The Torsion of the Group of Homeomorphisms of Powers of the Long Line

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Abstract

By blending techniques from Set Theory and Algebraic Topology we investigate the order of any homeomorphism of the nth power of the long ray or long line $\mathbb{L}$ having finite order, finding all possible orders when $n = 1, 2, 3$ or 4 in the first case and when $n = 1$ or 2 in the second. We also show that all finite powers of $\mathbb{L}$ are acyclic with respect to Alexander-Spanier cohomology.

1 Introduction

Topologists have adapted two powerful tools from other branches of Mathematics to assist them in solving topological problems: Algebra and Set Theory. Each has made major contributions but in distinct areas of topology. Algebraic Topology has been very effective in the context of compact spaces, essentially requiring finiteness. Set Theory has been effective in dealing with the large infinite. In the context of topological manifolds, Algebraic Topology has been invaluable in the study of compact manifolds while Set Theory has been of most use in the study of non-metrisable manifolds. Unfortunately it has been unusual for the two to be combined. In this paper we discuss one way in which these techniques can work together and as a result solve a problem in the theory of non-metrisable manifolds.

We denote by ω_1 the set of countable ordinals and by $\mathbb{L}$ the *long line*, which is obtained by inserting an open interval between each countable ordinal and its successor to obtain the *closed long ray* $\mathbb{L}_+$ then joining two copies of $\mathbb{L}_+$ (which we denote by $\mathbb{L}_+$ and $\mathbb{L}_-$) by identifying the ordinals $0 \in \mathbb{L}_+$ and $0 \in \mathbb{L}_-$. We denote the *open long ray*, which is L_+ with 0 removed, by L_o.

The primary goal of this paper is to determine the torsion of the group of homeomorphisms of low powers of the long line and long ray. Our main result shows that the only torsion in these groups is torsion corresponding to permutation of the coordinates and

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2-torsion. The computation involves a blend of notions of the Set Theory of the countable ordinals and Algebraic Topology, particularly the P A Smith theorems. Details appear in Section 4.

In order to apply the P A Smith theorems we find it necessary to calculate the Alexander-Spanier cohomology of powers of the long line. Of course it is trivial to show that in singular cohomology theory all powers of the long line are acyclic. We find in Section 3 that in Alexander-Spanier cohomology theory the powers are also acyclic. Although we will work with coefficients in an R-module G, we will suppress this in the notation.

In Section 2 we present a number of results which exhibit covering properties of powers of the long line. These results will be useful in Section 3.

We conclude this section by recalling some relevant facts from Algebraic Topology and Set Theory.

Definition 1.1 [6, Chapter 6] *Let X be a topological space. For each $q \in \omega$ set*

$$C^q(X) = \{\varphi : X^{q+1} \to G \ / \ \varphi \text{ is a function}\}$$

with pointwise addition and R-multiplication. Declare a function $\varphi \in C^q(X)$ to be locally zero *provided that there is an open cover $\mathcal{U}$ of X such that for each $x_0, \ldots, x_q \in U \in \mathcal{U}$ we have $\varphi(x_0, \ldots, x_q) = 0$. Let*

$$C^q_0(X) = \{\varphi \in C^q(X) \ / \ \varphi \text{ is locally zero}\},$$

and set $\bar{C}^q(X) = C^q(X)/C^q_0(X)$.

Define $\delta : \bar{C}^q(X) \to \bar{C}^{q+1}(X)$ by

$$(\delta[\varphi])(x_0, \ldots, x_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \varphi(x_0, \ldots, \hat{x}_i, \ldots, x_q).$$

Then $\langle \bar{C}^q(X), \delta \rangle$ is a cochain complex. The cohomology of this cochain complex is called the Alexander-Spanier cohomology *of X and its groups are denoted by $\bar{H}^q(X)$.*

An important class of subsets of ω_1 is the class of closed, unbounded sets. These sets possess the curious and invaluable property that the interesection of any two is always nonempty. More generally and more precisely, we have the following useful result, a proof of which is found in many books on Set Theory, for example [5, page 78]:

Proposition 1.2 *The intersection of a countable collection of closed, unbounded subsets of ω_1 is again closed and unbounded.*

We will also use the next result frequently. In this we use the following standard terminology. A function $f : A \to \omega_1$ is *regressive* if for each $\alpha \in A - \{0\} \subset \omega_1$ we have $f(\alpha) < \alpha$. A set $S \subset \omega_1$ is *stationary* provided that it meets every closed, unbounded subset of ω_1.

Proposition 1.3 (Pressing Down Lemma) *Let $S \subset \omega_1$ be a stationary set and $f : S \to \omega_1$ be a regressive function. Then there is $\alpha \in \omega_1$ so that $f^{-1}(\alpha)$ is stationary.*

2 Covering Properties of the Long Line and its Powers

Whenever x is a point in the finite product of sets we denote by x_i the ith coordinate of x. If $\alpha \in \mathbb{L}$ let $\tilde{\alpha} = (\alpha, \ldots, \alpha) \in \mathbb{L}^m$. For $t \in \mathbb{R}^l$ and $r > 0$ set

$$C[t;r] = \{s \in \mathbb{R}^k \ / \ \forall j = 1, \ldots, l, \ t_j - r \leq s_j \leq t_j + r\}.$$

For $x, y \in \mathbb{L}^m$ set

$$D[x,y] = \{z \in \mathbb{L}^m \ / \ \forall j = 1, \ldots, m, \ min\{x_j, y_j\} \leq z_j \leq max\{x_j, y_j\}\}.$$

Lemma 2.1 *Let $l, m \geq 0$ be integers and let $\mathcal{U}$ be a collection of open subsets of $\mathbb{R}^l \times \mathbb{L}_+^m$ which covers $[0,1]^l \times \mathbb{L}_+^m$. Then there are $\alpha \in \mathbb{L}_+$ and a cover $\{J_1, \ldots, J_p\}$ of $[0,1]$ by open intervals such that for each $s, t \in [0,1]^l$ and each $x, y \in \mathbb{L}_+^m$ if for each $j = 1, \ldots, l$ there is i_j with $s_j, t_j \in J_{i_j}$ and $x_j, y_j \geq \alpha$ for each $j = 1, \ldots, m$ then there is $U \in \mathcal{U}$ with $(s,x), (t,y) \in U$.*

Proof. For each $t \in [0,1]^l$ and each limit ordinal $\lambda \in \mathbb{L}_+$ we have $(t, \tilde{\lambda}) \in [0,1]^l \times \mathbb{L}_+^m$ so there is $U \in \mathcal{U}$ with $(t, \tilde{\lambda}) \in U$. Thus there are an integer $n > 0$ and an ordinal $f(\lambda) < \lambda$ so that $C[t; \frac{1}{n}] \times D[\widetilde{f(\lambda)}, \tilde{\lambda}] \subset U$.

