

DIFFERENTIABILITY AS CONTINUITY

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ABSTRACT. We characterize differentiability of a map $f : \mathbb{R} \rightarrow \mathbb{R}$ in terms of continuity of a canonically associated map \widehat{f} . To characterize pointwise differentiability of f , both the domain and range of \widehat{f} can be made topological. However, the global differentiability of f is characterized by the continuity of \widehat{f} whose domain is topological but whose range is a convergence space.

1. INTRODUCTION

A calculus student is well aware of the difference between continuity and differentiability of a map $f : \mathbb{R} \rightarrow \mathbb{R}$. For such a student, continuity is always understood as continuity for the usual topology of \mathbb{R} . After a first topology course, this same student may wonder if there is a way to find two topologies τ_d and τ_r such that the differentiability of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is characterized by the continuity of $f : (\mathbb{R}, \tau_d) \rightarrow (\mathbb{R}, \tau_r)$. This natural question was answered negatively by R. Geroch, E. Kronheimer and G. McCarty in [1]. This is in stark contrast with A. Machado's result [2, Propositions 2.2.1 and 2.2.2] that a map $f : \mathbb{C} \rightarrow \mathbb{C}$ is analytic if and only if $\widehat{f} : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is continuous, where the map \widehat{f} is canonically associated to f . However, the structure $\widehat{\mathbb{C}}$ used by Machado is not carried by \mathbb{C} but by \mathbb{C}^2 and is not a topology but a more general structure called *convergence* (see end of Section 2). Moreover Machado's result doesn't apply to the differentiability of a map $f : \mathbb{R} \rightarrow \mathbb{R}$. Hence, this could be seen both as yet another contrast between \mathbb{R} -differentiability and \mathbb{C} -differentiability and as a striking illustration that what fails in the realm of topologies can often be fixed within more general "topological-like" structures like convergence spaces.

A closer look at the proofs in [2] gives however a different picture: the arguments can be modified (and simplified) to apply to real functions and to use only topologies, at least for pointwise differentiability. Indeed, the convergence structure used in [2] can be split into a family of topologies $(\tau_\alpha)_{\alpha \in \mathbb{R}}$, the members of which are instrumental in characterizing *pointwise* differentiability of $f : \mathbb{R} \rightarrow \mathbb{R}$ in terms of continuity of the canonically associated map $\widehat{f} = Id \times f : (\mathbb{R}^2, \tau_1) \rightarrow (\mathbb{R}^2, \tau_\alpha)$.

2. DIFFERENTIABILITY AS CONTINUITY

Let α be a real number. We define on $\mathbb{R} \times \mathbb{R}$ the vector space topology τ_α in which a base of neighborhoods of $(0, 0)$ is given by the sets $N_\varepsilon^\alpha = \{(\lambda, \lambda\xi) : \sup(|\lambda|, |\xi - \alpha|) < \varepsilon\}$ for $\varepsilon > 0$.

Hence a typical neighborhood of (λ_0, x_0) in τ_α is of the form

$$(\lambda_0, x_0) + N_\varepsilon^\alpha = \{(\lambda_0 + \lambda, x_0 + \lambda\xi) : \sup(|\lambda|, |\xi - \alpha|) < \varepsilon\}.$$

If $\sup(|\lambda|, |\xi - 1|) < \frac{\gamma}{2}$ then $|\lambda\xi| < \gamma$ so that

$$\left| \frac{h(\lambda\xi)}{\lambda} \right| < \frac{\varepsilon|\xi|}{3} < \frac{2\varepsilon}{3}.$$

If $f'(x_0) = 0$, then $|\eta| \leq \left| \frac{h(\lambda\xi)}{\lambda} \right|$ and $\widehat{f}((\lambda_0, x_0) + (\lambda, \lambda\xi)) \in \widehat{f}(\lambda_0, x_0) + N_\varepsilon^0$ provided that $(\lambda, \lambda\xi) \in N_\delta^1$ where $\delta = \min\{\varepsilon, \frac{\gamma}{2}\}$. If $f'(x_0) \neq 0$ and $|\xi - 1| < \frac{\varepsilon}{3|f'(x_0)|}$, then $|f'(x_0)| \cdot |\xi - 1| < \frac{\varepsilon}{3}$. Now set $\delta = \min\{\varepsilon, \frac{\gamma}{2}, \frac{\varepsilon}{3|f'(x_0)|}\}$. If $(\lambda, \lambda\xi) \in N_\delta^1$ then it follows that $|\lambda| < \varepsilon$ and $|\eta - f'(x_0)| < \varepsilon$ so $\widehat{f}((\lambda_0, x_0) + (\lambda, \lambda\xi)) \in \widehat{f}(\lambda_0, x_0) + N_\varepsilon^{f'(x_0)}$.

