Metrisability of Manifolds

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June 29, 2006

Abstract

Manifolds have uses throughout and beyond Mathematics and it is not surprising that topologists have expended a huge effort in trying to understand them. In this article we are particularly interested in the question: 'when is a manifold metrisable?' We describe many conditions equivalent to metrisability.

1 Introduction.

By a *manifold* is meant a connected, Hausdorff space which is locally homeomorphic to euclidean space (we take our manifolds to have no boundary). Note that because of connectedness the dimension of the euclidean space is an invariant of the manifold (unless the manifold is empty!); this is the *dimension* of the manifold. A pair (U, h), where $U \subset M$ is open and $h : U \to \mathbb{R}^m$ is a homeomorphism, is called a *coordinate chart*.

The following notation is used. $\mathbb{R}$ denotes the real line with the usual (order) topology while $\mathbb{R}^n$ denotes the nth power of $\mathbb{R}$. $\mathbb{B}^n$ consists of all points of $\mathbb{R}^n$ at most 1 from the origin. The sets ω and ω_1 are, respectively, the finite and countable ordinals.

Clearly every manifold is Tychonoff. Of course manifolds share all of the local properties of euclidean space, including local compactness, local connectedness, local path or arc connectedness, first countability, local second countability, local hereditary separability, etc. As every manifold is locally compact and Hausdorff, hence completely regular, it follows that every manifold is uniformisable ([34, Proposition 11.5]). The following result shows that manifolds cannot be too big.

Proposition 1 *Let M be a (non-empty!) manifold. Then every countable subset of M is contained in an open subset which is homeomorphic to euclidean space. Hence every two points of M may be joined by an arc.*

Proof. Suppose that the dimension of M is m. Let $S \subset M$ be a countable subset, say $\langle S_n \rangle$ is such that $S = \cup_{n \geq 1} S_n$, $|S_n| = n$ and $S_n \subset S_{n+1}$.

By induction on n we choose open $V_n \subset M$ and compact $C_n \subset M$ such that

$$\text{(i) } S_n \cup C_{n-1} \subset \mathring{C}_n \text{ and (ii) } (V_n, C_n) \approx (\mathbb{R}^m, \mathbb{B}^m),$$

where $C_0 = \varnothing$.

For $n = 1$, S_1 is a singleton so V_1 may be any appropriate neighbourhood of that point while C_1 is a compact neighbourhood chosen to satisfy (ii) as well.

*Supported in part by a Marsden Fund Award, UOA611, from the Royal Society of New Zealand.

†The author thanks Abdul Mohamad, Wang Shuquan and Frédéric Mynard for useful comments.

Suppose that V_n and C_n have been constructed. Consider

$$S = \{x \in M \ / \ \exists \text{ open } U \subset M \text{ with } C_n \cup \{x\} \subset U \approx \mathbb{R}^m\}.$$

S is open. S is also closed, for suppose that $z \in \bar{S} - S$. Then we may choose open $O \subset M$ with $O \approx \mathbb{R}^m$ and $z \in O$. Choose $x \in O \cap S$. Then there is open $U \subset M$ with $C_n \cup \{x\} \subset U \approx \mathbb{R}^m$. We may assume that O is small enough that $O \cap C_n = \varnothing$. Using the euclidean space structure of O we may stretch U within O so as to include z but not uncover any of C_n. Thus $z \in S$.

As M is connected and $S \neq \varnothing$ we must have $S = M$. Thus there is open $V_{n+1} \subset M$ with $S_{n+1} \cup C_n = (S_{n+1} - S_n) \cup C_n \subset V_{n+1} \approx \mathbb{R}^m$. Because $S_{n+1} \cup C_n$ is compact we may find in V_{n+1} a compact subset C_{n+1} so that (i) and (ii) hold with n replaced by $n+1$.

Let $U_n = \mathring{C}_n$. Then U_n is open, $U_n \subset U_{n+1}$ and $U_n \approx \mathbb{R}^m$. Thus by [6], $U = \cup_{n \geq 1} U_n$ is also open with $U \approx \mathbb{R}^m$. Furthermore $S_n \subset U_n$ for each n so that $S \subset U$. ∎

There has been considerable study of metrisable manifolds, especially compact manifolds. In particular it is known that there are only two metrisable manifolds of dimension 1: the circle $\mathbb{S}^1$, where

$$\mathbb{S}^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \ / \ x_0^2 + \cdots + x_n^2 = 1\},$$

and the real line $\mathbb{R}$ itself. In dimension 2 the compact manifolds have also been classified, this time into two sequences: the orientable manifolds, which consist of the 2-sphere $\mathbb{S}^2$ with n handles ($n \in \omega$) sewn on, and the non-orientable manifolds, which consist of the 2-sphere with n cross-caps ($n \in \omega - \{0\}$) sewn on. See [20, Chapter 14 and Appendix B], for example. Despite considerable progress in the study of compact manifolds in higher dimensions there has been no classification even of compact manifolds of dimension 3. Indeed, the 3-dimensional Poincaré conjecture has only now apparently been resolved after about 100 years. The original conjecture, [47], differs slightly from that posed below and was found by Poincaré to be false. Poincaré's counterexample was published in [48], where the following version was also posed. The conjecture says that if a compact manifold M of dimension 3 is such that every continuous function $\mathbb{S}^1 \to M$ extends to a continuous function $\mathbb{B}^2 \to M$, where $\mathbb{B}^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, then M is homeomorphic to $\mathbb{S}^3$. The analogue of this conjecture in dimensions higher than 3 is known to be true, [17, Corollary 7.1B] in dimension 4 and [40, Proposition B, p109] in dimension 5 and higher.