By the Pressing Down Lemma there is an ordinal α_t such that $f^{-1}(\alpha_t)$ is uncountable, hence there are an unbounded set $S_t \subset f^{-1}(\alpha_t)$ and an integer $n_t > 0$ such that for each $\lambda \in S_t$ there is $U \in \mathcal{U}$ with $C[t; \frac{1}{n_t}] \times D[\widetilde{\alpha_t}, \tilde{\lambda}] \subset U$.

By compactness finitely many interiors of the hypercubes $\{C[t; \frac{1}{n_t}] \ / \ t \in [0,1]^l\}$ cover $[0,1]^l$; say $\{C[t_{(i)}; \frac{1}{n_i}] \ / \ i = 1, \ldots q\}$, where we abbreviate $n_{t_{(i)}}$ to n_i; we similarly abbreviate $\alpha_{t_{(i)}}$ and $S_{t_{(i)}}$. Let $\alpha = max\{\alpha_i \ / \ i = 1, \ldots, q\}$. From the hypercubes $\{C[t_{(i)}; \frac{1}{n_i}] \ / \ i = 1, \ldots q\}$ we may construct open intervals $\{J_1, \ldots, J_p\}$ covering $[0,1]$ such that each of the sets $\prod_{j=1}^l J_{i_j}$ lies in some $C[t_{(i)}; \frac{1}{n_i}]$.

Suppose that $s, t \in [0,1]^l$ and $x, y \in \mathbb{L}_+^m$ are such that for each $j = 1, \ldots, l$ there is i_j with $s_j, t_j \in J_{i_j}$ and $x_j, y_j \geq \alpha$ for each $j = 1, \ldots, m$. Then $s, t \in C[t_{(i)}; \frac{1}{n_i}]$ for some i. Choose any $\lambda \in S_i$ such that $x_j, y_j \leq \lambda$ for each j; then $x, y \in D[\widetilde{\alpha}_i, \tilde{\lambda}]$. Choose $U \in \mathcal{U}$ such that $C[t_{(i)}; \frac{1}{n_i}] \times D[\widetilde{\alpha}_i, \tilde{\lambda}] \subset U$. Then $(s,x), (t,y) \in U$. ∎

Corollary 2.2 *Let $\mathcal{U}$ be an open cover of $\mathbb{L}^m$, let $l \geq 0$ be an integer and let $\beta \in \mathbb{L}_+$. Then there are $\alpha \in \mathbb{L}$ and a cover $\{J_1, \ldots, J_p\}$ of $[-\beta, \beta]$ by open intervals such that for each $x, y \in \mathbb{L}^m$ with $x_j, y_j \in J_{i_j}$ for l values of j and $|x_j|, |y_j| \geq \alpha$ and x_j and y_j having the same sign for the remaining $m - l$ values of j, there is $U \in \mathcal{U}$ such that $x, y \in U$.*

Proof. Apply Lemma 2.1 to each of the $\binom{m}{l} 2^{m-l}$ regions in each of which some fixed l coordinates lie in $[-\beta, \beta]$, some of the remaining coordinates are restricted to $(-\omega_1, -\beta]$ and the remaining are restricted to $[\beta, \omega_1)$. After rearrangement of coordinates, each such region is of the form $[-\beta, \beta]^l \times \mathbb{L}_\pm^{m-l}$. Take the intersections of the resulting open intervals in say $(-\beta, \beta)$ and the maximum of the resulting ordinals $|\alpha|$. ∎

Proposition 2.3 *Let $m \geq 0$ be an integer and let $\mathcal{U}$ be an open cover of $\mathbb{L}^m$. Then there are open intervals $I_0, \ldots, I_n$ such that $\{I_0, \ldots, I_n\}$ covers $\mathbb{L}$ and for each $x, y \in \mathbb{L}^m$ such that for each $j = 1, \ldots, m$ there is an interval I_{i_j} such that $x_j, y_j \in I_{i_j}$, there is $U \in \mathcal{U}$ with $x, y \in U$.*

Proof. We apply Corollary 2.2 inductively over $l = 0, \dots, m$ to the cover $\mathcal{U}$. In the case $l = 0$ we take $\beta = 1$, and obtain a point $\alpha_0 \in \mathbb{L}$ such that for each $x, y \in \mathbb{L}^m$ with $|x_j|, |y_j| \geq \alpha_0$ and with x_j and y_j having the same sign for each $j = 1, \dots, m$, there is $U \in \mathcal{U}$ such that $x, y \in U$. Set $p_0 = 0$.

Suppose that for some l between 0 and m we have constructed $\alpha_l \in \mathbb{L}$ and open intervals $J_{l,1}, \dots, J_{l,p_l}$ covering $[-\alpha_{l-1}, \alpha_{l-1}]$ such that for each $x, y \in \mathbb{L}^m$ with $x_j, y_j \in J_{i_j}$ for l values of j (this condition is vacuous when $l = 0$ so in that case we do not need any intervals $J_{0,i}$) and $|x_j|, |y_j| \geq \alpha_l$ and x_j and y_j having the same sign for the remaining $m - l$ values of j, there is $U \in \mathcal{U}$ such that $x, y \in U$. We have already exhibited these in the case $l = 0$.

If $l < m$, apply Corollary 2.2 to the integer $l + 1$ and $\beta = \alpha_l$. Then there are $\alpha_{l+1} \in \mathbb{L}$ and a cover $\{J_{l+1,1}, \dots, J_{l+1,p_{l+1}}\}$ of $[-\alpha_l, \alpha_l]$ by open intervals such that for each $x, y \in \mathbb{L}^m$ with $x_j, y_j \in J_{i_j}$ for $l + 1$ values of j and $|x_j|, |y_j| \geq \alpha_{l+1}$ and x_j and y_j having the same sign for the remaining $m - l - 1$ values of j, there is $U \in \mathcal{U}$ such that $x, y \in U$. We will assume when $l > 0$ that each of the intervals $J_{l+1,j}$ is small enough that if it meets $[-\alpha_{l-1}, \alpha_{l-1}]$ then it lies in at least one of the intervals $J_{l,k}$.

Now set $n = p_m + 1$, let $I_j = J_{m,j}$ when $j = 1, \dots, p_m$ and let $I_0 = (-\omega_1, -\alpha_m)$ and $I_n = (\alpha_m, \omega_1)$. Suppose that $x, y \in L^m$ are such that for each $j = 1, \dots, m$ there is an interval I_{i_j} such that $x_j, y_j \in I_{i_j}$. Suppose that i_j is neither 0 nor n for exactly l values of j. Because for the other values of j we have $|x_j|, |y_j| > \alpha_m \geq \alpha_l$ and because of the nesting of the intervals $J_{l,j}$ with l, it follows from the lth inductive step that there is $U \in \mathcal{U}$ such that $x, y \in U$. ∎

3 The Cohomology of Powers of the Long Line

Lemma 3.1 *Let $A \subset X$ be a non-empty subspace of a topological space and $\varphi \in C^q(X)$ be a cochain, $q \geq 1$, such that $\delta\varphi$ is identically zero on A^{q+2}. Then $\varphi|A^{q+1}$ is a q-coboundary.*