Conversely, assume that $\widehat{f} = Id \times f : (\mathbb{R} \times \mathbb{R}, \tau_1) \rightarrow (\mathbb{R} \times \mathbb{R}, \tau_\alpha)$ is continuous at $(0, x_0)$ for some $\alpha \in \mathbb{R}$. We want to show that $\alpha = f'(x_0)$. By continuity, for every $\varepsilon > 0$, there exists δ such that $\varepsilon > \delta > 0$ and $\widehat{f}((0, x_0) + N_\delta^1) \subset \widehat{f}(0, x_0) + N_\varepsilon^\alpha$. In other words,

$$\{(\lambda, f(x_0 + \lambda\xi)) : \sup(|\lambda|, |\xi - 1|) < \delta\} \subset \{(\lambda, f(x_0) + \lambda\eta) : \sup(|\lambda|, |\eta - \alpha|) < \varepsilon\}.$$

In particular, $f(x_0 + \lambda) = f(x_0) + \lambda\eta$ for some η verifying $|\eta - \alpha| < \varepsilon$, provided that $|\lambda| < \delta$. Therefore

$$\left| \frac{f(x_0 + \lambda) - f(x_0)}{\lambda} - \alpha \right| < \varepsilon$$

provided that $|\lambda| < \delta$, so that $\alpha = \lim_{\lambda \rightarrow 0} \frac{f(x_0 + \lambda) - f(x_0)}{\lambda} = f'(x_0)$. \square

A dissatisfying aspect of this result is that the topology τ_α depends on x_0 . This can be remedied by using a *convergence structure* instead of a topology on the range. Recall that a *filter on X* is a family \mathcal{F} of subsets of X that is stable by finite intersections ($A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$) and by supersets ($A \in \mathcal{F}$ and $A \subset B \implies B \in \mathcal{F}$) and that does not contain the empty set. A family \mathcal{A} of subsets of X that does not contain the empty set and is stable by finite intersections is called a *filter base*. It *generates* a filter $\mathcal{A}^\uparrow = \{B : \exists A \in \mathcal{A}, A \subset B\}$. For instance, the family $\mathcal{N}(x)$ of neighborhoods of a fixed point x of a topological space X is a filter. The family of tails $\{\{x_n : n \geq k\} : k \in \mathbb{N}\}$ of a sequence $(x_n)_{n \in \mathbb{N}}$ on X is a filter-base on X . Filters on a given set are ordered by inclusion, that is, \mathcal{F} is *finer than* \mathcal{G} , in symbols $\mathcal{F} \geq \mathcal{G}$, if $\mathcal{F} \supset \mathcal{G}$. It is an easy exercise to verify that the sequence $(x_n)_{n \in \mathbb{N}}$ converges to x if and only if the filter generated by the filter-base of its tails is finer than $\mathcal{N}(x)$. More generally, a filter \mathcal{F} on a topological space X *converges to x* if $\mathcal{F} \geq \mathcal{N}(x)$. A *convergence ξ on a set X* defines what are the filters convergent to each point. Formally, it is a relation between X and the set of filters on X , denoted $x \in \lim_\xi \mathcal{F}$ or $\mathcal{F} \rightarrow_\xi x$ whenever $(x, \mathcal{F}) \in \xi$ and verifying:

- (1) $\{x\}^\uparrow \rightarrow x$ for every $x \in X$;
- (2) $\mathcal{F} \rightarrow x$ and $\mathcal{G} \geq \mathcal{F} \implies \mathcal{G} \rightarrow x$.