In contrast to the compact situation, where it is known that there are only countably many manifolds [11], in the nonmetrisable case there are $2^{\aleph_1}$ manifolds, even of dimension 2 ([43], p 669). However, there are only two nonmetrisable manifolds of dimension 1, the simpler being the open *long ray*, $\mathbb{L}^+$, [7]. To ease the description of $\mathbb{L}^+$ we firstly give a way of constructing the positive real numbers from the non-negative integers and copies of the unit interval. Between any integer and its successor we insert a copy of the open unit interval. More precisely, let $\mathbb{R}^+ = \omega \times [0, 1) - \{(0, 0)\}$, order $\mathbb{R}^+$ by the lexicographic order from the natural orders on ω and $[0, 1)$ and then topologise $\mathbb{R}^+$ by the order topology. To get $\mathbb{L}^+$ we do the same thing but replace ω by ω_1. More precisely, let $\mathbb{L}^+ = \omega_1 \times [0, 1) - \{(0, 0)\}$, order $\mathbb{L}^+$ by the lexicographic order from the natural orders on ω_1 and $[0, 1)$ and then topologise $\mathbb{L}^+$ by the order topology. The other nonmetrisable manifold of dimension 1 is the *long line*, which is obtained by joining together two copies of $\mathbb{L}^+$ at their (0,0) ends in much the same way as one may reconstruct the real line $\mathbb{R}$ by joining together two copies of $(0, \infty)$ at their 0 ends, thinking of one copy as giving the positive reals and the other the negative reals. More precisely, let $\mathbb{L}$ be the disjoint union of two copies of $\mathbb{L}^+$ (call them $\mathbb{L}^+$ and $\mathbb{L}^-$ respectively, with ordering $<^+$ and $<^-$ respectively) as well as a single point, which we denote by 0, order $\mathbb{L}$ by $x < y$ when $x <^+ y$ in $\mathbb{L}^+$, when $x = 0$ and $y \in \mathbb{L}^+$, when $x \in \mathbb{L}^-$ and $y \in \mathbb{L}^+$ or $y = 0$ and when $x, y \in \mathbb{L}^-$ and $y <^- x$, and topologise $\mathbb{L}$ by the order topology.

The survey articles [43] and [45] are good sources of information about nonmetrisable manifolds.

A significant question in topology is that of deciding when a topological space is metrisable, there being many criteria which have now been developed to answer the question. Perhaps the most natural is the following: a topological space is metrisable if and only if it is paracompact, Hausdorff and locally metrisable, see [54] and [33, Theorem 2.68]. Note that manifolds are always Hausdorff and locally metrisable so this criterion gives a criterion for the metrisability of a manifold, viz that a manifold is metrisable if and only if it is paracompact. Many metrisation criteria have been discovered for manifolds, as seen by Theorem 2 below, which lists criteria which require at least some of the extra properties possessed by manifolds. Of course one must not be surprised if conditions which in general topological spaces are considerably weaker than metrisability are actually equivalent to metrisability in the presence of the extra topological conditions which always hold for a manifold: such a condition is that of being nearly meta-Lindelöf, 10 in Theorem 2 below. Similarly one should not be surprised to find conditions which are normally stronger than metrisability: such a condition is that M may be embedded in euclidean space, 32 in Theorem 2. Finally one may expect to find conditions which in a general topological space have no immediate connection with metrisability: such a condition is second countability, 26 in Theorem 2.

2 Definitions.

In this section we list numerous definitions relevant to the question of metrisability..

Definitions: Let X be a topological space and $\mathcal{F}$ a family of subsets of X. Then:

- X is *submetrisable* if there is a metric topology on X which is contained in the given topology;

- X is *Polish* if X is a separable, complete metric space;

- X is *paracompact* (respectively *metacompact, paraLindelöf* and *metaLindelöf*) if every open cover $\mathcal{U}$ has a locally finite (respectively point finite, locally countable, and point countable) open refinement, ie there is another open cover $\mathcal{V}$ such that each member of $\mathcal{V}$ is a subset of some member of $\mathcal{U}$ and each point of X has a neighbourhood meeting only finitely (respectively lies in only finitely, has a neighbourhood meeting only countably, and lies in only countably) many members of $\mathcal{V}$;

- X is *finitistic* (respectively *strongly finitistic*) if every open cover of X has an open refinement $\mathcal{V}$ and there is an integer m such that each point of X lies in (respectively has a neighbourhood which meets) at most m members of $\mathcal{V}$ (finitistic spaces have also been called *boundedly metacompact* and strongly finitistic spaces have also been called *boundedly paracompact*);

- X is *strongly paracompact* if every open cover $\mathcal{U}$ has a star-finite open refinement $\mathcal{V}$, ie for any $V \in \mathcal{V}$ the set $\{W \in \mathcal{V} \ / \ V \cap W \neq \varnothing\}$ is finite. If in addition, given $\mathcal{U}$, there is an integer m such that $\{W \in \mathcal{V} \ / \ V \cap W \neq \varnothing\}$ contains at most m members then X is *star finitistic*;

- X is *screenable* (respectively *σ-metacompact* and *σ-paraLindelöf*) if every open cover $\mathcal{U}$ has an open refinement $\mathcal{V}$ which can be decomposed as $\mathcal{V} = \cup_{n\in\omega}\mathcal{V}_n$ such that each $\mathcal{V}_n$ is disjoint (respectively point finite and locally countable);

- X is *(linearly) [ω_1-]Lindelöf* if every open cover (which is a chain) [which has cardinality ω_1] has a countable subcover;

- X is *(nearly) [linearly ω_1-]metaLindelöf* if every open cover $\mathcal{U}$ of X [with $|\mathcal{U}| = \omega_1$] has an open refinement which is point-countable (on a dense subset);

- X is *almost metaLindelöf* if for every open cover $\mathcal{U}$ there is a collection $\mathcal{V}$ of open subsets of X such that each member of $\mathcal{V}$ lies in some member of $\mathcal{U}$, that each point of X lies in at most countably many members of $\mathcal{V}$, and that $X = \bigcup\{\overline{V} \ / \ V \in \mathcal{V}\}$.

- X is *(strongly) hereditarily Lindelöf* if every subspace (of the countably infinite power) of X is Lindelöf;

- X is *k-Lindelöf* provided every open k-cover (ie every compact subset of X lies in some member of the cover) has a countable k-subcover;

- X is *(strongly) hereditarily separable* if every subspace (of the countably infinite power) of X is separable;

- X is *Hurewicz* if for each sequence $\langle \mathcal{U}_n \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n \rangle$ such that $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ for each $n \in \omega$ and $\cup_{n\in\omega}\mathcal{V}_n$ covers X (note the alternative definition of Hurewicz, [14]: X is *Hurewicz* if for each sequence $\langle \mathcal{U}_n \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n \rangle$ such that $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$ we have $x \in \cup\mathcal{V}_n$ for all but finitely many $n \in \omega$. For a manifold these two conditions are equivalent.);

- X is *hemicompact* if there is an increasing sequence $\langle K_n \rangle$ of compact subsets of X such that for any compact $K \subset X$ there is n such that $K \subset K_n$;

