf{P}_{kl}\mathsf{P}^{ik}\mathsf{P}^{jl}\\
&+\frac{48\left(-32+20\,n-8\,n^2+n^3\right)}{(-6+n)\,(-4+n)}\mathsf{P}_{ij}\mathsf{P}^{i}{}_{k|l}\mathsf{P}^{jl}{}_{|}{}^{k}\\
&+\frac{6\,(-8+n)\left(256-32\,n-18\,n^2+3\,n^3\right)}{(-6+n\ )\,(-4+n)^2}\mathsf{P}_{ij}\mathsf{P}_{kl}\mathsf{P}^{ij}\mathsf{P}^{kl}\\
&+\frac{24\left(-176+80\,n-14\,n^2+n^3\right)}{(-6+n)\,(-4+n)}\mathsf{P}_{ij}\mathsf{P}_{kl|}{}^{i}\mathsf{P}^{kl}{}_{|}{}^{j}+\frac{4\left(48+80\,n-46\,n^2+5\,n^3\right)}{(-6+n)\,(-4+n)}\mathsf{P}_{ij|k}\mathsf{P}^{ij}{}_{|}{}^{k}\mathsf{J}\\
&+\frac{4\left(12-20\,n+5\,n^2\right)}{-6+n}\mathsf{P}_{ij}\mathsf{P}^{ij}{}_{|k}{}^{k}\mathsf{J}-\frac{384\,n}{-6+n}\mathsf{P}_{ij}\mathsf{P}^{i}{}_{k}\mathsf{P}^{jk}\mathsf{J}\\
&-\frac{96-80\,n-30\,n^2+5\,n^3}{-6+n}\mathsf{P}_{ij}\mathsf{P}^{ij}\mathsf{J}^2+\frac{(-4+n)\,n\,(4+n)}{8}\mathsf{J}^4\\
&+\left(2+13\,n-2\,n^2\right)\mathsf{J}\mathsf{J}_{|i}\mathsf{J}_{|}{}^{i}-\frac{(-2+n)\left(200-38\,n+3\,n^2\right)}{-4+n}\mathsf{P}_{ij}\mathsf{J}_{|}{}^{i}\mathsf{J}_{|}{}^{j}\\
&+\frac{24\,(-2+n)\left(88-20\,n+n^2\right)}{(-6+n)\,(-4+n)}\mathsf{P}_{ij}\mathsf{P}^{i}{}_{k|}{}^{j}\mathsf{J}_{|}{}^{k}+\frac{16\left(-48+70\,n-17\,n^2+n^3\right)}{(-6+n)\,(-4+n)}\mathsf{P}_{ij}\mathsf{P}^{ij}{}_{|k}\mathsf{J}_{|}{}^{k}\\
&+\frac{16+4\,n-3\,n^2}{2}\mathsf{J}^2\mathsf{J}_{|i}{}^{i}+\frac{-8+3\,n}{2}\mathsf{J}_{|i}{}^{i}\mathsf{J}_{|j}{}^{j}+\frac{2\left(32-40\,n+5\,n^2\right)}{-4+n}\mathsf{P}_{ij}\mathsf{P}^{ij}\mathsf{J}_{|k}{}^{k}\\
&-\frac{16\left(4-8\,n+n^2\right)}{(-4+n)^2}\mathsf{P}_{ij|k}{}^{k}\mathsf{J}_{|}{}^{ij}+\frac{4\,n\,(-46+5\,n)}{-6+n}\mathsf{P}_{ij}\mathsf{J}\mathsf{J}_{|}{}^{ij}\\
&+\frac{2\left(-336+136\,n-20\,n^2+n^3\right)}{(-4+n)^2}\mathsf{J}_{|ij}\mathsf{J}_{|}{}^{ij}+\frac{24\left(1088-608\,n+144\,n^2-18\,n^3+n^4\right)}{(-6+n)\,(-4+n)^2}\mathsf{P}_{ij}\mathsf{P}^{i}{}_{k}\mathsf{J}_{|}{}^{jk}\\
&+(-38+5\,n)\,\mathsf{J}_{|i}\mathsf{J}_{|}{}^{i}{}_{j}{}^{j}-\frac{32\,(-7+n)}{-4+n}\mathsf{P}_{ij|k}\mathsf{J}_{|}{}^{ijk}+2\,(-1+n)\,\mathsf{J}\mathsf{J}_{|i}{}^{i}{}_{j}{}^{j}\\
&-\frac{4\,(-54+7\,n)}{-6+n}\mathsf{P}_{ij}\mathsf{J}_{|}{}^{ij}{}_{k}{}^{k}-\mathsf{J}_{|i}{}^{i}{}_{j}{}^{j}{}_{k}{}^{k}-\frac{192\left(4-6\,n+n^2\right)}{(-6+n)\,(-4+n)^2}\mathsf{P}_{ij}\mathsf{P}_{kl|m}{}^{m}C^{ikjl}\\
&+\frac{96\,n}{-6+n}\mathsf{P}_{ij}\mathsf{P}_{kl}\mathsf{J}C^{ikjl}-\frac{24\,(-8+n)\left(-16-2\,n+n^2\right)}{(-6+n)\,(-4+n)^2}\mathsf{P}_{ij}\mathsf{J}_{|kl}C^{ikjl}\\
&-\frac{384\,(-5+n)}{(-6+n)\,(-4+n)}\mathsf{P}_{ij}\mathsf{P}_{kl|m}C^{ikjl}{}_{|}{}^{m}-\frac{192}{-4+n}\mathsf{P}_{ij|k}\mathsf{P}_{lm|}{}^{k}C^{iljm}\\
&+\frac{192\left(112-16\,n-6\,n^2+n^3\right)}{(-6+n)\,(-4+n)^2}\mathsf{P}_{ij}\mathsf{P}_{kl}\mathsf{P}^{i}{}_{m}C^{jklm}-\frac{96\,n}{-4+n}\mathsf{P}_{ij|k}\mathsf{P}^{i}{}_{l|m}C^{jlkm}\\
&+\frac{192}{-6+n}\mathsf{P}_{ij}\mathsf{P}_{kl}C^{i}{}_{m}{}^{k}{}_{p}C^{jmlp}-\frac{384}{-6+n}\mathsf{P}_{ij}\mathsf{P}_{kl}C^{i}{}_{m}{}^{k}{}_{p}C^{jplm}\\
&+\frac{192\,(-6+n)}{(-4+n)^2}\mathsf{P}_{ij}\mathsf{P}_{kl}C^{i}{}_{m}{}^{j}{}_{p}C^{kmlp}
\end{aligned}
$$

FIGURE 5. The invariant $Q^g_{8,n}$

$\frac{n-2k}{2}Q^g_{2k}\Omega^{\frac{n-2k}{2}}+P^{g,1}_{2k}\Omega^{\frac{n-2k}{2}}$. Expanding this out yields a universal transformation formula for Q^g_{2k}. Since $P^{g,1}_{2k}$ is a composition with d it is clear that we can divide this formula by $\frac{n-2k}{2}$. Then in dimension $n_0=2k$ we obtain

$$\Omega^{n_0}Q^{\hat{g}}_{2k}=Q^g_{2k}+\delta S^g_{2k}d\Upsilon$$

where $\Upsilon = \log\Omega$. Then, if we denote by $\boldsymbol{\epsilon}_g$ the volume form associated with a metric g, we have $\boldsymbol{\epsilon}_{\hat{g}} = \Omega^{n_0}\boldsymbol{\epsilon}_g$, and the conformal invariance of $\int_M Q^g_{2k}$ is clear. Recall that a choice of metric g determines a canonical section ξ^g of $\mathcal{E}[1]$ by $(\xi^g)^{-2}\boldsymbol{g} = g$. It is convenient to redefine Q^g_{2k} to be $(\xi^g)^{-n_0}$ times Q^g_{2k} as above. Then Q^g_{2k} is valued in $\mathcal{E}[-n]$ and the transformation law simplifies to

$$Q^{\hat{g}}_{2k} = Q^g_{2k} + \delta S^g_{2k} d\Upsilon, \tag{24}$$

where δ and S^g_{2k} are now also density valued. Note that we can also write this as $Q^{\hat{g}}_{2k} = Q^g_{2k} + P_{2k}\Upsilon$ as $P^{g,1}_{2k}$ agrees with P^g_{2k} in dimension $2k$.

The discussion above for $Q^g_{n_0}$ and its properties are a minor adaption of the arguments presented in Branson's [5]. It is clear that given explicit formulae for the P_{2k} as in the previous section we can extract a formula for the Q-curvature as follows: Take the order 0 part of the formula, divide by $(n-2k)/2$ and then set $n = 2k$. For example Q^g_8 is obtained by setting $n = 8$ in the formula given in figure 5. In [5] it is also shown that the global invariant is not trivial. In fact, it is established there that, on conformally flat structures, $Q^g_{n_0}$ is given by a multiple of the Pfaffian plus a divergence, and so $\int_M Q^g_{n_0}$ is a multiple of the Euler characteristic $\chi(M)$.

One of the keys to the importance of Q_{2k} is the remarkable transformation formula (24). We will describe a new definition and construction for Q_{2k} and proof of this transformation formula. This leads to a direct formula for Q_{2k}. Here to prove the transformation law we will use a dimensional continuation argument. (This plays a minor role and can in fact can be replaced by a direct proof [8]). The construction is then adapted to proliferate other curvature quantities with transformation formulae of the general form (24), and in these cases dimensional continuation is not used at all. See Proposition 2.8. (Since the original writing of this Fefferman and Graham [27] have given another alternative construction and generalisation of the Q-curvature which involves the Poincaré metric.)

We work on a conformal manifold of dimension $n_0 = 2k$. For a choice of metric g from the conformal structure let I^g_A be the section of $\mathcal{E}_A[-1]$ defined by $I^g_A := (n-2)Y_A - \mathsf{J}X_A$, where, recall, $Y_A \in \mathcal{E}_A[-1]$ gives the splitting of the tractor bundle corresponding to the metric g (as in section 2). We can write this as a triple $[I^g_A]_g = ((n-2),\ 0,\ -\mathsf{J})$. According to this definition, if $\hat{g} = \Omega^2 g$ then we have $I^{\hat{g}}_A := (n-2)\widehat{Y}_A - \widehat{\mathsf{J}}X_A$ or $[I^{\hat{g}}_A]_{\hat{g}} = ((n-2),\ 0,\ -\widehat{\mathsf{J}})$. In terms of the splitting determined by g, $I^{\hat{g}}_A$ is given by $[I^{\hat{g}}_A]_g = ((n-2), -(n-2)\Upsilon_a,\ -\widehat{\mathsf{J}} - (n/2-1)\Upsilon^b\Upsilon_b)$. By (2) and $\Upsilon_a = \nabla_a\Upsilon$ this becomes $[I^{\hat{g}}_A]_g = ((n-2), -(n-2)\nabla_a\Upsilon,\ -\mathsf{J} + \Delta\Upsilon)$, and so

$$I^{\hat{g}}_A = I^g_A - D_A\Upsilon. \tag{25}$$

This observation is due to Eastwood who also pointed out [20] that on conformally flat structures this yields Branson's curvature as follows. For each metric define Q^g_B by $Q^g_B := \Box_{2k-2} I^g_B$. Then, by (25), $Q^{\hat{g}}_B = Q^g_B - \Box_{2k-2}D_B\Upsilon$. Now since the structure is conformally flat, $\Box_{2k-2}D_A\Upsilon = -X_A P_{2k}\Upsilon$ (see Theorem 2.5 or e.g. [28]). Thus we have

$$Q^{\hat{g}}_B = Q^g_B + X_B P_{2k}\Upsilon.$$

It follows that $X^B Q_B^{\hat{g}} = X^B Q_B^g$ is a conformal invariant of weight $n_0 - 2$. On a conformally flat structure there are no conformal invariants of the structure and so this vanishes. Since this vanishes, $Z^{Bc} Q_B^g$ is also conformally invariant and so must vanish. This shows that for any conformally flat metric, $Q_B^g = X_B Q^g$ for some Riemannian invariant Q^g and also that $Q^{\hat{g}} = Q^g + P_{2k}\Upsilon$. That is, it transforms according to (24). On conformally flat structures one can always locally choose a metric that is flat whence all Riemannian invariants vanish. Using this we deduce that Q^g is Branson's curvature, that is $Q^g = Q_{2k}^g$.