A topology is a particular convergence in which $\mathcal{F} \rightarrow x$ if and only if \mathcal{F} is finer than the filter $\mathcal{N}(x)$ of neighborhoods of x and $\mathcal{N}(x)$ has a base of open sets, where $O \subset X$ is *open* if

$$\mathcal{F} \rightarrow x \in O \implies O \in \mathcal{F}.$$

A map $f : (X, \xi) \rightarrow (Y, \sigma)$ between two convergence spaces is *continuous* if $f(\mathcal{F}) \xrightarrow[\tau]{\sigma} f(x)$ whenever $\mathcal{F} \xrightarrow[\xi]{\sigma} x$. If $Id : (X, \xi) \rightarrow (X, \sigma)$ is continuous, we say that ξ

is finer than σ , in symbols $\xi \geq \sigma$. If $(\xi_i)_{i \in I}$ is a family of convergences on X , the supremum and infimum of the family with respect to this order are defined by:

$$\begin{aligned} \lim_{\vee_{i \in I} \xi_i} \mathcal{F} &= \bigcap_{i \in I} \lim_{\xi_i} \mathcal{F}; \\ \lim_{\wedge_{i \in I} \xi_i} \mathcal{F} &= \bigcup_{i \in I} \lim_{\xi_i} \mathcal{F}. \end{aligned}$$

We call the convergence $\Gamma_c = \bigwedge_{\alpha \in \mathbb{R}} \tau_\alpha$ the *convergence along cones*. Notice that even though each τ_α is a topology, Γ_c is not.

An immediate corollary of the definitions and of Theorem 1 is:

Corollary 2. $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable if and only if

$$\widehat{f} = Id \times f : (\mathbb{R}^2, \tau_1) \rightarrow (\mathbb{R}^2, \Gamma_c)$$

is continuous.

3. CALCULUS TOPOLOGICALLY

A cornerstone of calculus in one variable is Fermat's theorem stating that if f has a local extremum at a , then either f is not differentiable at a or $f'(a) = 0$. We show that Fermat's theorem can be proved topologically via theorem 1.

Proposition 3. Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ has a local extremum at $a \in \mathbb{R}$. Then for each α and λ_0 in \mathbb{R} with $\alpha \neq 0$, the function $\widehat{f} : (\mathbb{R}^2, \tau_1) \rightarrow (\mathbb{R}^2, \tau_\alpha)$ is not continuous at (λ_0, a) .

Proof. Assume that f has a local maximum at a .

Given $\alpha > 0$ and λ_0 , we will show that $\widehat{f}((\lambda_0, a) + N_\delta^1) \not\subseteq \widehat{f}(\lambda_0, a) + N_{\alpha/2}^\alpha$ for each $\delta > 0$, by exhibiting $h \in (0, \delta)$ such that $\widehat{f}((\lambda_0, a) + (h, h)) - \widehat{f}(\lambda_0, a) \notin N_{\alpha/2}^\alpha$.

As f has a local maximum at a , we may choose $h \in (0, \delta)$ small enough that $f(a+h) \leq f(a)$. Then

$$\begin{aligned} \widehat{f}((\lambda_0, a) + (h, h)) - \widehat{f}(\lambda_0, a) &= \widehat{f}(\lambda_0 + h, a + h) - \widehat{f}(\lambda_0, a) \\ &= (\lambda_0 + h, f(a+h)) - (\lambda_0, f(a)) \\ &= (h, f(a+h) - f(a)) \\ &= h \left(1, \frac{f(a+h) - f(a)}{h} \right). \end{aligned}$$

Therefore, $\widehat{f}((\lambda_0, a) + (h, h)) - \widehat{f}(\lambda_0, a) \notin N_{\alpha/2}^\alpha$ because $\frac{f(a+h) - f(a)}{h} \leq 0$ cannot be between $\alpha - \frac{\alpha}{2}$ and $\alpha + \frac{\alpha}{2}$.

A similar argument applies for $\alpha < 0$ using $h \in (-\delta, 0)$, and the proof for a local minimum is analogous. \square

In view of theorem 1, we obtain:

Corollary 4. (Fermat): If $f : \mathbb{R} \rightarrow \mathbb{R}$ has a local extremum at a , then either f is not differentiable at a or $f'(a) = 0$.

Interestingly, many calculus results follow from that fact - now interpreted topologically - combined with a topological argument. For instance, Rolle's theorem (hence the mean value theorem) follows immediately from corollary 4, given the existence of extrema of the function on $[a, b]$, which follows by continuity of the

function and compactness of $[a, b]$. The same is true, among others, for the fact that derivatives have the intermediate value property. In turn, a myriad of results are based on the mean value theorem without further use of the derivative.

REFERENCES

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