- X is *cosmic* if there is a countable family $\mathcal{C}$ of closed subsets of X such that for each point $x \in X$ and each open set U containing x there is a set $C \in \mathcal{C}$ such that $x \in C \subset U$;

- X is an *$\aleph_0$-space* ([29, page 493]) provided that it has a countable k-network, i.e. a countable collection $\mathcal{N}$ such that if $K \subset U$ with K compact and U open then $K \subset N \subset U$ for some $N \in \mathcal{N}$;

- X is an *$\aleph$-space* ([29, page 493]) provided that it has a σ-locally finite k-network;

- X has the *Moving Off Property*, [31], provided that every family $\mathcal{K}$ of non-empty compact subsets of X large enough to contain for each compact $C \subset X$ a disjoint $K \in \mathcal{K}$ has an infinite subfamily with a discrete open expansion;

- X is a *q-space* if each point admits a sequence of neighbourhoods Q_n such that $x_n \in Q_n$ implies that $\langle x_n \rangle$ clusters;

- X is *Fréchet* or *Fréchet-Urysohn* if whenever $x \in \overline{A}$ there is a sequence $\langle x_n \rangle$ in A that converges to x;

- X is a *k-space* if A is closed whenever $A \cap K$ is closed for every compact subset $K \subset X$;

- X is *Lašnev* if it is the image of a metrisable space under a closed map;

- X is *analytic* if it is the continuous image of a Polish space (equivalently of the irrational numbers);

- X is M_1 if it has a σ-closure preserving base (ie a base $\mathcal{B}$ such that there is a decomposition $\mathcal{B} = \cup_{n=1}^{\infty}\mathcal{B}_n$ where for each n and each $\mathcal{F} \subset \mathcal{B}_n$ we have $\overline{\cup\mathcal{F}} = \cup\{\bar{F} \ / \ F \in \mathcal{F}\}$);

- X is *stratifiable* or M_3 if there is a function G which assigns to each $n \in \omega$ and closed set $A \subset X$ an open set $G(n, A)$ containing A such that $A = \cap_n \overline{G(n, A)}$ and if $A \subset B$ then $G(n, A) \subset G(n, B)$;

- X is *perfectly normal* if for every pair A, B of disjoint closed subsets of X there is a continuous function $f : X \to \mathbb{R}$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$;

- X is *monotonically normal* if for each open $U \subset X$ and each $x \in U$ it is possible to choose an open set $\mu(x, U)$ such that $x \in \mu(x, U) \subset U$ and such that if $\mu(x, U) \cap \mu(y, V) \neq \varnothing$ then either $x \in V$ or $y \in U$;

- X is *extremely normal* if for each open $U \subset X$ and each $x \in U$ it is possible to choose an open set $\nu(x, U)$ such that $x \in \nu(x, U) \subset U$ and such that if $\nu(x, U) \cap \nu(y, V) \neq \varnothing$ and $x \neq y$ then either $\nu(x, U) \subset V$ or $\nu(y, V) \subset U$;

- X is *weakly normal* if for every pair A, B of disjoint closed subsets of X there is a continuous function $f : X \to S$, for some separable metric space S, such that $f(A) \cap f(B) = \varnothing$;

- X is a *Moore space* if it is regular and has a development, ie a sequence $\langle \mathcal{U}_n \rangle$ of open covers such that for each $x \in X$ the collection $\{st(x, \mathcal{U}_n) \ : \ n \in \omega\}$ forms a neighbourhood basis at x;

- X has a *regular G_δ-diagonal* if the diagonal Δ is a *regular G_δ-subset* of X^2, ie there is a sequence $\langle U_n \rangle$ of open subsets of X^2 such that $\Delta = \cap U_n = \cap \overline{U_n}$.

- X has a *quasi-regular G_δ-diagonal* if there is a sequence $\langle U_n \rangle$ of open subsets of X^2 such that for each $(x, y) \in X^2 - \Delta$ there is n with $(x, x) \in U_n$ but $(x, y) \notin \overline{U_n}$.

- X has a *G_δ^*-diagonal* if there is a sequence $\langle \mathcal{G}_n \rangle$ of open covers of X such that for each $x, y \in X$ with $x \neq y$ there is n with $\overline{\mathrm{st}(x, \mathcal{G}_n)} \subset X - \{y\}$.

- X has a *quasi-G_δ^*-diagonal* if there is a sequence $\langle \mathcal{G}_n \rangle$ of families of open subsets of X such that for each $x, y \in X$ with $x \neq y$ there is n with $x \in \overline{\mathrm{st}(x, \mathcal{G}_n)} \subset X - \{y\}$.

- X is *θ-refinable* if every open cover can be refined to an open θ-cover, ie a cover $\mathcal{U}$ which can be expressed as $\cup_{n\in\omega}\mathcal{U}_n$ where each $\mathcal{U}_n$ covers X and for each $x \in X$ there is n such that $ord(x, \mathcal{U}_n) < \omega$;

- X is *subparacompact* if every open cover has a σ-discrete closed refinement;

- X has *property pp*, [38], provided that each open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ such that for each choice function $f : \mathcal{V} \to X$ with $f(V) \in V$ for each $V \in \mathcal{V}$ the set $f(\mathcal{V})$ is closed and discrete in X;

- X has *property (a)*, [38], provided that for each open cover $\mathcal{U}$ of X and each dense subset $D \subset X$ there is a subset $C \subset D$ such that C is closed and discrete in X and $st(C, \mathcal{U}) = X$;

- X has a *base of countable order*, $\mathcal{B}$, if whenever $\mathcal{C} \subset \mathcal{B}$ is a collection such that each member of $\mathcal{C}$ contains a particular point $p \in X$ and for each $C \in \mathcal{C}$ there is $D \in \mathcal{C}$ with D a proper subset of C then $\mathcal{C}$ is a local base at p;

- X is *pseudocomplete* provided that it has a sequence $\langle \mathcal{B}_n \rangle$ of π-*bases* ($\mathcal{B} \subset 2^X - \{\varnothing\}$ is a π-*base* if every non-empty open subset of X contains some member of $\mathcal{B}$) such that if $B_n \in \mathcal{B}_n$ and $\overline{B_{n+1}} \subset B_n$ for each n, then $\bigcap_{n\in\omega} B_n \neq \varnothing$;

- X has the *countable chain condition* (abbreviated *ccc*) if every pairwise disjoint family of open subsets is countable;