Via the theorem we can generalise Eastwood's cunning construction to the curved case. Note (25) holds on any conformal manifold. Let us define the operator $F_C{}^B : \mathcal{E}_B[k-1-n/2] \to \mathcal{E}_C[1-k-n/2]$ by $F_C{}^B := (k-2)^{-1}(n-2k+2)^{-1} D^K E_{CK}{}^{AB} D_A$ where E is the operator defined in Theorem 2.5. Now on a dimension $n_0 = 2k$ manifold we simply define $Q_C^g := F_C{}^B I_B^g$. From (25) and the theorem we have immediately

$$Q_C^{\hat{g}} = Q_C^g + X_C P_{2k}\Upsilon. \tag{26}$$

It remains to verify that for any metric g in the conformal class, Q_C^g is indeed $X_C Q_{2k}^g$. In any dimension n and given any metric g, if $f \in \mathcal{E}[w]$ let us write $D_A f = D_A^1 f + w D_A^0 f$, where $D_A^1 f := (n+2w-2) Z_A{}^a \nabla_a f - X_A \Delta f$ and $w D_A^0 f$ is the remaining order zero part. That is, $D_A^0 f = (n+2w-2) Y_A f - X_A \mathsf{J} f$. Let $w = k - n/2$, and assume n is odd or $n \leq 2k$. Then $-X_A P_{2k} f = F_A{}^B D_B f = F_A{}^B D_B^1 f + w F_A{}^B D_B^0 f$. Let ξ^g be the section of $\mathcal{E}[1]$ corresponding to g. Recall that ξ^g is parallel for the Levi-Civita connection of g. Since $(\xi^g)^w$ is a section of $\mathcal{E}[w]$, we have

$$X_A w Q_{2k,n}^g (\xi^g)^w = w F_A{}^B D_B^0 (\xi^g)^w.$$

Now $X^A F_A{}^B D_B^0 (\xi^g)^w$ can be expressed as a universal expression which is polynomial in $\boldsymbol{g}$, $\boldsymbol{g}^{-1}$, C, P, ∇, and the ∇ covariant derivatives of C and P. The coefficients in this universal expression are real rational functions of the dimension n which are regular for all odd n and all $n \geq 2k$. Furthermore, from the left-hand-side of the display, this expression vanishes in even dimensions $n < 2k$ and for all odd dimensions. Thus it must vanish in dimension $n_0 = 2k$. Similarly we can conclude $Z^{Aa} F_A{}^B D_B^0 (\xi^g)^w$ vanishes if n is odd or $n \leq 2k$. Thus in dimension n_0, $w = 0$, $D_B^0 (\xi^g)^w = D_B^0 1 = I_B^g$, and as $Q_{2k}^g := Q_{n_0,n_0}^g$, we have

$$Q_A^g = F_A{}^B I_B^g = X_A Q_{2k}^g.$$

Thus we have the following.

Proposition 2.7. *$Y^A F_A{}^B I_B^g$ is a formula for Branson's Q-curvature Q_{2k}^g.*

Note that if we take this as a definition for Q then the transformation property (24) arises from (26) which in turn is an immediate consequence of (25). The formula itself is direct and requires no dimensional continuation. The only subtlety in the construction was in establishing that Q_A^g has the form $X_A Q_{2k}^g$. We employed a dimensional continuation argument to establish this above, but it turns out that there is an elementary direct proof using the ambient construction [8]. Finally note that if we write $I^A_g = h^{AB} I^g_B$, then $I^A_g F_A{}^B I^g_B = (n-2)Q^g_{2k}$.

It is straightforward to convert this tractor formula for Q^g_{2k} into a formula in terms of ∇, C, P, the metric, and its inverse. We simply expand $Y^A F_A{}^B I^g_B$ using the formula for $F_A{}^B$ as a partial contraction polynomial in D_A, W_{ABCD}, X_A, h_{AB}, and its inverse h^{AB}, as obtained from Proposition 4.5, and apply (8), (9), (11), and (13) along the same lines as the calculations in Section 2.2. In fact, as a means of checking against formulae or calculational errors it is prudent to calculate the entire tractor valued expression $F_A{}^B I^g_B$ and verify from this that only the bottom slot is not zero. That is that $X^A F_A{}^B I^g_B = 0 = Z^{Aa} F_A{}^B I^g_B$. Doing this for Q^g_4 we obtain the known formula

$$Q_4 = -2\mathsf{P}_{ij}\mathsf{P}^{ij} + 2\mathsf{J}^2 - \mathsf{J}_{|j}{}^j.$$

In terms of the Ricci curvature Rc and the scalar curvature Sc, this becomes $Q_4 = -\frac{1}{2}\mathrm{Rc}_{ij}\mathrm{Rc}^{ij} + \frac{1}{6}\mathrm{Sc}^2 - \frac{1}{6}\mathrm{Sc}_{|i}{}^i$. For Q_6 the formulae are more severely tested by calculating $E_{AB}{}^{CE} D_C I^g_E$. For this case $E_{AB}{}^{CE} D_C I^g_E = \Box D_A I^g_B + \frac{2}{n-4} W_A{}^C{}_B{}^E D_C I^g_E$ and the calculation verifies all components vanish except for the coefficient of $X_A X_B$, the negative of which is

$$Q^g_6 = -(8\mathsf{P}_{ij|k}\mathsf{P}^{ij}{}_|{}^k + 16\mathsf{P}_{ij}\mathsf{P}^{ij}{}_{|k}{}^k - 32\mathsf{P}_{ij}\mathsf{P}^i{}_k\mathsf{P}^{jk} - 16\mathsf{P}_{ij}\mathsf{P}^{ij}\mathsf{J} +$$

$$8\mathsf{J}^3 - 8\mathsf{J}_{|k}{}^k\mathsf{J} + \mathsf{J}_{|j}{}^j{}_k{}^k + 16\mathsf{P}_{ij}\mathsf{P}_{kl}\mathsf{C}^{ikjl}).$$

Note that these examples agree with setting $n=4$ in (21) and $n=6$ in (23).

Using I^g_A it is easy to construct examples of other functionals of the metric that have transformation laws of the same form as (24). We state this as a proposition.

Proposition 2.8. *In dimension $n_0 = 2k$, for each natural conformally invariant operator $G_A{}^B : \mathcal{E}_B[-1] \to \mathcal{E}_A[1-n_0]$ there is a Riemannian invariant $D^A G_A{}^B I^g_B$ with a conformal transformation of the form*

$$D^A G_A{}^B I^{\hat g}_B = D^A G_A{}^B I^g_B + \delta T^g_{2k} d\Upsilon,$$

where T^g_{2k} a Riemannian invariant differential operator such that the composition $\delta T^g_{2k} d$ is a conformally invariant operator between functions and densities of weight $-n$.

If $G_A{}^B$ is formally self-adjoint, then $\delta T^g_{2k} d$ is formally self-adjoint.

Proof. It is clear from (25) that $D^A G_A{}^B I^{\hat g}_B = D^A G_A{}^B I^g_B - D^A G_A{}^B D_B \Upsilon$. Note that $D^A G_A{}^B D_B$ is a composition of conformally invariant operators. Since Υ is a function (i.e. is a density of weight 0), $D_B\Upsilon$ factors through $d\Upsilon$. From Proposition 2.2 it follows that the formal adjoint of $D^A G_A{}^B D_B$ also factors through the exterior derivative d. Thus the conformally invariant operator $D^A G_A{}^B D_B$ has the form $-\delta T^g_{2k} d$.

Since $\delta T^g_{2k} d\Upsilon = -D^A G_A{}^B D_B \Upsilon$, the last part of the proposition is immediate from Proposition 2.2. □

An example in dimension 6 is to take $G_A{}^B$ to be the order zero operator $|C|^2 \delta_A{}^B$, where $|C|^2 = C^{abcd}C_{abcd}$. Then $D^A G_A{}^B I^g_B = -4\Delta|C|^2$, and $D^A G_A{}^B I^{\hat g}_B = -4\Delta|C|^2 + 16\nabla^a|C|^2\nabla_a\Upsilon$. We can easily make many other examples via the tractor objects already seen above. Other examples in dimension 6 are to take $G_A{}^B$ to be $W_{ACDE}W^{BCDE}$ or $D^E W_A{}^B{}_E{}^F D_F$. In dimension 8 we could take $G_A{}^B$ to be $\delta_A^B D^P W_{PCDE} W^{ECDQ} D_Q$ and so on. Note all these examples have G formally self-adjoint.

It is a trivial exercise to verify that $D^A G_A{}^B I_B^g$ is always a divergence, and so none of the invariants from the proposition yield non-trivial global invariants. Thus we could adjust the definition of Q_{2k}^g by adding such functions without affecting it as a representative of n_0^{th} de Rham cohomology and also without affecting the form of the transformation law (24). Such changes would of course alter what we meant by S_{2k}^g, but in any case $\delta S_{2k}^g d$ would remain an invariant operator on functions. Such potential modifications are important from several points of view. The transformation law (24) is satisfied in dimension 2 by the scalar curvature, or more precisely by $-\mathrm{Sc}/2$. In this context it is usually called the Gauss curvature prescription equation. As mentioned earlier, Q_{2k} lends itself to higher dimensional analogues of this curvature prescription problem. For the same reason, in any case where $D^A G_A{}^B I_B^g$ is non-trivial, $Q_{2k}^g + D^A G_A{}^B I_B^g$ yields a distinct, and apparently equally natural, curvature prescription problem. Of course then $Q_{2k}^g + D^A G_A{}^B I_B^g$ does not arise by Branson's construction from the GJMS operator P_{2k}. But it is easily verified that it does arise via Branson's argument applied to the conformally invariant operator

$$P'_{2k} := P_{2k} - D^A G_A{}^B D_B : \mathcal{E}[k - n/2] \to \mathcal{E}[-k - n/2],$$

and according to either either construction the conformal transformation formula in dimension $n_0 = 2k$ is

$$Q_{2k}^{\hat{g}} + D^A G_A{}^B I_B^{\hat{g}} = Q_{2k}^g + D^A G_A{}^B I_B^g + P'_{2k}\Upsilon.$$

It is possible, for example, that there are settings where such natural modifications to the GJMS operators will yield operators which are positive but the relevant GJMS operator fails to be positive.