Proof. Fix a point $z \in A$. For each $p \geq 1$ define a function $D : C^p(A) \to C^{p-1}(A)$ by setting

$$(D\psi)(a_0, \dots, a_{p-1}) = \psi(z, a_0, \dots, a_{p-1}).$$

Then for each $(p + 1)$-tuple $(a_0, \dots, a_p) \in A$ we have:

$$\begin{aligned}
(D\delta\psi + \delta D\psi)(a_0, \dots, a_p) &= \delta\psi(z, a_0, \dots, a_p) + \sum_{i=0}^{p}(-1)^i (D\psi)(a_0, \dots, \hat{a}_i, \dots, a_p) \\
&= \psi(a_0, \dots, a_p) - \sum_{i=0}^{p}(-1)^i \psi(z, a_0, \dots, \hat{a}_i, \dots, a_p) \\
&\quad + \sum_{i=0}^{p}(-1)^i \psi(z, a_0, \dots, \hat{a}_i, \dots, a_p) \\
&= \psi(a_0, \dots, a_p).
\end{aligned}$$

Thus on A^{q+1} we have $D\delta\psi + \delta D\psi = \psi$. Because $\delta\varphi$ vanishes on A^{q+2}, it follows that $\varphi = \delta D\varphi$ on A^{q+1}, ie φ is a coboundary. ∎

Proposition 3.2 *$\tilde{\bar{H}}^q(\mathbb{L}_+) = 0$ for each $q \geq 0$.*

Proof. Since $\mathbb{L}_+$ is a connected space, by [6, Corollary 6.4.7] it follows that $\overset{\simeq}{H}{}^0(\mathbb{L}_+)$ is trivial, so we will assume that $q \geq 1$.

Let $[\varphi] \in \tilde{\bar{H}}^q(\mathbb{L}_+) = \bar{H}^q(\mathbb{L}_+)$ be an arbitrary cohomology class, where $\varphi : \mathbb{L}_+^{q+1} \to G$ is a q-cocycle. Then $\delta\varphi$ is locally zero on $\mathbb{L}_+$ so there is an open covering $\mathcal{U}$ of $\mathbb{L}_+$ such that whenever $x_0, \ldots, x_{q+1} \in U \in \mathcal{U}$ then $\delta\varphi(x_0, \ldots, x_{q+1}) = 0$. By Lemma 2.1 there is $\alpha \in \mathbb{L}_+$ such that for any $\beta \in (\alpha, \omega_1)$ the interval $[\alpha, \beta]$ is contained in some member of $\mathcal{U}$. Given any $x_0, \ldots, x_{q+1} \in [\alpha, \omega_1)$ we can choose β so that $x_0, \ldots, x_{q+1} < \beta$; so $x_0, \ldots, x_{q+1} \in U \in \mathcal{U}$ and hence $\delta\varphi(x_0, \ldots, x_{q+1}) = 0$. Thus by Lemma 3.1 there is $\psi \in C^{q-1}(\mathbb{L}_+)$ such that on $[\alpha, \omega_1)^{q+1}$ we have $\delta\psi = \varphi$. It follows that $[\varphi]|(\alpha, \omega_1) = 0$.

Now $V = [0, \alpha + 1)$ and $W = (\alpha, \omega_1)$ are open subsets of $\mathbb{L}_+$ whose union is $\mathbb{L}_+$ and intersection is an open interval, therefore acyclic. Consider the following Mayer-Vietoris sequence for reduced Alexander-Spanier cohomology:

$$\cdots \to \tilde{\bar{H}}^{q-1}(V \cap W) \to \tilde{\bar{H}}^q(\mathbb{L}_+) \to \tilde{\bar{H}}^q(V) \oplus \tilde{\bar{H}}^q(W) \to \tilde{\bar{H}}^q(V \cap W) \to \cdots.$$

It follows that the homomorphism $\tilde{\bar{H}}^q(\mathbb{L}_+) \to \tilde{\bar{H}}^q(V) \oplus \tilde{\bar{H}}^q(W)$, which takes $[\varphi]$ to $([\varphi]|V, [\varphi]|W)$, is an isomorphism. Since V is contractible we have that $[\varphi]|V = 0$ and by what we showed in the previous paragraph we also have that $[\varphi]|W = 0$, so that $[\varphi] = 0$. ∎

Corollary 3.3 *The Alexander-Spanier cohomology of the long line is given by:*

$$\bar{H}^q(\mathbb{L}) = \begin{cases} G & \text{if } q = 0 \\ 0 & \text{if } q > 0. \end{cases}$$

Proof. We may use the Mayer-Vietoris sequence for reduced Alexander-Spanier cohomology as applied to the space $\mathbb{L}$ and its closed subspaces $\mathbb{L}_+$ and $\mathbb{L}_-$, noting that $\mathbb{L}_+ \cap \mathbb{L}_-$ consists of a single point. Proposition 3.2 now tells us that $\tilde{\bar{H}}^q(\mathbb{L}) \approx \tilde{\bar{H}}^q(\mathbb{L}_+) \oplus \tilde{\bar{H}}^q(\mathbb{L}_-) = 0$. ∎

Proposition 3.4 *For each $n \geq 1$ and each $q \geq 0$ the reduced Alexander-Spanier cohomology groups $\tilde{\bar{H}}^q(\mathbb{L}_+^n)$ are all trivial.*

Proof. We use induction on n, the case $n = 1$ having been shown in Proposition 3.2. Assume the result true for powers less than n.

Let $[\varphi] \in \tilde{\bar{H}}^q(\mathbb{L}_+^n)$, where φ is a cocycle. Then there is an open cover $\mathcal{U}$ of $\mathbb{L}_+^n$ such that $\delta\varphi$ is locally zero on $\mathcal{U}^{q+2}$. Hence by Lemma 2.1 there is $\alpha < \omega_1$ such that $\delta\varphi$ is identically zero on $([\alpha, \omega_1)^n)^{q+2}$. Hence by Lemma 3.1 there is $\psi \in C^{q-1}([\alpha, \omega_1)^n)$ such that $\delta\psi = \varphi|[\alpha, \omega_1)^n$. Thus $[\varphi]|[\alpha, \omega_1)^n = 0$.

We now prove by induction on m that $[\varphi]|[0, \omega_1)^m \times [\alpha, \omega_1)^{n-m} = 0$, having already begun the induction at $m = 0$. Suppose that $[\varphi]|[0, \omega_1)^{m-1} \times [\alpha, \omega_1)^{n-m+1} = 0$. Note that

$$\begin{aligned} [0, \omega_1)^m \times [\alpha, \omega_1)^{n-m} &= [0, \omega_1)^{m-1} \times ([0, \alpha] \cup [\alpha, \omega_1)) \times [\alpha, \omega_1)^{n-m} \\ &= ([0, \omega_1)^{m-1} \times [0, \alpha] \times [\alpha, \omega_1)^{n-m}) \cup ([0, \omega_1)^{m-1} \times [\alpha, \omega_1)^{n-m+1}). \end{aligned}$$

Thus we have expressed $[0,\omega_1)^m \times [\alpha,\omega_1)^{n-m}$ as a union of two closed subsets. By inductive hypotheses with respect to n, $[\varphi]|[0,\omega_1)^{m-1} \times [0,\alpha] \times [\alpha,\omega_1)^{n-m} = 0$ because the compact interval $[0,\alpha]$ is contractible, and by inductive hypothesis with respect to m we have $[\varphi]|[0,\omega_1)^{m-1} \times [\alpha,\omega_1)^{n-m+1} = 0$. Furthermore the intersection of these two closed subsets is $[0,\omega_1)^{m-1} \times \{\alpha\} \times [\alpha,\omega_1)^{n-m}$, which is homeomorphic to $[0,\omega_1)^{m-1} \times [\alpha,\omega_1)^{n-m}$ so again by inductive hypothesis is acyclic. Thus by the reduced Mayer-Vietoris sequence for these two closed sets we may conclude that $[\varphi]|[0,\omega_1)^m \times [\alpha,\omega_1)^{n-m} = 0$.