- X is *countably tight* if for each $A \subset X$ and each $x \in \bar{A}$ there is a countable $B \subset A$ for which $x \in \bar{B}$;

- X is *countably fan tight* if whenever $x \in \cap_{n\in\omega}\overline{A_n}$ there are finite sets $B_n \subset A_n$ such that $x \in \overline{\cup_{n\in\omega} B_n}$;

- X is *countably strongly fan tight* if whenever $x \in \cap_{n\in\omega}\overline{A_n}$ there is a sequence $\langle a_n \rangle$ such that $a_n \in A_n$ for each n and $x \in \overline{\{a_n \ / \ n \in \omega\}}$;

- X is *sequential* if for each $A \subset X$, the set A is closed whenever for each sequence of points of A each limit point is also in A;

- X is *weakly α-favourable* if there is a winning strategy for player α in the Banach-Mazur game (defined below);

- X is *strongly α-favourable* if there is a stationary winning strategy for player α in the Choquet game (defined below);

- for each $x \in X$ the *star* of x in $\mathcal{F}$ is $st(x, \mathcal{F}) = \cup\{F \in \mathcal{F} \ : \ x \in F\}$;

- X is *Baire* provided that the intersection of any countable collection of dense G_δ subsets is dense;

- X is *Volterra*, [25], provided that the intersection of any two dense G_δ subsets is dense;

- X is *strongly Baire* provided that X is regular and there is a dense subset $D \subset X$ such that β does not have a winning strategy in the game $\mathcal{G}_S(D)$ played on X.

- $\mathcal{F}$ is *point-star-open* if for each $x \in X$ the set $st(x, \mathcal{F})$ is open.

- The *Banach-Mazur game* has two players α and β whose play alternates. Player β begins by choosing a non-empty open subset of X. After that the players choose successive non-empty open subsets of their opponent's previous move. *Player α wins* iff the intersection of the sets is non-empty; otherwise *player β wins*.

- The *Choquet game* has two players α and β whose play alternates. Player β begins by choosing a point in an open subset of X, say $x_0 \in V_0 \subset X$. After that the players alternate with α choosing an open set $U_n \subset X$ with $x_n \in U_n \subset V_n$ then β chooses a point x_{n+1} and an open set V_{n+1} with $x_{n+1} \in V_{n+1} \subset U_n$. *Player α wins* iff the intersection of the sets is non-empty; otherwise *player β wins*.

- Gruenhage's game $G^o_{K,L}(X)$, [30], has, at the n^{th} stage, player K choose a compactum $K_n \subset X$ after which player L chooses another compactum $L_n \subset X$ so that $L_n \cap K_i = \varnothing$ for each $i \leq n$. *Player K wins* if $\langle L_n \rangle_{n\in\omega}$ has a discrete open expansion, ie there is a sequence $\langle U_n \rangle_{n\in\omega}$ of open sets such that $L_n \subset U_n$ and $\forall x \in X, \exists U \subset M$ open such that $x \in U$ and U meets at most one of the sets U_n.

- For a dense subset $D \subset X$ the game $\mathcal{G}_S(D)$ has two players α and β whose play alternates. Player β begins by choosing a non-empty open subset V_n of X. After that the players choose successive non-empty open subsets of their opponent's previous move, β choosing sets V_n and α choosing sets U_n. *Player α wins* iff the intersection of the sets is non-empty and each sequence $\langle x_n \rangle$, for which $x_n \in U_n \cap D$, clusters in X; otherwise *player β wins.*

- When players α and β play a topological game, a *strategy for α* is a function which tells α what points or sets to select given all the previous points and sets chosen by β. A *stationary strategy for α* is a function which tells α what points or sets to select given only the most recent choice of points and sets chosen by β. A *winning (stationary) strategy* for α is a (stationary) strategy which guarantees that α will win whatever moves β might make.

We will denote by $C_k(X,Y)$ (respectively $C_p(X,Y)$) the space of all continuous functions from X to Y with the compact-open topology (respectively the topology of pointwise convergence).

We will denote by \$ the space $\{0,1\}$ with the Sierpinski topology $\{\varnothing, \$, \{0\}\}$. Then for any space X we denote by $[X, \$]$ the space of continuous functions from X to \$ with the *upper Kuratowski topology*, i.e. that in which a subset $\mathcal{F} \subset [X, \$]$ is open if and only if

(i) for each $f \in \mathcal{F}$ and each $g \in [X, \$]$ if $g \leq f$ then $g \in \mathcal{F}$;

(ii) if $\mathcal{G} \subset [X, \$]$ is such that $\inf \mathcal{G} \in \mathcal{F}$ then there is a finite subfamily $\mathcal{G}' \subset \mathcal{G}$ with $\inf \mathcal{G}' \in \mathcal{F}$.

In this definition we are using the usual ordering on $\{0,1\}$ when discussing $\leq$ and inf. Of course identifying a closed subset of X with its characteristic function gives a bijective correspondence between $[X, \$]$ and the collection of closed subsets of X.

3 Criteria for metrisability.

We now state and outline the proof of the main theorem. It is believed that no two conditions are equivalent in a general topological space, though, as will be noticed at the start of the proof of Theorem 2, there may be a chain of implications holding in a general space.

Theorem 2 *Let M be a manifold. Then the following are equivalent:*

1. *M is metrisable;*
2. *M is paracompact;*
3. *M is strongly paracompact;*
4. *M is screenable;*
5. *M is metacompact;*
6. *M is σ-metacompact;*
7. *M is paraLindelöf;*
8. *M is σ-paraLindelöf;*
9. *M is metaLindelöf;*

10. *M is nearly metaLindelöf;*

11. *M is Lindelöf;*

12. *M is linearly Lindelöf;*

13. *M is ω_1-Lindelöf;*

14. *M is ω_1-metaLindelöf;*

15. *M is nearly linearly ω_1-metaLindelöf;*

16. *M is almost metaLindelöf;*

17. *M is hereditarily Lindelöf;*

18. *M is strongly hereditarily Lindelöf;*

19. *M is k-Lindelöf;*

20. *M is an $\aleph_0$-space;*

21. *M is cosmic;*

22. *M is an $\aleph$-space;*

23. *M has a star-countable k-network;*

24. *M has a point-countable k-network;*

25. *M has a k-network which is point-countable on some dense subset of M;*

26. *M is second countable;*

27. *M is hemicompact;*

28. *M is σ-compact;*

29. *M is Hurewicz;*

30. *M satisfies the selection criterion $\mathsf{S}_1(\mathcal{K}, \Gamma)$: for each sequence $\langle \mathcal{U}_n \rangle$ of open k-covers of X there is a sequence $\langle U_n \rangle$ with $U_n \in \mathcal{U}_n$ for each n, infinitely many of the sets U_n are distinct and each finite subset of X lies in all but finitely many of the sets U_n;*