3. The ambient metric construction

The ambient metric construction of Fefferman-Graham associates to a conformal manifold M of signature (p, q) a pseudo-Riemannian so-called *ambient manifold* $\tilde{M}$ of signature $(p+1, q+1)$. The ambient manifold $\tilde{M}$ is $\mathcal{Q} \times I$, where $I = (-1, 1)$. Henceforth we identify $\mathcal{Q}$ with its natural inclusion $\iota : \mathcal{Q} \to \tilde{M}$ given by $\mathcal{Q} \ni q \mapsto (q, 0) \in \tilde{M}$. Observe that $\mathcal{Q}$ carries a tautological symmetric 2-tensor g_0 given by $g_0 = \pi^* g$ at the point $(p, g) \in \mathcal{Q}$. This satisfies $\delta_s^* g_0 = s^2 g_0$, where δ_s is the natural $\mathbb{R}_+$-action on $\mathcal{Q}$ given by $\delta_s(p, g) = (p, s^2 g)$. We will also write δ_s for natural extension of this action to $\tilde{M}$ and denote by $\boldsymbol{X}$ the infinitesimal generator of this, i.e., for a smooth function f on $\tilde{M}$, $\boldsymbol{X} f(q) = \frac{d}{ds} f(\delta_s q)|_{s=1}$. The metric on the ambient manifold $\tilde{M}$ will be denoted $\boldsymbol{h}$ and is required to be a homogeneous extension of g_0 in the sense that

$$\iota^* \boldsymbol{h} = g_0 \quad \delta_s^* \boldsymbol{h} = s^2 \boldsymbol{h} \ \text{ for } \ s > 0. \tag{27}$$

The idea of the Fefferman-Graham construction is to attempt to find a formal power series solution along $\mathcal{Q}$ for the Cauchy problem of an ambient metric $\boldsymbol{h}$ satisfying (27) and the condition that it be Ricci-flat, i.e. $\mathrm{Ric}(\boldsymbol{h}) = 0$. It turns out that only a weaker curvature condition can be satisfied in the even dimensional case. The main results we need are contained in Theorem 2.1 of [26]: If n is odd then, up to a $\mathbb{R}_+$-equivariant diffeomorphism fixing $\mathcal{Q}$, there is a unique power series solution for $\boldsymbol{h}$ satisfying (27) and $\mathrm{Ric}(\boldsymbol{h}) = 0$. If n is even then, up to a $\mathbb{R}_+$-equivariant diffeomorphism fixing $\mathcal{Q}$ and the addition of terms vanishing to order $n/2$, there is a unique power series solution for $\boldsymbol{h}$ satisfying (27) and such that, along $\mathcal{Q}$, $\mathrm{Ric}(\boldsymbol{h})$ vanishes to order $n/2-2$ and that the tangential components of $\mathrm{Ric}(\boldsymbol{h})$ vanish to order $n/2-1$. We should point out that we only use the existence part of the Fefferman-Graham construction. The uniqueness of the GJMS operators, the covariant derivatives of the ambient curvature and so forth are a consequence of the existence of tractor formulae for these objects.

By choosing a metric g from the conformal class on M we determine a fibre variable on $\mathcal{Q}$ by writing a general point of $\mathcal{Q}$ in the form $(p, t^2 g(p))$, where $p \in M$ and $t > 0$. Local coordinates x^i on M then correspond to coordinates (t, x^i) on $\mathcal{Q}$. These extend [26, 32] to coordinates (t, x^i, ρ) on $\tilde{M}$, where ρ is a defining function for $\mathcal{Q}$ and such that the curves $\rho \mapsto (t, x^i, \rho)$ are geodesics for $\boldsymbol{h}$. In these coordinates the ambient metric takes the form

$$\boldsymbol{h} = t^2 g_{ij}(x, \rho) dx^i dx^j + 2\rho dt dt + 2t dt d\rho. \tag{28}$$

This form is forced to all orders in odd dimensions. In even dimensions it is forced up to the addition of terms vanishing to order $n/2$. In order, in even dimensions, to recover the order n GJMS operators via the procedure of [32] we need also to assume that the metric has this form up to the addition of terms vanishing to order $n/2+1$. Although we only need this form to that order, to simplify our discussion we will assume that the form (28) holds to all orders in even dimensions too. This simply involves some choice of extension for the Taylor series of the components g_{ij}, and then with this assumption the identities discussed in the remainder of this subsection hold to all orders in all dimensions. We write $\boldsymbol{\nabla}$ for the ambient Levi-Civita connection determined by $\boldsymbol{h}$.

In terms of the coordinates one has $\boldsymbol{X} = t\frac{\partial}{\partial t}$, and if we let $Q := \boldsymbol{h}(X, X)$, then $Q = 2\rho t^2$ and is a defining function for $\mathcal{Q}$. In terms of this we have that, when n is even, the ambient construction determines $\boldsymbol{h}$ up to $\mathrm{O}(Q^{n/2})$. Let us use upper case abstract indices $A, B, \cdots$ for tensors on $\tilde{M}$. For example, if v^B is a vector field on $\tilde{M}$, then the ambient Riemann tensor will be denoted $\boldsymbol{R}_{AB}{}^C{}_D$ and defined by $[\boldsymbol{\nabla}_A, \boldsymbol{\nabla}_B] v^C = \boldsymbol{R}_{AB}{}^C{}_D v^D$. Indices will be raised and lowered using the ambient metric $\boldsymbol{h}_{AB}$ and its inverse $\boldsymbol{h}^{AB}$ in the usual way. We will soon see that this index convention is consistent with our use of these indices for tractor bundles.

The homogeneity property of $\boldsymbol{h}$ in (27) means that $\boldsymbol{X}$ is a conformal Killing vector, and in particular $\mathcal{L}_{\boldsymbol{X}} \boldsymbol{h} = 2\boldsymbol{h}$, where $\mathcal{L}$ is the Lie derivative. It follows that $\boldsymbol{\nabla}_{(A} \boldsymbol{X}_{B)} = \boldsymbol{h}_{AB}$. On the other hand, from the explicit coordinate form of the metric, we have that $\boldsymbol{\nabla}_B Q = 2\boldsymbol{X}_B$, and so $\boldsymbol{\nabla}_A \boldsymbol{X}_B$ is symmetric. Thus

$$\boldsymbol{\nabla}_A \boldsymbol{X}_B = \boldsymbol{h}_{AB}$$

which, in turn, implies

$$\boldsymbol{X}^A \boldsymbol{R}_{ABCD} = 0. \tag{29}$$

In terms of our notation the theorem of [26] (mentioned above) means that in even dimensions the ambient Ricci curvature $\boldsymbol{R}_{BF}$ can be written in the form

$$\boldsymbol{R}_{BF}=Q^{n/2-2}\boldsymbol{X}_{(B}\boldsymbol{K}_{F)}+Q^{n/2-1}\boldsymbol{L}'_{BF}$$

for appropriately homogeneous ambient tensors $\boldsymbol{K}_F$ and $\boldsymbol{L}'_{BF}$. In fact the choice to extend the metric $\boldsymbol{h}$ so that it has the form (28) restricts $\boldsymbol{K}_A$ significantly. From (29) we have that $\boldsymbol{X}^A\boldsymbol{R}_{AC}$ vanishes to all orders. With the contracted Bianchi identity $2\boldsymbol{\nabla}^A\boldsymbol{R}_{AC}=\boldsymbol{\nabla}_C\boldsymbol{S}$ (where $\boldsymbol{S}$ denotes the ambient Ricci scalar curvature) this implies that $\boldsymbol{K}_A=\boldsymbol{X}_A\boldsymbol{K}$ for an ambient homogeneous function $\boldsymbol{K}$. Although it is not strictly necessary, it will simplify our subsequent calculations to restrict the ambient metric a little more. An elementary calculation verifies that we can adjust the components g_{ij} in (28) so that $\boldsymbol{K}=\mathrm{O}(Q)$. Thus finally we have that that in even dimensions the metric has the form (28) and

$$\boldsymbol{R}_{BF}=Q^{n/2-1}\boldsymbol{L}_{BF}. \tag{30}$$

for an appropriately homogeneous ambient tensor $\boldsymbol{L}_{BF}$. (The authors are appreciative of discussions with A. Čap and C.R. Graham in relation to this point.)

3.1. **Recovering tractor calculus.** Recall that a section of $\mathcal{E}[w]$ corresponds to a real-valued function f on $\mathcal{Q}$ with the homogeneity property $f(p,s^2g)=s^wf(p,g)$, where $p\in M$ and g is a metric from the conformal class $[g]$. Let $\tilde{\mathcal{E}}_{\mathcal{Q}}(w)$ denote the space of smooth functions on $\mathcal{Q}$ which are homogeneous of degree w in this way. We write $\tilde{\mathcal{E}}(w)$ for the smooth functions on $\tilde{M}$ which are similarly homogeneous, i.e. $\tilde{f}\in\tilde{\mathcal{E}}(w)$ means $\boldsymbol{X}^A\boldsymbol{\nabla}_Af=w\tilde{f}$. The construction of the GJMS operators in [32] exploits this relationship between $\mathcal{E}[w]$ and $\tilde{\mathcal{E}}(w)$. We will use here the analogous idea at the level of tensors on $\tilde{M}$. This is developed more fully in [12], and here we just summarise the basic ideas needed presently.

Writing δ'_s for the derivative of the action δ_s, let us define an equivalence relation on the ambient tangent bundle by $U_{q_1}\sim V_{q_2}$ if and only if there is $s\in\mathbb{R}_+$ such that $V_{q_2}=s^{-1}\delta'_sU_{q_1}$. Corresponding to this we have the equivalence relation on $\tilde{M}$ by $q_1\sim q_2$ if and only if $q_2=\delta_sq_1$. It is straightforward to verify that the space $T\tilde{M}/\sim$ is a rank $n+2$ vector bundle over $\tilde{M}/\sim$. Sections of this bundle correspond to smooth sections $V:\tilde{M}\to T\tilde{M}$ with the homogeneity property $V(\delta_sp)=s^{-1}\delta'_sV(p)$, or they could be alternatively characterised by their commutator with the Euler field $\boldsymbol{X}$, $[\boldsymbol{X},V]=-V$. We will let $\tilde{\mathcal{E}}^A(0)$ ($\tilde{\mathcal{E}}^A_{\mathcal{Q}}(0)$) denote the space of sections of $T\tilde{M}$ ($T\tilde{M}|_{\mathcal{Q}}$) which are homogeneous in this way, and we will write $\tilde{\mathcal{E}}^{AB}(w)$ ($\tilde{\mathcal{E}}^{AB}_{\mathcal{Q}}(w)$) to mean $\tilde{\mathcal{E}}^A\otimes\tilde{\mathcal{E}}^B\otimes\tilde{\mathcal{E}}(w)$ ($\tilde{\mathcal{E}}^A_{\mathcal{Q}}\otimes\tilde{\mathcal{E}}^B_{\mathcal{Q}}\otimes\tilde{\mathcal{E}}_{\mathcal{Q}}(w)$ respectively) and so forth. (The reason for the weight convention will soon be obvious.) We will write $\tilde{\mathcal{E}}^\Phi(w)$ to mean an arbitrary tensor power of $\tilde{\mathcal{E}}^A(0)$ (or symmetrization thereof and so forth) tensored with $\tilde{\mathcal{E}}(w)$ and we will say sections of $\tilde{\mathcal{E}}^\Phi(w)$ are tensors homogeneous of *weight* w. (We use the term 'weight' here to distinguish from the homogeneity 'degree' [12] as exposed by the Lie derivative along the field $\boldsymbol{X}$.) Of course this construction is formal at the same order as the construction of $\tilde{M}$, but upon restriction to $\mathcal{Q}$, $T\tilde{M}/\sim$ yields a genuine rank $n+2$ vector bundle over $M=\mathcal{Q}/\sim$ that will be denoted by $\mathcal{T}$ or $\mathcal{T}^A$.

It is immediate from the homogeneity property of $\boldsymbol{h}$ that if U and V are sections of $\tilde{\mathcal{E}}^A(0)$, then the function $\boldsymbol{h}_{AB}U^AV^B$ is in $\tilde{\mathcal{E}}(0)$. Restricting to $\mathcal{Q}$ we see that $\boldsymbol{h}_{AB}U^AV^B$ descends to a function on M. From the bilinearity and signature of $\boldsymbol{h}$ it follows that $\boldsymbol{h}$ descends to give a signature $(p+1,q+1)$ metric $h^{\mathcal{T}}$ on the bundle $\mathcal{T}$. We can use this to raise and lower indices in the usual way.