Taking $m = n$ in the statement above, we conclude that $[\varphi] = 0$. Thus $\tilde{\bar{H}}^q(\mathbb{L}_+^n) = 0$. ∎

Theorem 3.5 *The Alexander-Spanier cohomology of powers of the long line is given by:*

$$\bar{H}^q(\mathbb{L}^n) = \begin{cases} G & \text{if } q = 0 \\ 0 & \text{if } q > 0. \end{cases}$$

Proof. Using induction on n and $m \le n$ we show that each of the groups $\tilde{\bar{H}}^q(\mathbb{L}^m \times \mathbb{L}_+^{n-m})$ is trivial for each $q \ge 0$. When $m = 0$ we already know that $\tilde{\bar{H}}^q(\mathbb{L}^0 \times \mathbb{L}_+^n)$ is trivial for any n from Proposition 3.4. When $n = 0$ we know that $\tilde{\bar{H}}^q(\mathbb{L}^0 \times \mathbb{L}_+^0)$ is trivial by [6, Lemma 6.4.3].

Suppose that $\tilde{\bar{H}}^q(\mathbb{L}^m \times \mathbb{L}_+^{n'-m})$ is trivial for each $n' < n$ and each m and that $\tilde{\bar{H}}^q(\mathbb{L}^{m-1} \times \mathbb{L}_+^{n-m+1})$ is trivial for each $q \ge 0$. Write

$$\begin{aligned} \mathbb{L}^m \times \mathbb{L}_+^{n-m} &= \mathbb{L}^{m-1} \times (\mathbb{L}_+ \cup \mathbb{L}_-) \times \mathbb{L}_+^{n-m} \\ &= (\mathbb{L}^{m-1} \times \mathbb{L}_+^{n-m+1}) \cup (\mathbb{L}^{m-1} \times \mathbb{L}_- \times \mathbb{L}_+^{n-m}). \end{aligned}$$

Note that $\mathbb{L}^{m-1} \times \mathbb{L}_- \times \mathbb{L}_+^{n-m}$ is homeomorphic to $\mathbb{L}^{m-1} \times \mathbb{L}_+^{n-m+1}$ and each of the subsets $\mathbb{L}^{m-1} \times \mathbb{L}_+^{n-m+1}$ and $\mathbb{L}^{m-1} \times \mathbb{L}_- \times \mathbb{L}_+^{n-m}$ is closed so by inductive hypothesis

$$\tilde{\bar{H}}^q(\mathbb{L}^{m-1} \times \mathbb{L}_+^{n-m+1}) = \tilde{\bar{H}}^q(\mathbb{L}^{m-1} \times \mathbb{L}_- \times \mathbb{L}_+^{n-m}) = 0.$$

Furthermore the intersection of these two sets is $\mathbb{L}^{m-1} \times \{0\} \times \mathbb{L}_+^{n-m}$, which is homeomorphic to $\mathbb{L}^{m-1} \times \mathbb{L}_+^{n-m}$, so again by inductive hypothesis

$$\tilde{\bar{H}}^q((\mathbb{L}^{m-1} \times \mathbb{L}_+^{n-m+1}) \cap (\mathbb{L}^{m-1} \times \mathbb{L}_- \times \mathbb{L}_+^{n-m})) = 0.$$

Now apply the reduced Mayer-Vietoris for Alexander-Spanier cohomology to the closed sets $\mathbb{L}^{m-1} \times \mathbb{L}_+^{n-m+1}$ and $\mathbb{L}^{m-1} \times \mathbb{L}_- \times \mathbb{L}_+^{n-m}$ to deduce that $\tilde{\bar{H}}^q(\mathbb{L}^m \times \mathbb{L}_+^{n-m})$ is trivial.

The induction above continues until $m = n$, at which stage we conclude that $\tilde{\bar{H}}^q(\mathbb{L}^n)$ is trivial for each $q \ge 0$, as required. ∎

4 Torsion of the Group of Homeomorphisms of Powers of the Long Line

In this section we study the group of homeomorphisms, $\mathcal{H}(X)$, of a space X, where X is a finite power of $\mathbb{L}$ or $\mathbb{L}_+$. Although Theorems 4.2 and 4.3 are subsumed in later results of this section we present them independently because the proofs which we present here show a strong interplay between Algebraic Topology and Set Theory.

Proposition 4.1 *The cohomological dimension of the long line, $dim_{\mathbb{Z}}(\mathbb{L})$ is 1.*

Proof. As a 1-manifold, $\mathbb{L}$ is locally compact. The cohomological dimension of a locally compact space is determined locally and, as the local cohomological dimension of $\mathbb{L}$ is 1, so is the (global) cohomological dimension. ■

Theorem 4.2 *Let L denote either $\mathbb{L}_+$ or $\mathbb{L}_o$. Then the group $\mathcal{H}(L)$ has no torsion.*

Proof. It is enough to show that $\mathcal{H}(L)$ has no elements of prime order. Suppose to the contrary that there is a homeomorphism $h : L \to L$ of prime order p. Let $G = [h]$ be the cyclic group of order p generated by h, and $L^h = \{x \in L \ / \ h(x) = x\}$ be the fixed point set of h. Evidently $L^h = L^G$, the fixed point set of the group G.

The space L is a locally compact Hausdorff space which, by Proposition 3.2, is acyclic (mod$p$) with respect to Alexander-Spanier cohomology, and G is a group of order p. Hence by the P. A. Smith theorem, [3], the fixed point set L^G must also be acyclic (mod$p$). Hence L^G must be connected by [6, Corollary 6.4.7]. We also know from the proof of [4, Lemma 2] that L^h must contain an unbounded subset of ω_1. It follows that $L^h = L^G = L$, ie that h is the identity, a contradiction. ■

Theorem 4.3 *The group $\mathcal{H}(\mathbb{L})$ has only 2-torsion, ie any nontrivial element of finite order must be of order 2.*

Proof. Let $h \in \mathcal{H}(\mathbb{L})$ be a nontrivial homeomorphism of finite prime power order, say $h^{p^k} = e$, the identity. It suffices to show that $p^k = 2$, ie that $p = 2$ and $k = 1$.