31. *only countably many coordinate charts are needed to cover M;*

32. *M may be embedded in some euclidean space;*

33. *M may be embedded properly in some euclidean space;*

34. *M is completely metrisable;*

35. *there is a continuous discrete map $f : M \to X$ where X is Hausdorff and second countable;*

36. *M is Lašnev;*

37. *M is an M_1-space;*

38. *M is stratifiable;*

39. M *is finitistic;*

40. M *is strongly finitistic;*

41. M *is star finitistic;*

42. *there is an open cover* $\mathcal{U}$ *of* M *such that for each* $x \in M$ *the set* $st(x,\mathcal{U})$ *is homeomorphic to an open subset of* $\mathbb{R}^m$*;*

43. *there is a point-star-open cover* $\mathcal{U}$ *of* M *such that for each* $x \in M$ *the set* $st(x,\mathcal{U})$ *is Lindelöf;*

44. *there is a point-star-open cover* $\mathcal{U}$ *of* M *such that for each* $x \in M$ *the set* $st(x,\mathcal{U})$ *is metrisable;*

45. *the tangent microbundle on* M *is equivalent to a fibre bundle;*

46. M *is a normal Moore space;*

47. M *is a normal* θ*-refinable space;*

48. M *is a normal subparacompact space;*

49. M *is a normal space which has a* σ*-discrete cover by compact subsets;*

50. $M \times M$ *is perfectly normal;*

51. M *is a normal space which has a sequence* $\langle \mathcal{U}_n \rangle_{n\in\omega}$ *of open covers with* $\cap_n \overline{st(x,\mathcal{U}_n)} = \{x\}$ *for each* $x \in M$*;*

52. M *is perfectly normal and there is a sequence* $\langle \mathcal{U}_n \rangle_{n\in\omega}$ *of families of open sets such that* $\cap_{n\in C(x)} \overline{st(x,\mathcal{U}_n)} = \{x\}$ *for each* $x \in M$*, where*

$$C(x) = \{n \in \omega \;/\; \exists U \in \mathcal{U}_n \text{ with } x \in U\};$$

53. M *is separable and there is a sequence* $\langle \mathcal{C}_n \rangle_{n\in\omega}$ *of point-star-open covers such that* $\cap_n \overline{st(x,\mathcal{C}_n)} = \{x\}$ *for each* $x \in M$ *and for each* $x, y \in M$ *and each* $n \in \omega$ *we have* $y \in \overline{st(x,\mathcal{C}_n)}$ *if and only if* $x \in \overline{st(y,\mathcal{C}_n)}$*;*

54. M *is separable and there is a sequence* $\langle \mathcal{C}_n \rangle_{n\in\omega}$ *of point-star-open covers such that* $\cap_n \overline{st(x,\mathcal{C}_n)} = \{x\}$ *for each* $x \in M$ *and for each* $x \in M$ *and each* $n \in \omega$*,* $ord(x,\mathcal{C}_n)$ *is finite;*

55. M *is separable and hereditarily normal and there is a sequence* $\langle \mathcal{C}_n \rangle_{n\in\omega}$ *of point-star-open covers such that* $\cap_n \overline{st(x,\mathcal{C}_n)} = \{x\}$ *for each* $x \in M$*;*

56. M *is separable and there is a sequence* $\langle \mathcal{U}_n \rangle_{n\in\omega}$ *of families of open sets such that* $\cap_{n\in C(x)} \overline{st(x,\mathcal{U}_n)} = \{x\}$ *for each* $x \in M$*, and* $ord(x,\mathcal{C}_n)$ *is countable for each* $x \in M$ *and each* $n \in \omega$*;*

57. $M \times M$ *has a countable sequence* $\langle U_n : n \in \omega \rangle$ *of open subsets, such that for all* $(x,y) \in M \times M - \Delta$*, there is* $n \in \omega$ *such that* $(x,x) \in U_n$ *but* $(x,y) \notin \overline{U_n}$*;*

58. *For every subset* $A \subset M$ *there is a continuous injection* $f : M \to Y$*, where* Y *is a metrisable space, such that* $f(A) \cap f(M - A) = \varnothing$*;*

59. *For every subset $A \subset M$ there is a continuous $f : M \to Y$, where Y is a space with a quasi-regular-G_δ-diagonal, such that $f(A) \cap f(M - A) = \varnothing$;*

60. *M is weakly normal with a G_δ^*-diagonal;*

61. *M has a quasi-G_δ^*-diagonal and for every closed subset $A \subset M$ there is a countable family $\mathcal{G}$ of open subsets such that, for every $x \in A$ and $y \in X - A$, there is a $G \in \mathcal{G}$ with $x \in G, y \notin \overline{G}$;*

62. *M has a regular G_δ-diagonal;*

63. *M is submetrisable;*

64. *M is separable and monotonically normal;*

65. *$M \times M$ is monotonically normal;*

66. *M is monotonically normal and of dimension ≥ 2 or $M \approx \mathbb{S}^1$ or $\mathbb{R}$;*

67. *M is extremely normal;*

68. *M has the Moving Off Property;*

69. *M has property pp;*

70. *every open cover of M has an open refinement $\mathcal{V}$ such that for every choice function $f : \mathcal{V} \to M$ the set $f(\mathcal{V})$ is closed in M;*

71. *every open cover of M has an open refinement $\mathcal{V}$ such that for every choice function $f : \mathcal{V} \to M$ the set $f(\mathcal{V})$ is discrete in M;*

72. *M is a point-countable union of open subspaces each of which is metrisable;*