Observe that $\boldsymbol{X}^A\in\tilde{\mathcal{E}}^A(1)$. Thus if $\varphi\in\tilde{\mathcal{E}}(-1)$, then $\varphi\boldsymbol{X}^A\in\tilde{\mathcal{E}}^A(0)$. The same is true upon restriction to $\mathcal{Q}$, so we have a canonical inclusion $\mathcal{E}[-1]\hookrightarrow\mathcal{T}$ with image denoted by $\mathcal{T}^1$. We write $X^A_{\mathcal{T}}$ for the natural section of $\mathcal{T}^A[1]:=\mathcal{T}^A\otimes\mathcal{E}[1]$ giving this map, and so on $\mathcal{Q}$, $\boldsymbol{X}^A$ is the homogeneous section representing $X^A_{\mathcal{T}}$. Clearly then $V^A\mapsto h^{\mathcal{T}}_{AB}X^A_{\mathcal{T}}V^B$ determines a canonical homomorphism $\mathcal{T}\to\mathcal{E}[1]$, and we let $\mathcal{T}^0$ denote the kernel. Recall that Q was defined to be $\boldsymbol{h}_{AB}\boldsymbol{X}^A\boldsymbol{X}^B$ and that this was a defining function for $\mathcal{Q}$. Thus $X^A_{\mathcal{T}}$ is a null vector for the metric $h^{\mathcal{T}}$, and it follows immediately that $\mathcal{T}^1\subset\mathcal{T}^0$. There is a simple geometric interpretation of $\mathcal{T}^0$ and $\mathcal{T}^1$. Observe $\mathcal{T}^0[1]$ corresponds to sections of $\tilde{\mathcal{E}}^A_{\mathcal{Q}}(1)$ that are annihilated by contraction with $\boldsymbol{X}_A$. On $\mathcal{Q}$ we have that $\boldsymbol{X}_A=\frac{1}{2}\boldsymbol{\nabla}_AQ$, so along $\mathcal{Q}$ the sections of $\tilde{\mathcal{E}}^A_{\mathcal{Q}}(1)$ corresponding to $\mathcal{T}^0[1]$ are precisely those taking values in $T\mathcal{Q}\subset T\tilde{M}|_{\mathcal{Q}}$ and which are invariant under the action of δ'_s. Then, since $\boldsymbol{X}$ is the Euler vector field, it follows that $\mathcal{T}^1[1]$ corresponds to functions in $\tilde{\mathcal{E}}^A_{\mathcal{Q}}(1)$ taking values in the vertical subbundle of $T\mathcal{Q}$. Of course the map $\mathcal{Q}\to M$ is a submersion, and so $\mathcal{T}^0[1]/\mathcal{T}^1[1]$ is naturally isomorphic to $\mathcal{E}^a=TM$. Tensoring by $\mathcal{E}[-1]$ we have $\mathcal{T}^0/\mathcal{T}^1\cong\mathcal{E}^a[-1]$, and we can summarise the filtration of $\mathcal{T}$ by the composition series

$$\mathcal{T}=\mathcal{E}[1]\lpluscirc\mathcal{E}^a[-1]\lpluscirc\mathcal{E}[-1].$$

It is now straightforward to observe that the ambient Levi-Civita connection $\boldsymbol{\nabla}$ also descends to give a connection on $\mathcal{T}$. First, from the defining property that $\boldsymbol{\nabla}$ preserves the metric it follows that if $U^A\in\tilde{\mathcal{E}}^A(w)$ and $V^A\in\tilde{\mathcal{E}}^A(w')$, then $U^A\boldsymbol{\nabla}_AV^B\in\tilde{\mathcal{E}}^B(w+w'-1)$. Then since $\boldsymbol{\nabla}$ is torsion free, we have that $\boldsymbol{\nabla}_{\boldsymbol{X}}U-\boldsymbol{\nabla}_U\boldsymbol{X}-[\boldsymbol{X},U]=0$ for any tangent vector field U. So if $U\in\tilde{\mathcal{E}}^A(0)$, then $\boldsymbol{\nabla}_{\boldsymbol{X}}U=0$, as, in that case, $[\boldsymbol{X},U]=-U$. So sections of $\tilde{\mathcal{E}}^A(0)$ may be characterised as those which are covariantly parallel along the vertical Euler vector field. These two results imply that $\boldsymbol{\nabla}$ determines a connection $\nabla^{\mathcal{T}}$ on $\mathcal{T}$. For $U\in\mathcal{T}$ let $\tilde{U}$ be the corresponding section of $\mathcal{E}^A_{\mathcal{Q}}(0)$. Similarly a tangent vector field V on M has a lift to field $\tilde{V}\in\tilde{\mathcal{E}}^A(1)$, on $\mathcal{Q}$, which is everywhere tangent to $\mathcal{Q}$. This is unique up to adding $f\boldsymbol{X}$, where $f\in\tilde{\mathcal{E}}(0)$. We extend $\tilde{U}$ and $\tilde{V}$ homogeneously to fields on $\tilde{M}$. Then we can form $\boldsymbol{\nabla}_{\tilde{V}}\tilde{U}$. This is clearly independent of the extensions. Since $\boldsymbol{\nabla}_{\boldsymbol{X}}\tilde{U}=0$, it is also independent of the choice of $\tilde{V}$ as a lift of V. Finally, it is a section of $\tilde{\mathcal{E}}^A(0)$ and so determines a section $\nabla^{\mathcal{T}}_VU$ of $\mathcal{T}$ which only depends on U and V. It is easily verified that this defines a covariant derivative on $\mathcal{T}$.

Let us summarise. By the above construction the ambient manifold and metric construction of Fefferman and Graham naturally determines a rank

$(n+2)$ vector bundle $\mathcal{T}$ on M. This vector bundle comes equipped with a signature $(p+1,q+1)$ metric $h^{\mathcal{T}}$, a connection $\nabla^{\mathcal{T}}$, and a filtration determined by a canonical section $X_{\mathcal{T}}$ of $\mathcal{T}[1]$. Furthermore if v^a is a smooth tangent field on M and φ is a smooth section of $\mathcal{E}[1]$, one easily verifies from the above that the image of $v^a\nabla^{\mathcal{T}}_a(\varphi X^B)$ lies in $\mathcal{T}^0$ and that composing with the map to the quotient $\mathcal{T}^0/\mathcal{T}^1$ recovers φv^b. This is a non-degeneracy property of the connection. This with the fact that $\nabla^{\mathcal{T}}$ preserves the metric means that $\mathcal{T}$ is a tractor bundle with a tractor connection in the sense of [14]. Since $\boldsymbol{\nabla}$ is Ricci flat it follows that $\nabla^{\mathcal{T}}$ satisfies the curvature normalisation condition described in [13, 14]. (This is shown explicitly in [12].) From this and the non-degeneracy we can conclude that $\mathcal{T}^A$ and $\nabla^{\mathcal{T}}_a$ are a normal tractor bundle and connection corresponding to the defining representation of $\mathrm{SO}(p+1,q+1)$. That is we can take, $\mathcal{T}^A=\mathcal{E}^A$, $X^A_{\mathcal{T}}=X^A$, and $\nabla^{\mathcal{T}}_a$ to be the usual tractor connection as in Section 2. We henceforth drop the notation $\mathcal{T}$.

We can also recover the operators introduced in the tractor setting. Observe that the operator $\boldsymbol{D}_{AP}:=2\boldsymbol{X}_{[P}\boldsymbol{\nabla}_{A]}$ annihilates the function Q on $\tilde{M}$. Thus $\boldsymbol{D}_{AP}$ gives an operator $\tilde{\mathcal{E}}^{\Phi}_{\mathcal{Q}}(w)\to\tilde{\mathcal{E}}_{[AP]}\otimes\tilde{\mathcal{E}}^{\Phi}(w)$, and it is a trivial matter to show that this descends to $D_{AP}:\mathcal{E}^{\Phi}[w]\to\mathcal{E}_{[AP]}\otimes\mathcal{E}^{\Phi}[w]$ as defined in Section 2. (Here, of course, $\mathcal{E}^{\Phi}[w]$ is *the* weight w tractor bundle corresponding to $\tilde{\mathcal{E}}^{\Phi}(w)$.) Now we can formally follow the construction of D_A. First one calculates that, for $\tilde{V}\in\tilde{\mathcal{E}}^{\Phi}(w)$, and using (29), we have $\boldsymbol{h}^{AB}\boldsymbol{D}_{A(Q}\boldsymbol{D}_{|B|P)_0}V=-\boldsymbol{X}_{(Q}\boldsymbol{D}_{P)_0}V$, where

$$\boldsymbol{D}_AV=(n+2w-2)\boldsymbol{\nabla}_AV-\boldsymbol{X}_A\boldsymbol{\Delta}V,\quad \boldsymbol{\Delta}:=\boldsymbol{\nabla}^B\boldsymbol{\nabla}_B. \tag{31}$$

Then we observe the map $\tilde{\mathcal{E}}_P(w-1)\to\tilde{\mathcal{E}}_{(PQ)_0}(w)$ given by $\tilde{S}_P\mapsto\boldsymbol{X}_{(Q}\tilde{S}_{P)_0}$ is injective. It follows immediately that, along $\mathcal{Q}$, (31) is determined by the equation $\boldsymbol{h}^{AB}\boldsymbol{D}_{A(Q}\boldsymbol{D}_{|B|P)_0}V=-\boldsymbol{X}_{(Q}\boldsymbol{D}_{P)_0}V$ and so is precisely the operator $\boldsymbol{D}_A:\tilde{\mathcal{E}}^{\Phi}_{\mathcal{Q}}(w)\to\tilde{\mathcal{E}}_A\otimes\tilde{\mathcal{E}}^{\Phi}_{\mathcal{Q}}(w-1)$, which descends to $D_A:\mathcal{E}^{\Phi}[w]\to\mathcal{E}_A\otimes\mathcal{E}^{\Phi}[w-1]$. In particular this is true when $w=1-n/2$, and so $\boldsymbol{\Delta}:\tilde{\mathcal{E}}^{\Phi}(1-n/2)\to\tilde{\mathcal{E}}^{\Phi}(-1-n/2)$ descends to the generalised Yamabe operator $\Box:\mathcal{E}^{\Phi}[1-n/2]\to\mathcal{E}^{\Phi}[-1-n/2]$. We will take (31) as the definition of $\boldsymbol{D}_A$ on $\tilde{M}$. Although we will not need it here, let us point out that $\boldsymbol{D}_{AP}$ as defined above acts more generally on sections of tensor bundles on $\tilde{M}$ and not just sections which are homogeneous. Following through the argument above in this more general setting yields a generalisation of the operator $\boldsymbol{D}_A$ on tensor bundles given by $\boldsymbol{D}_A=n\boldsymbol{\nabla}_A+2\boldsymbol{X}^B\boldsymbol{\nabla}_B\boldsymbol{\nabla}_A-\boldsymbol{X}_A\boldsymbol{\Delta}$. This still has the property that along $\mathcal{Q}$ it acts tangentially.

Observe that $\boldsymbol{h}^{AB}\boldsymbol{D}_{A(Q}\boldsymbol{D}_{|B|P)_0}V$ is only of the form $-\boldsymbol{X}_{(Q}\boldsymbol{D}_{P)_0}V$ to order Q^0 along $\mathcal{Q}$ and that although $\boldsymbol{D}_A$ acts tangentially to $\mathcal{Q}$ to this order, it does not commute with Q. In fact for any tensor field V, homogeneous of weight w on $\tilde{M}$, from (31) we have

$$\boldsymbol{D}_AQV=Q\boldsymbol{D}_AV+4Q\boldsymbol{\nabla}_AV. \tag{32}$$

So, along the $Q=0$ surface $\mathcal{Q}$, $\boldsymbol{D}_A$ acts tangentially, but, $\boldsymbol{D}_A$ does not act tangentially to other $Q=$ constant surfaces. Nevertheless this allows us to conclude that if U and V are tensors of the same rank (and with $U+QV$

homogeneous of some weight), then

$$\boldsymbol{D}_{A_1}\cdots\boldsymbol{D}_{A_\ell}(U+QV)=(\boldsymbol{D}_{A_1}\cdots\boldsymbol{D}_{A_\ell}U)+QW$$

for some tensor W. Thus, along $\mathcal{Q}$, $\boldsymbol{D}_{A_1}\cdots\boldsymbol{D}_{A_\ell}U$ is independent of how U is extended off $\mathcal{Q}$. The identities

$$\boldsymbol{X}_A\boldsymbol{D}^AV=w(n+2w-2)V-Q\boldsymbol{\Delta}V \tag{33}$$

and

$$\boldsymbol{D}^A\boldsymbol{X}_AV=(n+2w+2)(n+w)V-Q\boldsymbol{\Delta}V \tag{34}$$

will also be useful. Here V is a tensor which is homogeneous of weight w.