We remark that the P A Smith theorems are also valid for prime-power order groups, for any prime p. It follows from Corollary 3.3 that the fixed point set $\mathbb{L}^h$ is a connected closed subset of $\mathbb{L}$. Since connected subsets of $\mathbb{L}$ are intervals, it follows that $\mathbb{L}^h$ is either a point or a nontrivial interval.

If $\mathbb{L}^h$ is a nontrivial interval then h maps $X = \mathbb{L} - \mathring{\mathbb{L}}^h$ to itself. Note that X is either connected or has two components.

If X is connected then $h|X \in \mathcal{H}(X) \approx \mathcal{H}(\mathbb{L}_+)$, which, by Theorem 4.2, has no torsion. Thus $h|X$ is the identity, so that $h = e$, a contradiction.

If X is disconnected then X has two components, say A and B, each of which is homeomorphic to $\mathbb{L}_+$. Continuity of h at the end points of $\mathbb{L}^h$ ensures that $h(A) = A$ and $h(B) = B$, and, as in the previous paragraph, we conclude that $h = e$, a contradiction.

It follows that $\mathbb{L}^h$ is a single point. Hence $h(A) = B$ and $h(B) = A$, where A and B are the two components of $\mathbb{L} - \mathbb{L}^h$. Thus $h^2(A) = A$, and, as above, we conclude that $h^2 = e$. ■

For each $\alpha \in \omega_1$ we denote by $S_\alpha = \{x \in \mathbb{L}^n \ / \ |x| \leq \alpha\}$ the "square" of sides 2α.

Lemma 4.4 *Let $h : \mathbb{L}^n \to \mathbb{L}^n$ be a homeomorphism. Set $D = \{\alpha \in \omega_1 \ / \ h(S_\alpha) = S_\alpha\}$. Then D is closed and unbounded.*

Proof. *D is closed.* Suppose that $\langle \alpha_n \rangle$ is a sequence in D converging upwards to α. If $y \in \mathring{S}_\alpha$ then there is $n \in \omega$ such that $y \in S_{\alpha_n}$ so $h(y) \in h(S_{\alpha_n}) = S_{\alpha_n} \subset S_\alpha$. This shows that $h(\mathring{S}_\alpha) \subset S_\alpha$ so that $h(S_\alpha) \subset S_\alpha$. On the other hand, if $h(y) \in \mathring{S}_\alpha$ then there is $n \in \omega$ such that $h(y) \in S_{\alpha_n} = h(S_{\alpha_n})$ so $y \in S_{\alpha_n} \subset S_\alpha$ and as before $S_\alpha \subset h(S_\alpha)$. Thus $h(S_\alpha) = S_\alpha$ and hence $\alpha \in D$.

D is unbounded. Suppose that $\beta \in \omega_1$ and set $\alpha_0 = \beta$. Suppose that α_n has been constructed. Because $h(S_{\alpha_n}) \cup h^{-1}(S_{\alpha_n})$ is compact, we may choose $\alpha_{n+1} > \alpha_n$ so that $h(S_{\alpha_n}) \cup h^{-1}(S_{\alpha_n}) \subset S_{\alpha_{n+1}}$. Suppose that $\alpha_n \nearrow \alpha$.

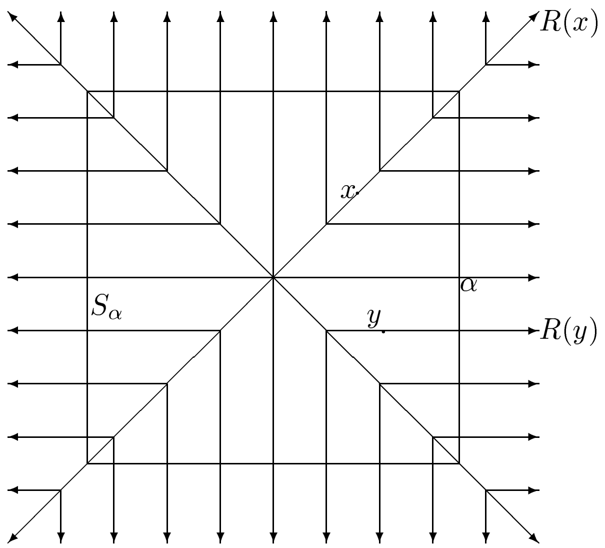
We claim that $\alpha \in D$. Suppose that $y \in S_\alpha$. As $\alpha_n \nearrow \alpha$ it follows that there is n so that $y \in S_{\alpha_n}$. Then $h^{-1}(y) \in S_{\alpha_{n+1}} \subset S_\alpha$ so $y \in h(S_\alpha)$, ie $\mathring{S}_\alpha \subset h(S_\alpha)$ and hence $S_\alpha \subset h(S_\alpha)$. Similarly $S_\alpha \subset h^{-1}(S_\alpha)$, ie $h(S_\alpha) \subset S_\alpha$. Thus $S_\alpha = h(S_\alpha)$ so $\alpha \in D$. ∎

Now let L be either $\mathbb{L}$ or $\mathbb{L}_+$. Given $x \in L^n - \{0\}$ define the (open) ray through x in the following way. Let

$$|x| = \max\{|x_i| \ / \ i = 1, \dots n\},$$
$$\hat{x} = \{i \in \{1, \dots, n\} \ / \ |x_i| = |x|\}, \qquad \check{x} = \{1, \dots, n\} - \hat{x} \text{ and}$$
$$\|x\| = \max\{|x_i| \ / \ i \in \check{x}\} \text{ (with } \|x\| = 0 \text{ if } \hat{x} = \{1, \dots, n\}).$$

Then the *open ray through* x is

$$R(x) = \{y \in L^n \ / \ \frac{\|x\|}{|x|} < \frac{y_i}{x_i} = \frac{y_j}{x_j} \text{ for each } i, j \in \hat{x} \text{ and } y_i = x_i \text{ for each } i \in \check{x}\}.$$



Define an equivalence relation $\sim$ on $L^n - \{0\}$ by setting $x \sim y$ if and only if x and y belong to the same open ray in $L^n - \{0\}$. Let R_α be the quotient space $(L^n - S_\alpha)/\sim$ and denote the quotient map by $\pi_\alpha : L^n - S_\alpha \to R_\alpha$. Then R_α consists of the ends of all the open rays. For each $\alpha, \beta \in \omega_1$ with $\alpha \leq \beta$ let $p^\beta_\alpha : R_\beta \to R_\alpha$ be the bijection induced by the inclusion $L^n - S_\beta \subset L^n - S_\alpha$. Then p^β_α is continuous (but $(p^\beta_\alpha)^{-1}$ is not unless $\beta = \alpha$). Thus $\langle p^\beta_\alpha : R_\beta \to R_\alpha \rangle$ is an inverse system.