73. *M has a point-countable basis;*

74. *M is separable and M^ω is a countable union of metrisable subspaces;*

75. *$C_k(M, \mathbb{R})$ is Polish;*

76. *$C_k(M, \mathbb{R})$ is completely metrisable;*

77. *$C_k(M, \mathbb{R})$ is first countable;*

78. *$C_k(M, \mathbb{R})$ is second countable;*

79. *$C_k(M, \mathbb{R})$ is a q-space;*

80. *$C_k(M, \mathbb{R})$ is Fréchet;*

81. *$C_k(M, \mathbb{R})$ is countably tight;*

82. *$C_k(M)$ has countable strong fan tightness;*

83. *$C_k(M, \mathbb{R})$ is an $\aleph_0$-space;*

84. *$C_k(M, \mathbb{R})$ is cosmic;*

85. *$C_k(M, \mathbb{R})$ is analytic;*

86. $C(M,\mathbb{R})$ *satisfies the selection criterion* $\mathsf{S}_1(\Omega^k_{\underline{0}}, \Sigma^p_{\underline{0}})$*: for each sequence* $\langle F_n\rangle$ *of subsets of* $C(M,\mathbb{R})$ *whose compact-open closures contain the constant function* $\underline{0}$ *there is a sequence* $\langle f_n\rangle$*, infinitely many members of which are distinct, with* $f_n \in F_n$ *for all* n *and* $\langle f_n\rangle$ *converges pointwise to* $\underline{0}$*;*

87. $C_p(M,\mathbb{R})$ *has countable tightness;*

88. $C_p(M,\mathbb{R})$ *has countable fan tightness;*

89. $C_p(M,\mathbb{R})$ *is analytic;*

90. $C_p(M,\mathbb{R})$ *is hereditarily separable;*

91. $C_p(M,\mathbb{R})$ *(equivalently* $C_k(M,\mathbb{R})$*) is separable;*

92. $[M, \$]$ *is first countable;*

93. $[M, \$]$ *is countably tight;*

94. $[M, \$]$ *is sequential;*

95. K *has a winning strategy in Gruenhage's game* $G^o_{K,L}(M)$*;*

96. $C_k(M,\mathbb{R})$ *is strongly* α*-favourable;*

97. $C_k(M,\mathbb{R})$ *is weakly* α*-favourable;*

98. $C_k(M,\mathbb{R})$ *is pseudocomplete;*

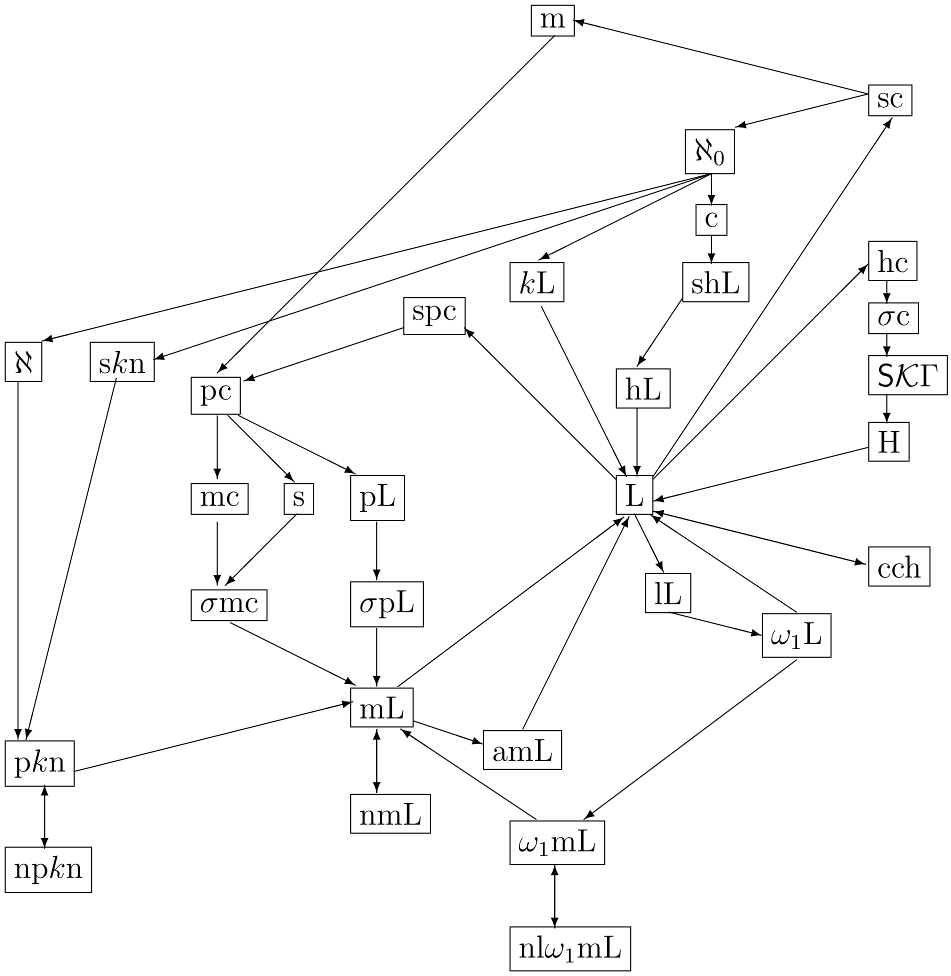
99. $C_k(M,\mathbb{R})$ *is strongly Baire;*

100. $C_k(M,\mathbb{R})$ *is Baire;*

101. $C_k(M,\mathbb{R})$ *is Volterra.*

Outline of the proof of theorem 2.

The following diagram shows how items 1-31 are related:



$\boxed{\text{m}}$=metrisable; $\boxed{\text{sc}}$=second countable; $\boxed{\text{shL}}$=strongly hereditarily Lindelöf; $\boxed{\text{hL}}$=hereditarily Lindelöf; $\boxed{\sigma\text{c}}$=σ-compact; $\boxed{\text{hc}}$=hemicompact; $\boxed{\text{H}}$=Hurewicz; $\boxed{\mathsf{S}\mathcal{K}\Gamma}$=satisfies $\mathsf{S}_1(\mathcal{K},\Gamma)$; $\boxed{\text{cch}}$=countably many charts cover; $\boxed{\aleph_0}$=$\aleph_0$-space; $\boxed{\aleph}$=$\aleph$-space; $\boxed{\text{s}k\text{n}}$=has a star-countable k-network; $\boxed{\text{p}k\text{n}}$=has a point-countable k-network; $\boxed{\text{np}k\text{n}}$=has a k-network which is point-countable on a dense subset; $\boxed{k\text{L}}$=k-Lindelöf; $\boxed{\text{c}}$=cosmic; $\boxed{\text{L}}$=Lindelöf; $\boxed{\text{lL}}$=linearly Lindelöf; $\boxed{\omega_1\text{L}}$=ω_1-Lindelöf; $\boxed{\text{spc}}$=strongly paracompact; $\boxed{\text{pc}}$=paracompact; $\boxed{\text{mc}}$=metacompact; $\boxed{\text{s}}$=screenable; $\boxed{\text{pL}}$=paraLindelöf; $\boxed{\sigma\text{mc}}$=σ-metacompact; $\boxed{\sigma\text{pL}}$=σ-paraLindelöf; $\boxed{\text{mL}}$=metaLindelöf; $\boxed{\text{amL}}$=almost metaLindelöf; $\boxed{\text{nmL}}$=nearly metaLindelöf; $\boxed{\omega_1\text{mL}}$=$\omega_1$-metaLindelöf; $\boxed{\text{nl}\omega_1\text{mL}}$=nearly linearly ω_1-metaLindelöf.