We are now in a position to show directly how the tractor field W_{ABCD} is represented in the ambient setting. Let us for the while restrict to $n\neq 4$. Note that the curvature of the ambient connection $\boldsymbol{R}_{ABCD}$ is a section of $\tilde{\mathcal{E}}_{ABCD}(-2)$ and so corresponds to a section of the tractor bundle $\mathcal{E}_{ABCD}[-2]$. We will write R_{ABCD} to denote this section. Let $\tilde{V}\in\tilde{\mathcal{E}}^\Phi(w)$. From (31) we obtain

$$[\boldsymbol{D}_A,\boldsymbol{D}_B]\tilde{V}=(n-2)(n-4)[\boldsymbol{\nabla}_A,\boldsymbol{\nabla}_B]\tilde{V}-2(n-2)\boldsymbol{X}_{[A}[\boldsymbol{\Delta},\boldsymbol{\nabla}_{B]}]\tilde{V}.$$

Now let $V^A\in\mathcal{E}^A$. We write $\tilde{V}=\tilde{V}^A\in\tilde{\mathcal{E}}^A_{\mathcal{Q}}(0)$ for the corresponding field on $\mathcal{Q}$, and extend this homogeneously to a field on $\tilde{M}$. Then, along $\mathcal{Q}$, we have (see remark below)

$$[\boldsymbol{D}_A,\boldsymbol{D}_B]\tilde{V}^C=(n-2)(n-4)\boldsymbol{R}_{AB}{}^C{}_E\tilde{V}^E+4(n-2)\boldsymbol{X}_{[A}\boldsymbol{R}_{B]F}{}^C{}_E\boldsymbol{\nabla}^F\tilde{V}^E.$$

Thus, since $\boldsymbol{X}^F\boldsymbol{R}_{BFCE}=0=\boldsymbol{X}_F\boldsymbol{\nabla}^F\tilde{V}^E$, this implies

$$[D_A,D_B]V^C=(n-2)(n-4)R_{AB}{}^C{}_EV^E+4(n-2)X_{[A}R_{B]F}{}^C{}_EZ^F{}_f\nabla^fV^E.$$

Comparing this with (12) (with w set to 0 in that expression) we can at once conclude that $X_{[A}W_{BC]DE}V^E=(n-4)X_{[A}R_{BC]DE}V^E$. Since this holds for any section V^A of $\mathcal{E}^A$, it follows from the definition of W_{ABCD} that $X_{[A}\Omega_{BC]DE}=X_{[A}R_{BC]DE}$. Contracting with $Z^F{}_f$ we have immediately $X_{[A}R_{B]F}{}^C{}_EZ^F{}_f=X_{[A}\Omega_{B]F}{}^C{}_EZ^F{}_f$. Substituting this in the above display and once again comparing to (12) we now have that $W_{BCDE}V^E=(n-4)R_{BCDE}V^E$ for all V^E, and so

$$W_{BCDE}=(n-4)R_{BCDE}. \tag{35}$$

Remark: Note that

$$[\boldsymbol{\Delta},\boldsymbol{\nabla}_B]\tilde{V}_C=2\boldsymbol{R}_{EBCF}\boldsymbol{\nabla}^E\tilde{V}^F+(\boldsymbol{\nabla}^E\boldsymbol{R}_{EBCF})\tilde{V}^F+\boldsymbol{R}_{BF}\boldsymbol{\nabla}^F\tilde{V}_C.$$

>From the contracted Bianchi identity $\boldsymbol{\nabla}^E\boldsymbol{R}_{EBCF}=2\boldsymbol{\nabla}_{[C}\boldsymbol{R}_{F]B}$, so in odd dimensions the last two terms of the display vanish to all orders. In even dimensions recall we have that, along $\mathcal{Q}$, $\boldsymbol{R}_{BF}$ vanishes to order $n/2-1$ and so in all even dimensions, other than 4, these last terms also vanish along $\mathcal{Q}$.

4. The GJMS operators

Using the properties of $\boldsymbol{D}_A$, we observed in the previous section that if V is a tensor homogeneous of weight $1-n/2$ then, along $\mathcal{Q}$, $\boldsymbol{\Delta} V$ is independent of how V is extended off $\mathcal{Q}$. So $\boldsymbol{\Delta}$ gives an operator $\boldsymbol{\Delta} : \tilde{\mathcal{E}}^\Phi(1-n/2) \to \tilde{\mathcal{E}}^\Phi(-1-n/2)$, and this descends to the generalised conformally invariant Laplacian (or Yamabe operator) as in (14). The observation that the conformally invariant Laplacian (on densities) can be obtained from an ambient Laplacian in this way goes back to [38] in the conformally flat dimension 4 setting and to [26] for the general curved case. For the generalised conformally invariant Laplacian we can also show this directly using the result

$$\boldsymbol{\nabla}_A Q = 2\boldsymbol{X}_A \tag{36}$$

from above. From this it follows that if U is a tensor field homogeneous of weight w (i.e. $U \in \tilde{\mathcal{E}}^\Phi(w)$), then

$$[\boldsymbol{\Delta}, Q]U = 2(n+2w+2)U. \tag{37}$$

Thus if $V \in \tilde{\mathcal{E}}^\Phi(1-n/2)$ and U is a tensor of the same rank and type but homogeneous of weight $-1-n/2$, then

$$\boldsymbol{\Delta}(V+QU) = \boldsymbol{\Delta} V + Q\boldsymbol{\Delta} U. \tag{38}$$

So clearly $\boldsymbol{\Delta} V$ is independent of how V extends off $\mathcal{Q}$. In [32], Graham, Jenne, Mason, and Sparling establish a remarkable generalisation of the result for densities which we state here in our current notation.

Proposition 4.1. *For n even and $k \in \{1, 2, \cdots, n/2\}$ or n odd and $k \in \mathbb{Z}_+$, let $f \in \tilde{\mathcal{E}}_{\mathcal{Q}}(k-n/2)$, and let $\tilde{f} \in \tilde{\mathcal{E}}(k-n/2)$ be a homogeneous extension of f. The restriction of $\boldsymbol{\Delta}^k \tilde{f}$ to $\mathcal{Q}$ depends only on f and the conformal structure on M but not on the choice of the extension $\tilde{f}$ or on any choices in the ambient metric. Thus there is a conformally invariant operator*

$$\boldsymbol{\Delta}^k : \tilde{\mathcal{E}}_{\mathcal{Q}}(k-n/2) \to \tilde{\mathcal{E}}_{\mathcal{Q}}(-k-n/2),$$

and this descends to a natural conformally invariant differential operator

$$P_{2k} : \mathcal{E}[k-n/2] \to \mathcal{E}[-k-n/2]$$

on M.

As mentioned in the introduction, we call the operators P_{2k} the GJMS operators.

In this section we will describe a way that one can directly rewrite these operators in terms of $\boldsymbol{D}_A$, $\boldsymbol{X}_A$, the curvature $\boldsymbol{R}$, and just one $\boldsymbol{\Delta}$. As observed above, each of these corresponds to an object in the tractor calculus. Before we begin we need one more result from [32]. (This follows from Proposition 2.2 and Section 3 from there).

Proposition 4.2. *For n even and $k \in \{1, 2, \cdots, n/2\}$ or n odd and $k \in \mathbb{Z}_+$, let $f \in \tilde{\mathcal{E}}_{\mathcal{Q}}(k-n/2)$. Then f has an extension $\tilde{f} \in \tilde{\mathcal{E}}(k-n/2)$ uniquely determined modulo $O(Q^k)$ by the requirement that $\boldsymbol{\Delta}\tilde{f} = 0$ modulo $O(Q^{k-1})$. The extension is independent of any choices in the ambient metric.*

We are ready to consider an example. Let $\tilde{f} \in \tilde{\mathcal{E}}(2-n/2)$, and let f denote the section of $\mathcal{E}[2-n/2]$ that it determines. Consider $\boldsymbol{\Delta D}_A \tilde{f} = \boldsymbol{\Delta}(2\boldsymbol{\nabla}_A \tilde{f} - \boldsymbol{X}_A \boldsymbol{\Delta} \tilde{f})$. Since in all dimensions the ambient Ricci curvature vanishes along $\mathcal{Q}$, we have $[\boldsymbol{\Delta}, \boldsymbol{\nabla}_A]\tilde{f} = 0$. So with the operator equality $[\boldsymbol{\Delta}, \boldsymbol{X}_A] = 2\boldsymbol{\nabla}_A$ we immediately see that

$$\boldsymbol{\Delta D}_A \tilde{f} = -\boldsymbol{X}_A \boldsymbol{\Delta}^2 \tilde{f}.$$

Thus $\Box D_A f = -X_A P_4 f$, where $P_4 f$ is the fourth-order GJMS operator (which agrees with the Paneitz operator). Note that according to the earlier proposition above, the right-hand side is independent of how f extends off $\mathcal{Q}$. So the left-hand side is likewise independent of the choice of extension. In fact this is already clear from (32) and (38).

This suggests attempting to recover the higher order GJMS operators from $\boldsymbol{\Delta D}_A \cdots \boldsymbol{D}_B \tilde{f}$. On conformally flat structures this is immediately successful.

Proposition 4.3. *On conformally flat structures, if $\tilde{f} \in \tilde{\mathcal{E}}(k-n/2)$, $k \in \mathbb{Z}_+$, then*

$$\boldsymbol{\Delta D}_{A_{k-1}} \cdots \boldsymbol{D}_{A_1} \tilde{f} = (-1)^{k-1} \boldsymbol{X}_{A_1} \cdots \boldsymbol{X}_{A_{k-1}} \boldsymbol{\Delta}^k \tilde{f}.$$

Proof. We are only interested in local results and differential operators. So without loss of generality we suppose that we are in the setting of the flat model for which the ambient space is simply $\mathbb{R}^{n+2}$ equipped with the flat metric $\boldsymbol{h}$ given by a fixed bilinear form of signature $(p+1, q+1)$ and the standard parallel transport. The latter also gives the ambient connection in this setting. In the standard coordinates, $\boldsymbol{X} = \boldsymbol{X}^I \partial/\partial \boldsymbol{X}^I$ at the point $\boldsymbol{X}^I$, and the identities of the previous section hold as genuine equalities rather than just formally.

We have the operator identity $[\boldsymbol{\Delta}, \boldsymbol{X}_A] = 2\boldsymbol{\nabla}_A$ on sections of $\tilde{\mathcal{E}}^\Phi(w)$. Since the structure is conformally flat, we also have $[\boldsymbol{\Delta}, \boldsymbol{\nabla}_A] = 0$. It follows that $[\boldsymbol{\Delta}^m, \boldsymbol{X}_A] = 2m\boldsymbol{\Delta}^{m-1}\boldsymbol{\nabla}_A$. Thus if $\tilde{f} \in \tilde{\mathcal{E}}^\Phi(m+1-n/2)$, we have

$$-\boldsymbol{\Delta}^m \boldsymbol{D}_A \tilde{f} = -\boldsymbol{\Delta}^m [2m\boldsymbol{\nabla}_A \tilde{f} - \boldsymbol{X}_A \boldsymbol{\Delta} \tilde{f}] = \boldsymbol{X}_A \boldsymbol{\Delta}^{m+1} \tilde{f}.$$

The proposition now follows by induction on k. $\Box$

To relate $\boldsymbol{\Delta D}_{A_{k-1}} \cdots \boldsymbol{D}_{A_1} \tilde{f}$ and $\boldsymbol{\Delta}^k \tilde{f}$ in the general case we must take account of the curvature of the ambient manifold. Since this is Ricci flat we have that if $\tilde{V}_B \in \tilde{\mathcal{E}}_A(w)$, then $[\boldsymbol{\Delta}, \boldsymbol{\nabla}_A]\tilde{V}_B = -2\boldsymbol{R}_A{}^P{}_B{}^Q \boldsymbol{\nabla}_P \tilde{V}_Q$. More generally if $\tilde{V}_{BC\cdots E} \in \tilde{\mathcal{E}}_{BC\cdots E}(w)$, then

$$\begin{aligned} &[\boldsymbol{\Delta}, \boldsymbol{\nabla}_A]\tilde{V}_{BC\cdots E} = \\ &\quad -2\boldsymbol{R}_A{}^P{}_B{}^Q \boldsymbol{\nabla}_P \tilde{V}_{QC\cdots E} - 2\boldsymbol{R}_A{}^P{}_C{}^Q \boldsymbol{\nabla}_P \tilde{V}_{BQ\cdots E} - \cdots \\ &\quad -2\boldsymbol{R}_A{}^P{}_E{}^Q \boldsymbol{\nabla}_P \tilde{V}_{BC\cdots Q}. \end{aligned} \tag{39}$$

In even dimensions the ambient metric is only Ricci flat and determined by the conformal structure on M to finite order, as described above. For example for even n (39) only holds mod $\mathrm{O}(Q^{n/2-2})$ (or mod $\mathrm{O}(Q^{n/2-1})$ if $\tilde{V}$ has rank 0). For simplicity in the following discussion we will often ignore this point and assume the given calculations do not involve sufficient transverse derivatives of the ambient metric to encounter this problem. We will return to a careful count of tranverse derivatives later in the section. We will also henceforth restrict to $n \neq 4$. This also simplifies matters. And there is no loss, as the results for $n = 4$ have been obtained above.