Definition 4.5 *The* ray space, *$R(L^n)$, of L^n is the limit of the inverse system $\langle p^\beta_\alpha : R_\beta \to R_\alpha \rangle$.*

We have the following facts about $R(L^n)$:

1. There is a natural continuous bijection, which we denote by $p_\alpha : R(L^n) \to R_\alpha$;
2. $R(L^n)$ has a natural stratification $R(L^n) = \cup_{i=0}^{n-1} R_i$, where each R_i is a finite disjoint union of subsets each of which is homeomorphic to $\mathbb{L}^i$ when $L = \mathbb{L}$ and $\mathbb{L}_o^i$ when $L = \mathbb{L}_+$;
3. $R(L^n)$ is a compact Hausdorff space.

$R(\mathbb{L}^n)$ may be thought of as the boundary of a hypercube in which each open face of dimension i has been replaced by a copy of $\mathbb{L}^i$. Furthermore, the closure of a face of dimension i is homeomorphic to the ith power of the 2-point compactification of $\mathbb{L}$. For $R(\mathbb{L}_+)$ the situation is similar except that the faces are replaced by copies of $\mathbb{L}_o^i$. One may carry out a similar construction in $\mathbb{R}^n$ and so get an ordinary hypercube in which each face is the 2-point compactification of $\mathbb{R}$. The following lemma shows that there is a major difference between the situations in $\mathbb{R}^n$ and L^n: the analogue of this lemma in $\mathbb{R}^n$ is false. As a result there is a major difference between the behaviours of homeomorphisms of L^n, as exhibited by Corollary 4.10, and $\mathbb{R}^n$.

Lemma 4.6 *If $e : \omega_1 \to L^n$ is an embedding then there is a unique point $r \in R(L^n)$ such that for each $\alpha \in \omega_1$ the set $e^{-1}(\overline{\pi_\alpha^{-1} p_\alpha(r)})$ is a closed unbounded subset of ω_1.*

Proof. Clearly for each $r \in R(L^n)$ and each $\alpha \in \omega_1$ we have that $e^{-1}(\overline{\pi_\alpha^{-1} p_\alpha(r)})$ is closed so we need only show that there is a unique r for which this set is unbounded. If we can show that there is at least one such r then it must be unique as any two closed unbounded subsets of ω_1 have non-empty intersection by Proposition 1.3. Thus we need only show the existence of such a point $r \in R(L^n)$.

Because ω_1, and hence $e(\omega_1)$, is not Lindelöf whereas every bounded subset of L^n is Lindelöf, it follows that at least 1 coordinate of $e(\omega_1)$ is unbounded; let us suppose that exactly k coordinates of $e(\omega_1)$ are unbounded, and for ease of notation we will suppose that the first k coordinates are unbounded. Thus:

1. for each $a \in \mathbb{L}_+$ there is $\alpha_a \in \omega_1$ such that for each $i \leq k$ we have $|e_i(\alpha_a)| \geq a$;
2. there is $b \in \mathbb{L}_+$ such that for each $\alpha \in \omega_1$ and each $i > k$ we have $|e_i(\alpha)| \leq b$.

Firstly we show that $E = \{\alpha \in \omega_1 \ / \ |e_i(\alpha)| = |e_j(\alpha)| \text{ for all } i, j \leq k\}$ is closed and unbounded. E is clearly closed, so we need only show that E is unbounded. Suppose that $\beta_0 \in \omega_1$. Given $\beta_n \in \omega_1$ choose $a_n \in \mathbb{L}_+$ so that $e([0, \beta_n]) \subset [-a_n, a_n]^k \times [-b, b]^{n-k}$. By assumption there is $\beta_{n+1} \in \omega_1$ such that $|e_i(\beta_{n+1})| \geq a_n$ for each $i \leq k$. Let $a_n \nearrow a$ and $\beta_n \nearrow \beta$. Then $|e_i(\beta)| = a$, so $\beta \in E$.

Now we show that there is a ray $r \in R(L^n)$ as described. For each $y \in \mathbb{L}^{n-k}$ consider $\mathbb{L}^k \times \{y\} \subset \mathbb{L}^k \times \mathbb{L}^{n-k} = \mathbb{L}^n$. It suffices to show that $e^{-1}(\mathbb{L}^k \times \{y\})$ is unbounded for some $y \in [-b, b]^{n-k}$, for then this set will form a closed, unbounded subset of ω_1, and this must intersect the closed, unbounded set E of the previous paragraph in a closed, unbounded set. Suppose instead that this is not the case. Then for each $y \in [-b, b]^{n-k}$ there is $\alpha_y \in \omega_1$

such that $e^{-1}(\mathbb{L}^k \times \{y\}) \subset [0, \alpha_y)$. By continuity of e it follows that $e^{-1}(\mathbb{L}^k \times \{\eta\}) \subset [0, \alpha_y)$ for each η in some neighbourhood of y in $[-b, b]^{n-k}$. Then by compactness of $[-b, b]^{n-k}$ we conclude that there is $\alpha \in \omega_1$ such that $e^{-1}(\mathbb{L}^k \times \{y\}) \subset [0, \alpha]$ for each $y \in [-b, b]^{n-k}$, which contradicts assumption 1. ∎

Lemma 4.7 *Suppose that X is a compact first countable space and $e : \omega_1 \times X \to L^n$ is an embedding. Then the induced map $\bar{e} : X \to R(L^n)$, where $\bar{e}(t)$ is that point of $R(L^n)$ given by Lemma 4.6 applied to e_t, is also an embedding.*

Proof. To show that $\bar{e}$ is continuous we must show that for each $\alpha \in \omega_1$ the composition $\bar{e}_\alpha : X \xrightarrow{\bar{e}} R(L^n) \xrightarrow{p_\alpha} R_\alpha$ is continuous. Suppose that $t \in X$. Then $\pi_\alpha^{-1}\bar{e}_\alpha(t)$ will consist of a homeomorph of the open long ray in which some coordinates are fixed and up to sign the remaining coordinates are equal and range through (α, ω_1): without loss of generality we will assume that the first k coordinates are equal and range through (α, ω_1) and the remaining coordinates are all non-negative, so that $\pi_\alpha^{-1}\bar{e}_\alpha(t)$ is of the form $\{(x, x, \ldots, x, b_1, \ldots, b_{n-k}) \ / \ x > \alpha\}$. Then a basic neighbourhood of $\bar{e}_\alpha(t)$ may be taken in the form $\pi_\alpha(N)$, where $N = (\alpha, \omega_1)^k \times \prod_{i=1}^{n-k}(b_i^-, b_i^+)$ and $b_i^- < b_i < b_i^+$.

Suppose that $\bar{e}_\alpha^{-1}\pi_\alpha(N)$ is not a neighbourhood of t. Then there is a sequence $\langle t_i \rangle$ of points of $X - \bar{e}^{-1}\pi_\alpha(N)$ converging to t. Thus there are closed, unbounded subsets $\langle C_i \rangle$ and C such that $e(C_i \times \{t_i\}) \cap N = \varnothing$ while $e(C \times \{t\}) \subset N$. As the countably many sets $\{C\} \cup \{C_i \ / \ i \in \omega\}$ are all closed and unbounded, by Proposition 1.3 so is their intersection; choose $\beta \in C \cap (\cap_{i \in \omega} C_i)$. Then $\langle(\beta, t_i)\rangle$ converges to (β, t) but $\langle e(\beta, t_i)\rangle$ does not converge to $e(\beta, t)$, contradicting continuity of e.