All arrows denote implications. Downward sloping arrows show an implication which holds in an arbitrary topological space. Upward sloping arrows require one or more properties of manifolds to realise the implication. $\boxed{\text{mL}}\Rightarrow\boxed{\text{L}}$ in every locally separable and connected space. $\boxed{\text{amL}}\Rightarrow\boxed{\text{L}}$ in every regular, locally separable and connected space, [23]. $\boxed{\text{nmL}}\Rightarrow\boxed{\text{mL}}$ in every locally hereditarily separable space. $\boxed{\text{L}}\Rightarrow\boxed{\text{spc}}$ in every T_3 space. $\boxed{\omega_1\text{L}}\Rightarrow\boxed{\text{L}}$ in every locally metrisable space, [4]. $\boxed{\text{L}}\Rightarrow\boxed{\text{sc}}$ in every locally second countable space. $\boxed{\text{L}}\Rightarrow\boxed{\text{hc}}$ in every locally compact space. $\boxed{\text{cch}}\Rightarrow\boxed{\text{L}}$ because a countable union of Lindelöf sets is Lindelöf. $\boxed{\text{sc}}\Rightarrow\boxed{\text{m}}$ in every T_3 space (Urysohn's metrisation theorem). $\boxed{\omega_1\text{mL}}\Rightarrow\boxed{\text{mL}}$ in every locally second countable space, [27]. $\boxed{\text{nl}\omega_1\text{mL}}\Rightarrow\boxed{\omega_1\text{mL}}$ in every locally hereditarily separable space, [27]. $\boxed{\text{p}k\text{n}}\Rightarrow\boxed{\text{mL}}$ in every regular Fréchet space. $\boxed{\text{np}k\text{n}}\Rightarrow\boxed{\text{p}k\text{n}}$ in every regular, locally compact, locally hereditarily separable space.

By [46, Proposition 7.3.9] we conclude that a metrisable n-manifold, being separable and of covering dimension n, embeds in $\mathbb{R}^{2n+1}$, so 1⇒32. By choosing a proper continuous real-valued function on M we can add a further coordinate to embed M in $\mathbb{R}^{2n+2}$ so that the image is closed, ie the embedding is proper, hence 1⇒33. It is clear that 33⇒34.

Every second countable Hausdorff space satisfies 35 so 26⇒35. Conversely, given the situation of 35, if $\mathcal{B}$ is a countable base for the topology on X then the Poincaré-Volterra Lemma of [16, Lemma 23.2] asserts that

$$\{U \subset M \quad / \quad U \text{ is second countable and there is } V \in \mathcal{B} \text{ such that } U \text{ is a component of } f^{-1}(V)\}$$

is a countable base for M.

Clearly every metrisable space is Lašnev so 1⇒36. The implication 36⇒37 is [29, Theorem 5.5]. It is easy to show that 37⇒38. The implication 38⇒2 is [29, Theorem 5.7].

The conditions 1, 39, 40 and 41 are shown to be equivalent in [13].

The equivalence of conditions 1 and 42-45 is established as follows: 1⇒42 is reasonably straightforward making use of the fact that metrisable manifolds are σ-compact. Then 42⇒43 is trivial. 43⇒44 requires use of Urysohn's metrisation theorem to deduce that the Lindelöf stars are metrisable. 44⇒11 requires some delicate manoeuvres; see [24]. 45⇒42 is also found in [24] while 1⇒45 is [36, Corollary 2].

The implication 1⇒46 holds in every topological space while its converse holds provided that the space is locally compact and locally connected, [49] or [50, Theorem 3.4]. The equivalence of 46 and 47 comes from [56, Theorem 3], while the equivalence of 46, 47, 48 and 49 is discussed in [44, Theorem 8.2].

The equivalence of conditions 1, 50 and 51 is referred to briefly in [21]. The implications 1⇒50⇒51 hold in any topological space and the implication 51⇒1 uses some properties of a manifold.

The equivalence of conditions 1 and 52-55 is discussed in [41].

Proofs of the equivalence of 1 and 56 may be found in [19] and of 1 and 57-61 may be found in [18].

The implication 62⇒1 holds in every locally compact, locally connected space ([29, Theorem 2.15(b)]) and, as noted in [29, p. 430], every submetrisable space has a regular G_δ-diagonal so 63⇒62.

Every metric space is monotonically normal and every metrisable manifold is second countable, hence separable, so 1⇒64. To get the converse implication 64⇒2 use is made of the fact that every monotonically normal space is hereditarily collectionwise normal ([32]), and hence no separable monotonically normal space contains a copy of ω_1. On the other hand in [5, Theorem I] it is shown that a monotonically normal space is paracompact if and only if it does not contain a stationary subset of a regular uncountable ordinal.

If M is metrisable, so is $M \times M$, so that $M \times M$ is monotonically normal and hence 1⇒65. The converse follows from a metrisability result of [32] as manifolds are locally countably compact.

The criterion 66 is [5, corollary 2.3(e)], except that we have listed all of the metrisable 1-manifolds.

Every metrisable space is extremely normal. The implication 67⇒2 is found in [57].

The equivalence of conditions 1, 68 and 100 is discussed in [8].