It follows from the last display that if $\tilde{f} \in \tilde{\mathcal{E}}(w)$ (and $\ell < n/2$ if n is even), then

$$\begin{aligned}
\boldsymbol{\Delta} \boldsymbol{\nabla}_{A_\ell} \cdots \boldsymbol{\nabla}_{A_1} \tilde{f} = \\
& -2\boldsymbol{R}_{A_\ell}{}^{P}{}_{A_{\ell-1}}{}^{Q} \boldsymbol{\nabla}_P \boldsymbol{\nabla}_Q \boldsymbol{\nabla}_{A_{\ell-2}} \cdots \boldsymbol{\nabla}_{A_1} \tilde{f} - \cdots \\
& -2\boldsymbol{R}_{A_\ell}{}^{P}{}_{A_1}{}^{Q} \boldsymbol{\nabla}_P \boldsymbol{\nabla}_{A_{\ell-1}} \cdots \boldsymbol{\nabla}_{A_2} \boldsymbol{\nabla}_Q \tilde{f} \\
& -2\boldsymbol{\nabla}_{A_\ell} \boldsymbol{R}_{A_{\ell-1}}{}^{P}{}_{A_{\ell-2}}{}^{Q} \boldsymbol{\nabla}_P \boldsymbol{\nabla}_Q \boldsymbol{\nabla}_{A_{\ell-3}} \cdots \boldsymbol{\nabla}_{A_1} \tilde{f} - \cdots \\
& -2\boldsymbol{\nabla}_{A_\ell} \cdots \boldsymbol{\nabla}_{A_3} \boldsymbol{R}_{A_2}{}^{P}{}_{A_1}{}^{Q} \boldsymbol{\nabla}_P \boldsymbol{\nabla}_Q \tilde{f} + \\
& \boldsymbol{\nabla}_{A_\ell} \cdots \boldsymbol{\nabla}_{A_1} \boldsymbol{\Delta} \tilde{f},
\end{aligned} \tag{40}$$

where here all $\boldsymbol{\nabla}_A$'s act on all tensors to their right and the result is mod $\mathrm{O}(Q^{n/2-\ell})$ if n is even. We may apply the Leibniz rule to (40). The term $\boldsymbol{\nabla}_{A_\ell} \boldsymbol{R}_{A_{\ell-1}}{}^{P}{}_{A_{\ell-2}}{}^{Q} \boldsymbol{\nabla}_P \boldsymbol{\nabla}_Q \boldsymbol{\nabla}_{A_{\ell-3}} \cdots \boldsymbol{\nabla}_{A_1} \tilde{f}$, for example, becomes

$$\begin{aligned}
&(\boldsymbol{\nabla}_{A_\ell} \boldsymbol{R}_{A_{\ell-1}}{}^{P}{}_{A_{\ell-2}}{}^{Q}) \boldsymbol{\nabla}_P \boldsymbol{\nabla}_Q \boldsymbol{\nabla}_{A_{\ell-3}} \cdots \boldsymbol{\nabla}_{A_1} \tilde{f} + \\
&\boldsymbol{R}_{A_{\ell-1}}{}^{P}{}_{A_{\ell-2}}{}^{Q} \boldsymbol{\nabla}_{A_\ell} \boldsymbol{\nabla}_P \boldsymbol{\nabla}_Q \boldsymbol{\nabla}_{A_{\ell-3}} \cdots \boldsymbol{\nabla}_{A_1} \tilde{f}.
\end{aligned}$$

Often we will not require the details of contractions or the value of coefficients, and so we might write the last result symbolically as $\boldsymbol{\nabla} \boldsymbol{R} \boldsymbol{\nabla}^{\ell-1} \tilde{f} = (\boldsymbol{\nabla} \boldsymbol{R}) \boldsymbol{\nabla}^{\ell-1} \tilde{f} + \boldsymbol{R} \boldsymbol{\nabla}^{\ell} \tilde{f}$. (In this informal notation we will write $\boldsymbol{\nabla}$ to indicate a $\boldsymbol{\nabla}_A$ which is not part of a $\boldsymbol{\Delta}$. For example, it may have a free index or be contracted to the ambient curvature $\boldsymbol{R}$.) We may repeatedly apply the Leibniz rule to (40) in this way until all of the terms on the right-hand side are of the form (omitting indices) $(\boldsymbol{\nabla}^p \boldsymbol{R}) \boldsymbol{\nabla}^q \tilde{f}$. We might write the result symbolically as

$$\boldsymbol{\Delta} \boldsymbol{\nabla}^{\ell} \tilde{f} = \boldsymbol{\nabla}^{\ell} \boldsymbol{\Delta} \tilde{f} + \sum (\boldsymbol{\nabla}^p \boldsymbol{R}) \boldsymbol{\nabla}^q \tilde{f}. \tag{41}$$

Note that each term of the second sort on the right-hand side has $q \geq 2$ and $p + q = \ell$. Although in these symbolic formulae we omit the details of the contractions and the coefficients, we really want to regard these expressions as representing precise formulae. The idea of this notation is simply to manifest explicitly only the aspects of the formulae that we need for our general discussion.

Now observe that

$$\begin{aligned}
&(n + 2w - 2\ell - 2) \boldsymbol{\nabla}_{A_{\ell+1}} \boldsymbol{\nabla}_{A_\ell} \cdots \boldsymbol{\nabla}_{A_1} \tilde{f} = \\
&\quad \boldsymbol{D}_{A_{\ell+1}} \boldsymbol{\nabla}_{A_\ell} \cdots \boldsymbol{\nabla}_{A_1} \tilde{f} + \boldsymbol{X}_{A_{\ell+1}} \boldsymbol{\Delta} \boldsymbol{\nabla}_{A_\ell} \cdots \boldsymbol{\nabla}_{A_1} \tilde{f},
\end{aligned}$$

or, in our symbolic notation, $(n + 2w - 2\ell - 2) \boldsymbol{\nabla}^{\ell+1} \tilde{f} = \boldsymbol{D} \boldsymbol{\nabla}^{\ell} \tilde{f} + \boldsymbol{X} \boldsymbol{\Delta} \boldsymbol{\nabla}^{\ell} \tilde{f}$. We can substitute (41) into the right-hand side of this and so observe that if $n + 2w - 2\ell - 2 \neq 0$, then we can replace a term $\boldsymbol{\nabla}^{\ell+1} \tilde{f}$ by the expression $\boldsymbol{D} \boldsymbol{\nabla}^{\ell} \tilde{f} + \boldsymbol{X} \sum (\boldsymbol{\nabla}^p \boldsymbol{R}) \boldsymbol{\nabla}^q \tilde{f} + \boldsymbol{X} \boldsymbol{\nabla}^{\ell} \boldsymbol{\Delta} \tilde{f}$. Suppose $w = k - n/2$. Then $n + 2w - 2\ell - 2 = 2(k - \ell - 1)$, and we have

$$2(k - \ell - 1) \boldsymbol{\nabla}^{\ell+1} \tilde{f} = \boldsymbol{D} \boldsymbol{\nabla}^{\ell} \tilde{f} + \boldsymbol{X} \sum (\boldsymbol{\nabla}^p \boldsymbol{R}) \boldsymbol{\nabla}^q \tilde{f} + \boldsymbol{X} \boldsymbol{\nabla}^{\ell} \boldsymbol{\Delta} \tilde{f}. \tag{42}$$

In each term of the sum we again have $q \geq 2$ and $p+q=\ell$. Note that the left-hand side of (42) has at most $\ell+1$ transverse derivatives of $\tilde{f}$. Apart from the term $\boldsymbol{X}\boldsymbol{\nabla}^{\ell}\boldsymbol{\Delta}\tilde{f}$, which we will deal with below, the right-hand side has at most ℓ transverse derivatives of $\tilde{f}$, as $\boldsymbol{D}$ acts tangentially to $\mathcal{Q}$. Our strategy below will be to replace $\boldsymbol{\nabla}$'s with $\boldsymbol{D}$'s beginning from the left.

We may apply similar reasoning to $\boldsymbol{R}$. Since $\boldsymbol{R}$ has weight -2, we have $(n-2m-4)\boldsymbol{\nabla}^m\boldsymbol{R} = \boldsymbol{D}\boldsymbol{\nabla}^{m-1}\boldsymbol{R} + \boldsymbol{X}\boldsymbol{\Delta}\boldsymbol{\nabla}^{m-1}\boldsymbol{R}$. Here we have used the same informal notation that we used with $\tilde{f}$, above. By (39) we may write this as $(n-2m-4)\boldsymbol{\nabla}^m\boldsymbol{R} = \boldsymbol{D}\boldsymbol{\nabla}^{m-1}\boldsymbol{R} + \boldsymbol{X}\sum(\boldsymbol{\nabla}^p\boldsymbol{R})\boldsymbol{\nabla}^q\boldsymbol{R} + \boldsymbol{X}\boldsymbol{\nabla}^{m-1}\boldsymbol{\Delta}\boldsymbol{R}$. Now note that since $\boldsymbol{R}$ is Ricci flat, we have

$$\begin{gathered}\boldsymbol{\Delta}\boldsymbol{R}_{BCDE} = \\ 2\left(\boldsymbol{R}^A{}_{CB}{}^F\boldsymbol{R}_{FADE} + \boldsymbol{R}^A{}_{CD}{}^F\boldsymbol{R}_{BAFE} + \boldsymbol{R}^A{}_{CE}{}^F\boldsymbol{R}_{BADF}\right),\end{gathered} \tag{43}$$

from the Bianchi identity. In odd dimensions this holds to all orders. In general we have $\boldsymbol{\Delta}\boldsymbol{R}_{BCDE} = 2(\boldsymbol{\nabla}_B\boldsymbol{\nabla}_{[D}\boldsymbol{R}_{E]C} - \boldsymbol{\nabla}_C\boldsymbol{\nabla}_{[D}\boldsymbol{R}_{E]B}) + \mathrm{O}(\boldsymbol{R}^2)$ where $\mathrm{O}(\boldsymbol{R}^2)$ indicates the quadratic term in the display. Using, once again, that in even dimensions $\boldsymbol{R}_{AB} = Q^{n/2-1}\boldsymbol{L}_{AB}$ it follows that $(\boldsymbol{\nabla}_B\boldsymbol{\nabla}_{[D}\boldsymbol{R}_{E]C} - \boldsymbol{\nabla}_C\boldsymbol{\nabla}_{[D}\boldsymbol{R}_{E]B})$ vanishes to order $n/2-3$, and so (43) holds to that order.

Thus we get the simplification

$$(n-2m-4)\boldsymbol{\nabla}^m\boldsymbol{R} = \boldsymbol{D}\boldsymbol{\nabla}^{m-1}\boldsymbol{R} + \boldsymbol{X}\sum(\boldsymbol{\nabla}^p\boldsymbol{R})\boldsymbol{\nabla}^q\boldsymbol{R}, \tag{44}$$

where in each term of the sum, $p+q=m-1$. In even dimensions we need $m < n/2-2$. This follows immediately from the previous paragraph. That is, we need $(n-2m-4)>0$.