Now e must be an injection, for if not then there will be two points $s, t \in X$ so that $\bar{e}(s) = \bar{e}(t)$. Hence there are closed, unbounded subsets $C_s, C_t \subset \omega_1$ so that $e_s(C_s)$ and $e_t(C_t)$ are mapped to the same ray. Then we can find two distinct points $(s', s), (t', t) \in \omega_1 \times X$ so that $e(s', s) = e(t', t)$, contrary to e being an embedding. As a continuous injection from a compact space to a Hausdorff space, e is then an embedding. ∎

Lemma 4.8 *If $f : I \to R(L^n)$ is a path then $f(I) \subset R_i$ for some stratum R_i.*

Proof. As noted, each component of the ith stratum of $R(L^n)$ is homeomorphic to $\mathbb{L}^i$ or $\mathbb{L}_o^i$, and the closure of this component is homeomorphic to the ith power of the 2-point compactification. The 2-point compactification of has 3 path components, viz $\mathbb{L}$ or $\mathbb{L}_+$ and each of the 2 extra points. Hence the closure of the component consists of 3^i path components, one of which is the component itself. Thus $f(I)$ must lie in one of these path components, which is a subset of some stratum. ∎

Corollary 4.9 *If $f : X \to R(L^n)$ is continuous and X is path connected then $f(X) \subset R_i$ for some stratum R_i.*

Corollary 4.10 *Any homeomorphism $h : L^n \to L^n$ induces a homeomorphism $\bar{h} : R(L^n) \to R(L^n)$. Moreover, $\bar{h}(R_i) = R_i$ for each stratum R_i.*

Proof. Given $r \in R(L^n)$ we apply Lemma 4.6 to $\pi_\alpha^{-1}p_\alpha(r)$, which contains a subset order equivalent to ω_1, to find the natural candidate for $\bar{h}(r)$. By Corollary 4.9 $\bar{h}$ takes each component of a stratum to a stratum and by Lemma 4.7 it embeds each such stratum; hence no stratum is taken by $\bar{h}$ to a stratum of lower dimension. Applying the same reasoning to h^{-1} we conclude that $\bar{h}$ carries a stratum of dimension i to one of the same dimension. Continuity of $\bar{h}$ and its inverse is similar to the proof of continuity in Lemma 4.7. ∎

Let $\mathcal{G}_n$ be the group of symmetries of the hypercube $[-1,1]^n$ and $\mathcal{G}_{+,n}$ be the group of symmetries of the hypercube $[0,1]^n$ which send any point with at least one coordinate equal to 0 to another such point and any point with no coordinate equal to 0 to another such point. Then the symmetric group of order n, $\mathcal{S}_n$, acts on $\mathcal{G}_n$ and $\mathcal{G}_{+,n}$ by permuting the coordinates and the group $\bigoplus_{i=1}^n \mathbb{Z}_2$ acts on $\mathcal{G}_n$ by letting the ith summand reverse the ith coordinate. Let $\mu : \mathcal{G}_n \to \mathcal{H}(\mathbb{L}^n)$ and $\mu_+ : \mathcal{G}_{+,n} \to \mathcal{H}(\mathbb{L}^n_+)$ be the natural monomorphisms. By Corollary 4.10 there are homomorphisms $\varphi : \mathcal{H}(\mathbb{L}^n) \to \mathcal{H}(R(\mathbb{L}^n))$ and $\varphi_+ : \mathcal{H}(\mathbb{L}^n_+) \to \mathcal{H}(R(\mathbb{L}^n_+))$. The compositions $\varphi\mu : \mathcal{G}_n \to \mathcal{H}(R(\mathbb{L}^n))$ and $\varphi_+\mu_+ : \mathcal{G}_{+,n} \to \mathcal{H}(R(\mathbb{L}^n_+))$ are also monomorphisms.

Theorem 4.11 *Suppose that $h \in \mathcal{H}(\mathbb{L}^n_+)$ is an element of finite order q. Then:*

1. $n = 1 \Longrightarrow q = 1$;
2. $n = 2 \Longrightarrow q = 1$ *or* 2;
3. $n = 3 \Longrightarrow q = 1, 2$ *or* 3;
4. $n = 4 \Longrightarrow q = 1, 2, 3, 4$ *or* 6.

Proof. For any $\alpha \in \omega_1$ set

$$V_\alpha = \{(x_1, \ldots, x_n) \in \mathbb{L}^n_+ \ / \ x_i = 0 \text{ for exactly one coordinate } i \text{ and } x_i = \alpha \text{ for all others}\}.$$

By Proposition 1.2, Lemma 4.4 and Corollary 4.10 there is a closed unbounded subset $A \subset \omega_1$ so that for each $\alpha \in A$ we have $h(S_\alpha \cap \mathbb{L}^n_+) = S_\alpha \cap \mathbb{L}^n_+$ and $h(V_\alpha) = V_\alpha$. Fix $\alpha \in A$: it suffices to show that the qth power of $h|S_\alpha \cap \mathbb{L}^n_+$ is the identity, where q is as in the theorem.

Case 1: h fixes V_α.

If necessary replace h by a power of h so that q is a power of some prime p.

Consider $h|\partial(\mathbb{S}_\alpha \cap \partial\mathbb{L}^n_+)$ when $n \geq 3$: because $\partial(\mathbb{S}_\alpha \cap \partial\mathbb{L}^n_+) \approx \mathbb{S}^{n-2}$ and h fixes n points therein, it follows from the P A Smith theorem [2, Theorem III.5.1] that the fixed point set of $h|\partial(\mathbb{S}_\alpha \cap \partial\mathbb{L}^n_+)$ is a mod p homology r-sphere, where $n-2-r$ is even if p is odd and h is an orientation-reversing involution if $p = 2$. If $n = 3$ then $\partial(\mathbb{S}_\alpha \cap \partial\mathbb{L}^n_+) \approx \mathbb{S}^1$ so that h must fix all of $\partial(\mathbb{S}_\alpha \cap \partial\mathbb{L}^n_+)$. If $n = 4$ then $\partial(\mathbb{S}_\alpha \cap \partial\mathbb{L}^n_+) \approx \mathbb{S}^2$ and h has at least 4 fixed points on this set. Furthermore if p is odd then the only way for $n-2-r$ to be even is for $r = 0$ or 2. Now it is not possible to have $r = 0$ by the P A Smith theorem [2, Theorem III.5.2] because there are more than 2 fixed points. On the other hand, $r = 2$ means that h fixes all of $\partial(\mathbb{S}_\alpha \cap \partial\mathbb{L}^n_+)$. Thus when $n \leq 3$ h fixes $\partial(\mathbb{S}_\alpha \cap \partial\mathbb{L}^n_+)$ and even when $n = 4$ h either fixes $\partial(\mathbb{S}_\alpha \cap \partial\mathbb{L}^n_+)$ or is an involution there, ie h^2 fixes $\partial(\mathbb{S}_\alpha \cap \partial\mathbb{L}^n_+)$.