It is readily shown that every T_1-space which is paracompact has property pp. We now obtain the implication 69⇒5. Suppose that $\mathcal{U}$ is an open cover of M. Use the property pp to find an open refinement $\mathcal{V}$ such that for each choice function $f : \mathcal{V} \to M$ with $f(V) \in V$ for each $V \in \mathcal{V}$ the set $f(\mathcal{V})$ is closed and discrete. We will show that $\mathcal{V}$ is point-finite. Suppose to the contrary that $x \in M$ is such that $\{V \in \mathcal{V} \ / \ x \in V\}$ is infinite; let $\langle V_n \rangle$ be a sequence of distinct members of $\mathcal{V}$ each of which contains x. Because M is a manifold, hence first countable, we may choose a countable neighbourhood basis $\{W_n \ / \ n \in \omega\}$ at x. Note that for each n, $V_n \cap W_n - \{x\} \neq \varnothing$ as M has no isolated points. Choose a function $f : \mathcal{V} \to M$ as follows:

if $V \in \mathcal{V}$ but $V \neq V_n$ for each n then choose $f(V) \in V - \{x\}$ arbitrarily; if $V = V_n$ choose $f(V_n) \in V_n \cap W_n - \{x\}$. Then $x \in \overline{f(\mathcal{V})} - f(\mathcal{V})$ so that $f(\mathcal{V})$ is not closed, contrary to the choice of $\mathcal{V}$. Thus $\mathcal{V}$ is point-finite so M is metacompact.

It is easy to show that conditions 70 and 71 are equivalent to each other, and hence also to 69; cf [22, Lemma 2.3].

Details for the implication 72⇒9 appear in [24], while details for the implication 73⇒1 appear in [15]. Of course 26⇒73.

The implication 74⇒1 is a consequence of the more general result that if the countable power of a topological space X is a countable union of metrisable subspaces and in X discrete families of open sets are countable then X is metrisable, [55].

The equivalence of conditions 1 and 75 to 94, excluding 77, 82 and 86, is shown in [26]. A number of properties of manifolds are required, including that every manifold is a q-space and a k-space, and some of the equivalences to metrisability already proved.

Conditions 77 and 82 are shown to be equivalent to condition 11 in [10, Theorem 6] using Hausdorffness, local compactness and first countability of manifolds.

In [10, Theorem 15] there is a proof that in a Tychonoff space 30 and 86 are equivalent.

The implication 1⇒96 follows from 75 and [35, Theorem 8.17]. 96⇒97 is trivial. 97⇒95 is [30, Lemma 4.3]. 95⇒2 is [30, Theorem 4.1].

Complete metrisability implies pseudocompleteness in any space and in turn pseudocompleteness implies α-favourability in a regular space, so 76⇒98⇒97.

The implications 34⇒99 and 99⇒28 are shown in [8, Theorem 2.2].

The equivalence of 100 was already considered above in the context of 68.

Clearly every Baire space is Volterra and the converse holds in any locally convex topological vector space, [9, Theorem 3.4] so 100⇔101. ∎

4 Other properties of manifolds.

In this section we collect a few more properties which we may hope a manifold to possess.

Theorem 3 *Every manifold has a base of countable order.*

Proof: By [56, Theorem 2] every metric space has a base of countable order. As every manifold is locally metrisable it follows from [56, Theorem 1] that every manifold has a base of countable order. ∎

Some standard conditions which manifolds may possess but which are weaker than metrisability are contained in the following theorem.

Theorem 4 *Suppose that the manifold M is metrisable. Then M is also normal, hereditarily normal, perfectly normal, separable, strongly hereditarily separable and has property (a).*

Proof outline: Every metrisable space is perfectly normal, normal and hereditarily normal. Every second countable space is separable and strongly hereditarily separable. Theorem 2(69) shows that metrisable manifolds satisfy property pp while in [22, Proposition 2.1] it is shown that every space having property pp has property (a). ∎

There are manifolds which are normal but not metrisable, for example the long ray. The long ray also has property (a) (and, as shown in [22], even the stronger properties a-favourable and strongly a-favourable found in [38]).

The observant reader may have noticed that separability is absent as a criterion for metrisability in Theorem 2. The following example shows that it must be.

Example 5 *There is a manifold which is separable but not metrisable.*

One can make such a manifold out of the plane by replacing each point of the y-axis by an interval as follows. Let $S = \{(x,y) \in \mathbb{R}^2 \ / \ x \neq 0\}$ with the usual topology from $\mathbb{R}^2$. Let $M = S \cup [\{0\} \times \mathbb{R}^2]$. For each $(0,\eta,\zeta) \in \mathbb{R}^3$ and each $r > 0$ let

$$W_{\eta,\zeta,r} = \{(x,y) \in S \ / \ \zeta - r < \frac{y-\eta}{|x|} < \zeta + r \text{ and } |x| < r\} \cup [\{0\} \times \{\eta\} \times (\zeta - r, \zeta + r)].$$

Topologise M by declaring $U \subset M$ to be open if and only if $U \cap S$ is open in S and for each $(0,\eta,\zeta) \in U \cap (M - S)$ there is $r > 0$ so that $W_{\eta,\zeta,r} \subset U$. Then M is a separable 2-manifold.

There are even manifolds which are both normal and separable but not metrisable, [52].

We need now some facts from Set Theory. The Continuum Hypothesis (CH), dating back to Cantor, states that any subset of $\mathbb{R}$ either has the same cardinality as $\mathbb{R}$ or is countable. Martin's Axiom (MA) can be expressed in various forms, the most topological of which is the following: in every compact, ccc, Hausdorff space the intersection of fewer than $2^{\aleph_0}$ dense open sets is dense. Recall the Baire Category Theorem which states that if X is Čech complete (ie X is a G_δ-set in βX; for example every locally compact, Hausdorff space or every complete metric space) and $\{U_n \ / \ n \in \omega\}$ is a collection of open dense subsets of X then $\cap_{n\in\omega} U_n$ is dense in X. From the Baire Category theorem it is immediate that CH⇒MA. Both CH and MA are independent of the axioms of ZFC and otherwise of each other: thus there are models of Set Theory satisfying ZFC in which CH (and hence MA) holds, models in which MA holds but CH fails (denoted MA+¬CH), and models in which MA (and hence CH) fails.

The question whether perfect normality is equivalent to metrisability for a manifold is an old one, dating back to [1]. It was shown in [51] that under MA+¬CH the two conditions are equivalent. On the other hand in [53] there is constructed an example of a perfectly normal non-metrisable manifold under CH. The same situation prevails when we consider strong hereditary separability. In [37] it is shown that under MA+¬CH every strongly hereditarily separable space is Lindelöf. On the other hand even when we combine the two notions the resulting manifold need not be metrisable in general; in [21] there is constructed under CH a manifold which is strongly hereditarily separable and perfectly normal but not metrisable. There are many other examples of conditions which are equivalent to metrisability for manifolds in some models of Set Theory but not equivalent in other models.

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