Our effort to relate $\boldsymbol{\Delta}\boldsymbol{D}_{A_{k-1}}\cdots\boldsymbol{D}_{A_1}\tilde{f}$ and $\boldsymbol{\Delta}^k\tilde{f}$ involves another identity, viz

$$\boldsymbol{\Delta}(\boldsymbol{\nabla}^t\boldsymbol{\Delta}^u\boldsymbol{R})E = (\boldsymbol{\Delta}\boldsymbol{\nabla}^t\boldsymbol{\Delta}^u\boldsymbol{R})E + (\boldsymbol{\nabla}^t\boldsymbol{\Delta}^u\boldsymbol{R})\boldsymbol{\Delta}E + 2(\boldsymbol{\nabla}^{t+1}\boldsymbol{\Delta}^u\boldsymbol{R})\boldsymbol{\nabla}E. \tag{45}$$

Here E is any expression (for a linear operator) which, in terms of our informal symbolic notation, is a polynomial in $\boldsymbol{\nabla}$, $\boldsymbol{\Delta}$, $\boldsymbol{R}$, and $\tilde{f}$. We also need the following fact which follows from the above:

Lemma 4.4. *Suppose n is odd or $t+u \leq n/2-3$. Then on $\mathcal{Q}$ there is an expression for $\boldsymbol{\nabla}^t\boldsymbol{\Delta}^u\boldsymbol{R}$ as a partial contraction polynomial in $\boldsymbol{D}_A$, $\boldsymbol{R}_{ABCD}$, $\boldsymbol{X}_A$, $\boldsymbol{h}_{AB}$, and its inverse $\boldsymbol{h}^{AB}$. This expression is rational in n, and each term is of degree at least 1 in $\boldsymbol{R}_{ABCD}$.*

Proof. Repeatedly use (39), (43), and (45) to rewrite $\boldsymbol{\nabla}^t\boldsymbol{\Delta}^u\boldsymbol{R}$ as a sum of terms of the form $(\boldsymbol{\nabla}^{v_1}\boldsymbol{R})\cdots(\boldsymbol{\nabla}^{v_j}\boldsymbol{R})$. In doing this we convert some $\boldsymbol{\Delta}$'s into pairs of $\boldsymbol{\nabla}$'s via (45), but at most one $\boldsymbol{\nabla}$ from each pair acts on any given $\boldsymbol{R}$. Thus in even dimensions, $v_i \leq n/2-3$, $i \in \{1,\cdots,j\}$, and we may construct the desired partial contraction polynomial by repeatedly applying (44) to the terms $(\boldsymbol{\nabla}^{v_1}\boldsymbol{R})\cdots(\boldsymbol{\nabla}^{v_j}\boldsymbol{R})$. In even dimensions, using (37) and $\boldsymbol{\nabla}_AQ = 2\boldsymbol{X}_A$ with the restriction $t+u \leq n/2-3$, we see that that (39), (43), and (44) all hold to sufficient order. ☐

Now let $f \in \tilde{\mathcal{E}}_{\mathcal{Q}}(k-n/2)$. Suppose $\tilde{f} \in \tilde{\mathcal{E}}(k-n/2)$ is any homogeneous extension of f as in Proposition 4.2. We will consider $\boldsymbol{\Delta}\boldsymbol{D}_{A_{k-1}}\cdots\boldsymbol{D}_{A_1}\tilde{f}$, where k is a positive integer. If n is even, we assume that $k \leq n/2$. Let

us systematically rewrite this in terms of $(-1)^{k-1}\boldsymbol{X}_{A_1}\cdots\boldsymbol{X}_{A_{k-1}}\boldsymbol{\Delta}^k\tilde{f}$ and curvature coupled terms via the following steps:
Step 1: Observe that

$$\boldsymbol{\Delta}\boldsymbol{D}_{A_{k-1}}\cdots\boldsymbol{D}_{A_1}\tilde{f}=\boldsymbol{\Delta}(2\boldsymbol{\nabla}_{A_{k-1}}-\boldsymbol{X}_{A_{k-1}}\boldsymbol{\Delta})\cdots(2(k-1)\boldsymbol{\nabla}_{A_1}-\boldsymbol{X}_{A_1}\boldsymbol{\Delta})\tilde{f}.$$

Expand this out via the distributive law without changing the order of any of the operators.
Step 2: Move all $\boldsymbol{X}$'s to the left of any $\boldsymbol{\nabla}$ or $\boldsymbol{\Delta}$ via the identities $[\boldsymbol{\nabla}_A,\boldsymbol{X}_B]=\boldsymbol{h}_{AB}$ and $[\boldsymbol{\Delta},\boldsymbol{X}_A]=2\boldsymbol{\nabla}_A$ (which hold to all orders).
Step 3: Move all $\boldsymbol{\Delta}$'s to the right of any $\boldsymbol{\nabla}$'s (other than those implicit in $\boldsymbol{\Delta}$) via (40), and (45). In even dimensions one of course needs to be careful, since (40) is valid only if $\ell<n/2$ and holds mod $\mathrm{O}(Q^{n/2-\ell})$. Elementary counting arguments (along similar lines to the discussion in the next paragraph) quickly establish that for terms encountered we have $\ell<k$ satisfied and with no more than $(k-\ell-1)$ transverse derivatives of the result. Since we assume $k\leq n/2$ when n is even the use of (40) is valid. Next by the proof of Proposition 4.3, we may cancel all terms not explicitly involving the curvature except for the term $(-1)^{k-1}\boldsymbol{X}_{A_1}\cdots\boldsymbol{X}_{A_{k-1}}\boldsymbol{\Delta}^k\tilde{f}$. (The proof of Proposition 4.3 involves only the identities used in steps 1 and 2 with just the difference that these are applied in a different order.) We thus obtain

$$(46)\quad (-1)^{k-1}\boldsymbol{X}^{k-1}\boldsymbol{\Delta}^k\tilde{f}+\sum\boldsymbol{h}^s\boldsymbol{X}^x(\boldsymbol{\nabla}^{p_1}\boldsymbol{\Delta}^{r_1}\boldsymbol{R})\cdots(\boldsymbol{\nabla}^{p_d}\boldsymbol{\Delta}^{r_d}\boldsymbol{R})\boldsymbol{\nabla}^q\boldsymbol{\Delta}^r\tilde{f},$$

where $d\geq 1$ in each term of the right-hand part.

At this point let us take stock of what we have. For each term in the result of Step 1, the sum of the number of $\boldsymbol{\Delta}$'s in the term and the number of $\boldsymbol{\nabla}$'s in the term is exactly k. In steps 2 and 3 some $\boldsymbol{\Delta}$'s may have been exchanged for $\boldsymbol{\nabla}$'s via the identity $[\boldsymbol{\Delta},\boldsymbol{X}_A]=2\boldsymbol{\nabla}_A$ or for $\boldsymbol{R}$'s via the commutator $[\boldsymbol{\Delta},\boldsymbol{\nabla}_A]$, and similarly we may have lost some $\boldsymbol{\nabla}$'s by $[\boldsymbol{\nabla}_A,\boldsymbol{X}_B]=\boldsymbol{h}_{AB}$. On the other hand, we may have converted some $\boldsymbol{\Delta}$'s into pairs of $\boldsymbol{\nabla}$'s via (45); note that at most one $\boldsymbol{\nabla}$ from each pair acts on $\tilde{f}$, and similarly at most one $\boldsymbol{\nabla}$ from each pair acts on any given $\boldsymbol{R}$. Thus for each term of the sum in (46) we must have $d+q+r\leq k$. Since $d\geq 1$, it follows that $k-q-r\geq 1$. Note that each $\boldsymbol{R}$ in (40) is followed by at least two $\boldsymbol{\nabla}$'s. Thus at each step in the construction of the right-hand part of (46), each of the rightmost two $\boldsymbol{\nabla}$'s of each term arose from steps 1 and 2, and not from the use of (45). It follows that at least two of the $\boldsymbol{\nabla}$'s in $\boldsymbol{\nabla}^q\boldsymbol{\Delta}^r\tilde{f}$ did *not* arise from (45). Thus $q\geq 2$, and for any $i\in\{1,\cdots,d\}$, $p_i+r_i+3\leq k$. Now suppose n is even. Then, by assumption, $k\leq n/2$, and for $i\in\{1,\cdots,d\}$ we have $p_i+r_i\leq n/2-3$. Since the ambient metric is determined modulo terms of $\mathrm{O}(Q^{n/2})$, it follows immediately that the metric connection $\boldsymbol{\nabla}$ is determined modulo terms of $\mathrm{O}(Q^{n/2-1})$. Its curvature $\boldsymbol{R}$ is similarly determined modulo $\mathrm{O}(Q^{n/2-2})$. Now when $\boldsymbol{\Delta}=\boldsymbol{\nabla}^A\boldsymbol{\nabla}_A$ acts on functions, its rightmost $\boldsymbol{\nabla}$ is really just the exterior derivative. Thus as an operator on functions, $\boldsymbol{\Delta}$ is determined modulo terms of $\mathrm{O}(Q^{n/2-1})$. It now follows from (36) and (37) that, as an operator on $\tilde{\mathcal{E}}(k-n/2)$, all terms in the sum of (46) are determined uniquely modulo $\mathrm{O}(Q)$. If n is odd, the ambient metric is determined to infinite order so certainly the same is true.

Next, by Proposition 4.2 we can assume that $\tilde{f}$ satisfies $\boldsymbol{\Delta}\tilde{f} = 0$ modulo $\mathrm{O}(Q^{k-1})$, and given f, this determines $\tilde{f}$ uniquely modulo $\mathrm{O}(Q^k)$. This will simplify our arguments. The end result will be independent of this choice. Since $k-q-r \geq 1$, we see immediately that all terms in (46) with $r \geq 1$ will vanish modulo $\mathrm{O}(Q)$. We will thus delete these terms. From the inequality $k-q-r \geq 1$ and, in even dimensions, the inequality $p_i + r_i \leq n/2 - 3$, it follows that we can carry out the next step.

Step 4: First rewrite (46) as

$$(-1)^{k-1}\boldsymbol{X}^{k-1}\boldsymbol{\Delta}^k\tilde{f} + \sum \boldsymbol{h}^s\boldsymbol{X}^x(\boldsymbol{\nabla}^{p_1}\boldsymbol{\Delta}^{r_1}\boldsymbol{R})\cdots(\boldsymbol{\nabla}^{p_d}\boldsymbol{\Delta}^{r_d}\boldsymbol{R})\boldsymbol{\nabla}^q\tilde{f}. \tag{47}$$

Then repeatedly use (42) and Lemma 4.4 to eliminate all $\boldsymbol{\nabla}$'s and $\boldsymbol{\Delta}$'s from the right-hand part of this expression. The use of (42) introduces additional $\boldsymbol{\Delta}$'s. But terms containing these $\boldsymbol{\Delta}$'s vanish modulo $\mathrm{O}(Q)$, and we cancel them as soon as they appear. We obtain as result

$$(-1)^{k-1}\boldsymbol{X}^{k-1}\boldsymbol{\Delta}^k\tilde{f} = \boldsymbol{\Delta}\boldsymbol{D}^{k-1}\tilde{f} + \sum \boldsymbol{h}^s\Psi\tilde{f}, \tag{48}$$

where, in terms of our informal symbolic notation, the operator Ψ is a polynomial in $\boldsymbol{X}$, $\boldsymbol{D}$, and $\boldsymbol{R}$. The exponent s here is not claimed to bear any relationship to the s from earlier. The only differential operator of non-zero order used in the formula is $\boldsymbol{D}$. Thus although we used the $\tilde{f}$ satisfying $\boldsymbol{\Delta}\tilde{f} = 0$ modulo $\mathrm{O}(Q^{k-1})$ to obtain (48), observe now that it follows immediately from (32) that each term depends only on f and is otherwise independent of the extension $\tilde{f}$. Thus for any extension $\tilde{f}$, (48) holds modulo $\mathrm{O}(Q)$.