Firstly consider the case where h fixes $\partial(\mathbb{S}_\alpha \cap \partial\mathbb{L}^n_+)$ $(n \geq 2)$. Consider $h|\mathbb{S}_\alpha \cap \partial\mathbb{L}^n_+$. As $\mathbb{S}_\alpha \cap \partial\mathbb{L}^n_+ \approx \mathbb{B}^{n-1}$ and h fixes the boundary of this set it follows from the P A Smith theorem [2, Theorem III.5.2] that h fixes all of $\mathbb{S}_\alpha \cap \partial\mathbb{L}^n_+$. Similarly h fixes all of $\overline{\partial(\mathbb{S}_\alpha \cap \mathbb{L}^n_+) - \partial\mathbb{L}^n_+}$. Thus h fixes all of $\partial(S_\alpha \cap \mathbb{L}^n_+)$, and this also holds if $n = 1$. Applying [2, Theorem III.5.2] once again, it follows that h fixes all of S_α, ie is the identity there as claimed.

Secondly consider the case where h is an involution and apply the argument to h^2 to conclude that h^2 is the identity.

General Case: As $h(V_\alpha) = V_\alpha$, it follows that some power of h fixes V_α, and this power must be 1 when $n = 1$, it must be 1 or 2 when $n = 2$, it must be 1, 2 or 3 when $n = 3$ and it must be 1, 2, 3 or 4 when $n = 4$; call it q. Thus we may apply Case 1 to h^q to conclude that either h^q or h^{2q} is the identity. However when $n = 4$ the involution subcase does not arise when $q = 2$ or 4 as then h^q is already orientation-preserving. ∎

Theorem 4.12 *For $n = 1, 2$ and for any element of $\mathcal{H}(\mathbb{L}^n)$ of finite order there is an element of $\mathcal{G}_n$ having the same order.*

Proof. We have already provided a proof of Theorem 4.12 in the case $n = 1$ in Theorem 4.3 but the proof below applies to both cases $n = 1$ and $n = 2$.

Suppose that $h : \mathbb{L}^n \to \mathbb{L}^n$ is a homeomorphism of finite order. Then the corresponding homeomorphism $\bar{h}$ from Corollary 4.10 must map the strata of $R(\mathbb{L}^n)$ to themselves. Furthermore if two points $r, s \in R_0$ are such that $\pi_\alpha^{-1}(s)$ is obtained from $\pi_\alpha^{-1}(r)$ by changing the signs on each coordinate (so they are on opposite ends of a diagonal when $n = 2$) then the same applies to their images. It follows that there is an element $\gamma \in \mathcal{G}_n$ such that $\varphi\mu(\gamma)|R_0 = \bar{h}|R_0$.

Suppose that γ has order l. We claim that h also has order l. By Lemma 4.4, Lemma 4.6, Corollary 4.10 and Proposition 1.2 there is a closed, unbounded subset of ω_1 such that for each α in this set we have $h(S_\alpha) = S_\alpha$ and $h\pi_0^{-1}p_0(R_i) \subset \pi_0^{-1}p_0(R_i)$ for each stratum R_i. It suffices to show that $h^l|S_\alpha$ is the identity. We firstly show that $h^l|\partial S_\alpha$ is the identity.

The set $\partial S_\alpha \cap \pi_0^{-1}p_0(R_0)$ consists of isolated points, the ends of the interval or the vertices of the square, so by choice of γ, it follows that h^l fixes these points. Thus h^l is the identity on ∂S_α for the case $n = 1$. For the case $n = 2$, because h^l sends ∂S_α to itself, fixes the corners and is of finite order, again it follows that h^l is the identity on ∂S_α.

Now consider $h^l|S_\alpha$. Because it is the identity on the boundary and has finite order, and S_α is homeomorphic to the unit ball in $\mathbb{R}^n$, it follows from [2, Theorem III 5.2] that $h^l|S_\alpha$ is the identity. ∎

Example 4.13 *It is not the case that the only homeomorphisms of $\mathbb{L}^2$ of finite order are the eight which possibly interchange the coordinates and possibly reverse the direction of one or both.*

For example suppose that $\theta_t : [-1, 1] \to [-1, 1]$ is any isotopy of order-preserving homeomorphisms with θ_1 the identity and define a homeomorphism $\psi : [1, 3] \times [-1, 1] \to [1, 3] \times [-1, 1]$ by $\psi(x, y) = (x, \theta_{|2-x|}(y))$: then ψ moves each vertical segment within itself, is like θ_0 when $x = 2$ and is the identity on the boundary. Let $\rho : \mathbb{L}^2 \to \mathbb{L}^2$ be the rotation defined by $\rho(x, y) = (y, -x)$. Now define the homeomorphism $h : \mathbb{L}^2 \to \mathbb{L}^2$ as follows.

- if $(x,y) \in \mathbb{L}^2 - [(1,3)\times(-1,1)\cup(-1,1)\times(1,3)\cup(-3,-1)\times(-1,1)\cup(-1,1)\times(-3,-1)]$ let $h(x,y) = (y,-x)$;
- if $(x,y) \in [1,3] \times [-1,1]$ let $h(x,y) = \rho\psi(x,y)$;
- if $(x,y) \in \rho([1,3] \times [-1,1])$ let $h(x,y) = \rho^2\psi\rho^3(x,y)$;
- if $(x,y) \in \rho^2([1,3] \times [-1,1])$ let $h(x,y) = \rho^3\psi^{-1}\rho^2(x,y)$;
- if $(x,y) \in \rho^3([1,3] \times [-1,1])$ let $h(x,y) = \psi^{-1}\rho(x,y)$.

Then h has order 4. If θ_t is the identity for all t then $h = \rho$, which just interchanges the coordinate axes and then reverses the sign of one. Otherwise this is certainly not the case for h. ■

Question 4.14 *Can we somehow be more specific concerning the homeomorphisms of finite order?*

For example are they all isotopic to a homeomorphism h of the form $h(x_1,\dots,x_n) = (\pm x_{\pi(1)},\dots,\pm x_{\pi(n)})$, where π is a permutation of $\{1,\dots,n\}$?

Question 4.15 *Do Theorems 4.11 and 4.12 hold for all n?*

We should be more specific with respect to Theorem 4.11: the intention of the question is that the torsion in $\mathcal{H}(\mathbb{L}^n_+)$ should be the same as that of $\mathcal{G}_{+,n}$. As the statement of Theorem 4.11 part 4 does not satisfy this condition, the question includes the case where $n = 4$.

Certainly large parts of the proofs are valid for the general case. The generalisation of the proof of Theorem 4.12 breaks down at the point where we show that $h^l|\partial S_\alpha$ is the identity because we cannot be sure that h sends each i-face of ∂S_α to an i-face except for $i = 0$. We have been able to show that for α in some closed, unbounded set there is a subset of each i-face which has non-empty interior in the face and which is mapped by h to an i-face but this does not appear to be sufficient to complete the proof.

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