Remark: At this point it is worthwhile to justify our use of (42) in Step 4. First note that in each term in (47) we have $q \leq k-1$, by the counting given above. Thus in (42), $\ell + 1$ will always be at most $k-1$, ℓ will be at most $k-2$, and $k-\ell-1$ will be nonzero. We may therefore solve for $\boldsymbol{\nabla}^{\ell+1}\tilde{f}$ in (42). On the other hand the use of (42) may generate additional curvature terms $(\boldsymbol{\nabla}^p\boldsymbol{R})\boldsymbol{\nabla}^q\tilde{f}$. But $p+q = \ell$, where $q \geq 2$. Thus in even dimensions, $p \leq \ell - 2 \leq k-4 \leq n/2 - 4$, and we may apply Lemma 4.4 to $\boldsymbol{\nabla}^p\boldsymbol{R}$.

In the final step we will use the fact that $(n-4)\boldsymbol{R}$ descends to the tractor field W, $\boldsymbol{X}$ descends to X, $\boldsymbol{h}$ descends to h, and that $\boldsymbol{\Delta} : \tilde{\mathcal{E}}^\Phi(1-n/2) \to \tilde{\mathcal{E}}^\Phi(-1-n/2)$ descends to $\Box : \mathcal{E}^\Phi[1-n/2] \to \mathcal{E}^\Phi[-1-n/2]$.

Step 5: In the right-hand side of (48) make the following formal replacements: $\tilde{f}$ with f, $\boldsymbol{\Delta}$ with $\Box$, $\boldsymbol{X}$ with X, $\boldsymbol{h}$ with h, $\boldsymbol{R}$ with $W/(n-4)$ (in dimensions $n \neq 4$) and $\boldsymbol{D}$ with D. The result is a tractor formula for $(-1)^{k-1}X^{k-1}P_{2k}f$. We state this as a proposition.

Proposition 4.5. *There is a tractor calculus expression for the GJMS operators of the form*

$$\begin{aligned} &X_{A_1}\cdots X_{A_{k-1}}P_{2k}f = \\ &\qquad (-1)^{k-1}\Box D_{A_{k-1}}\cdots D_{A_1}f + \Psi_{A_{k-1}\cdots A_1}{}^{PQ}D_PD_Qf, \end{aligned} \tag{49}$$

where $f \in \mathcal{E}[k-n/2]$ and Ψ is a linear differential operator

$$\Psi_{A_{k-1}\cdots A_1}{}^{PQ} : \mathcal{E}_{PQ}[k-2-n/2] \to \mathcal{E}_{A_{k-1}\cdots A_1}[-1-n/2],$$

expressed as a partial contraction polynomial in D_A, W_{ABCD}, X_A, h_{AB}, and its inverse h^{AB}. The expression for Ψ is rational in n, and each term is of degree at least 1 in W_{ABCD}.

Proof. It is clear from the argument of this section that

$$X_{A_1}\cdots X_{A_{k-1}}P_{2k}f = (-1)^{k-1}\Box D_{A_{k-1}}\cdots D_{A_1}f + \Psi_{A_{k-1}\cdots A_1}f,$$

where $\Psi_{A_{k-1}\cdots A_1}$ is a linear differential operator on f expressed as a partial contraction polynomial in D_A, W_{ABCD}, X_A, h_{AB}, and its inverse h^{AB}. It is also clear that that this expression for Ψ is rational in n and that each term is of degree at least 1 in W_{ABCD}. Furthermore, recall that in Step 4 we used (42) and Lemma 4.4 to convert the expression $\boldsymbol{\nabla}^q\tilde{f}$ of (47) into an expression in $\boldsymbol{D}$, $\boldsymbol{X}$, $\boldsymbol{R}$, $\boldsymbol{h}$, and $\boldsymbol{h}^{-1}$. Since $q\geq 2$ in (42) and (47), it follows that each term of this tractor expression ends in two consecutive $\boldsymbol{D}$'s. The result now follows. □

We conclude this section with examples.

4.1. **Examples.** The simplest example of our procedure is the Paneitz operator P_4, which we treated at the outset of this section. Recall that we obtained $\Box D_A f = -X_A P_4 f$, and it is clear that the tractor expression on the left-hand side of this is independent of any choices in the ambient construction. This is as guaranteed by the argument following Step 3.

The next simplest case is of course the operator P_6. By assumption then, $n\neq 4$. Let f denote a section of $\mathcal{E}[3-n/2]$. Let $\tilde{f}$ be a section of $\tilde{\mathcal{E}}(3-n/2)$ such that its restriction to $\mathcal{Q}$ agrees with f and such that, $\boldsymbol{\Delta}\tilde{f} = Q^2 g$ for some smooth $g\in\tilde{\mathcal{E}}(-3-n/2)$. Expanding out $\boldsymbol{\Delta D}_A\boldsymbol{D}_B\tilde{f}$ according to steps 1 and 2 gives

$$\begin{aligned}&\boldsymbol{\Delta D}_A\boldsymbol{D}_B\tilde{f} =\\ &\quad \boldsymbol{X}_A\boldsymbol{X}_B\boldsymbol{\Delta}^3\tilde{f} + 2\boldsymbol{X}_B[\boldsymbol{\nabla}_A,\boldsymbol{\Delta}]\boldsymbol{\Delta}\tilde{f}+\\ &\quad 2\boldsymbol{X}_A[\boldsymbol{\nabla}_B,\boldsymbol{\Delta}]\boldsymbol{\Delta}\tilde{f} + 4\boldsymbol{X}_A\boldsymbol{\Delta}[\boldsymbol{\nabla}_B,\boldsymbol{\Delta}]\tilde{f} - 8[\boldsymbol{\nabla}_A,\boldsymbol{\Delta}]\boldsymbol{\nabla}_B\tilde{f}.\end{aligned}$$

Since $[\boldsymbol{\Delta},\boldsymbol{\nabla}_A]$ vanishes on functions mod $\mathrm{O}(Q^2)$, the third step reduces to

$$\begin{aligned}\boldsymbol{\Delta D}_A\boldsymbol{D}_B\tilde{f} = \boldsymbol{X}_A\boldsymbol{X}_B\boldsymbol{\Delta}^3\tilde{f} - 8[\boldsymbol{\nabla}_A,\boldsymbol{\Delta}]\boldsymbol{\nabla}_B\tilde{f} =\\ \boldsymbol{X}_A\boldsymbol{X}_B\boldsymbol{\Delta}^3\tilde{f} - 16\boldsymbol{R}_A{}^C{}_B{}^E\boldsymbol{\nabla}_C\boldsymbol{\nabla}_E\tilde{f},\end{aligned}$$

along $\mathcal{Q}$. The fourth step is simply the observation that on $\mathcal{Q}$ (with $\tilde{f}$ as above) we have

$$8\boldsymbol{\nabla}_C\boldsymbol{\nabla}_E\tilde{f} = \boldsymbol{D}_C\boldsymbol{D}_E\tilde{f},$$

and so

$$\boldsymbol{X}_A\boldsymbol{X}_B\boldsymbol{\Delta}^3\tilde{f} = \boldsymbol{\Delta D}_A\boldsymbol{D}_B\tilde{f} + 2\boldsymbol{R}_A{}^C{}_B{}^E\boldsymbol{D}_C\boldsymbol{D}_E\tilde{f}.$$

As we have observed generally, at this stage none of the terms on either side depend on how $\tilde{f}$ extends off $\mathcal{Q}$. Thus finally we have

$$\Box D_A D_B f + \frac{2}{n-4}W_A{}^C{}_B{}^E D_C D_E f = X_A X_B P_6 f,$$

where P_6 is the sixth-order GJMS operator. Thus as promised, we have recovered the tractor formula found by other means in Section 2.1.

Our final example is P_8. By following steps 1 through 4, above, and by applying (29), we obtain

$$
\begin{aligned}
&-\boldsymbol{X}_A\boldsymbol{X}_B\boldsymbol{X}_C\boldsymbol{\Delta}^4\tilde{f} = \\
&\quad \boldsymbol{\Delta}\boldsymbol{D}_A\boldsymbol{D}_B\boldsymbol{D}_C\tilde{f} + 2\boldsymbol{R}_A{}^P{}_B{}^Q\boldsymbol{D}_P\boldsymbol{D}_Q\boldsymbol{D}_C\tilde{f} + 2\boldsymbol{R}_A{}^P{}_C{}^Q\boldsymbol{D}_P\boldsymbol{D}_B\boldsymbol{D}_Q\tilde{f} \\
&\quad -\tfrac{2}{(n-6)}\boldsymbol{X}_A(\boldsymbol{D}_E\boldsymbol{R}_B{}^P{}_C{}^Q)\boldsymbol{D}^E\boldsymbol{D}_P\boldsymbol{D}_Q\tilde{f} \\
&\quad +4\boldsymbol{X}_A\boldsymbol{R}_B{}^P{}_C{}^Q\boldsymbol{R}_P{}^E{}_Q{}^F\boldsymbol{D}_E\boldsymbol{D}_F\tilde{f} - 2\boldsymbol{X}_A\boldsymbol{U}_B{}^P{}_C{}^Q\boldsymbol{D}_P\boldsymbol{D}_Q\tilde{f} \\
&\quad -\tfrac{2}{n-6}\boldsymbol{X}_A\boldsymbol{X}_E\boldsymbol{U}_B{}^P{}_C{}^Q\boldsymbol{D}^E\boldsymbol{D}_P\boldsymbol{D}_Q\tilde{f} \\
&\quad +\tfrac{4}{(n-6)}\boldsymbol{X}_A\boldsymbol{X}^E(\boldsymbol{D}_E\boldsymbol{R}_B{}^P{}_C{}^Q)\boldsymbol{R}_P{}^F{}_Q{}^G\boldsymbol{D}_F\boldsymbol{D}_G\tilde{f}.
\end{aligned}
$$

Here $\boldsymbol{U}_B{}^P{}_C{}^Q$ denotes the tractor field

$$2\left(\boldsymbol{R}^{AP}{}_B{}^F\boldsymbol{R}_{FAC}{}^Q + \boldsymbol{R}^{AP}{}_C{}^F\boldsymbol{R}_{BAF}{}^Q + \boldsymbol{R}^{APQF}\boldsymbol{R}_{BACF}\right).$$

To demonstrate explicitly that P_8 is formally self-adjoint, a variation on this formula is preferred. It is a straightforward exercise to rewrite the above equation as follows.

$$
\begin{aligned}
&\boldsymbol{X}_A\boldsymbol{X}_B\boldsymbol{X}_C\boldsymbol{\Delta}^4\tilde{f} = \\
&\quad -\boldsymbol{\Delta}\boldsymbol{D}_A\boldsymbol{D}_B\boldsymbol{D}_C\tilde{f} - 2\boldsymbol{R}_A{}^P{}_B{}^Q\boldsymbol{D}_P\boldsymbol{D}_Q\boldsymbol{D}_C\tilde{f} - 2\boldsymbol{R}_A{}^P{}_C{}^Q\boldsymbol{D}_P\boldsymbol{D}_B\boldsymbol{D}_Q\tilde{f} \\
&\quad -\tfrac{4}{n-6}\boldsymbol{X}_A(\boldsymbol{U}_B{}^P{}_C{}^Q)\boldsymbol{D}_P\boldsymbol{D}_Q\tilde{f} + \tfrac{2}{n-6}\boldsymbol{X}_A\boldsymbol{D}^E\boldsymbol{R}_B{}^P{}_C{}^Q\boldsymbol{D}_E\boldsymbol{D}_P\boldsymbol{D}_Q\tilde{f} \\
&\quad +4\tfrac{n-2}{n-6}\boldsymbol{X}_A\boldsymbol{R}_B{}^P{}_C{}^Q\boldsymbol{R}_P{}^S{}_Q{}^T\boldsymbol{D}_S\boldsymbol{D}_T\tilde{f}.
\end{aligned}
$$

This together with (43) yields the tractor formula of Proposition 2.3.

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