Spaces with property pp[*]

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Abstract

This paper continues the study of pp spaces. It is shown that under a wide variety of circumstances, pp spaces are paracompact. However, examples of pp non-paracompact spaces are given, some have strong separation and covering properties, other examples fail dramatically to be paracompact. The stability of pp is examined and contrasted with paracompactness. Various strengthenings of pp are briefly examined.

1 Introduction

This paper is concerned with a family of topological properties related to paracompactness.

Definition 1 *Let $\mathcal{P}$ be a property of subsets of a space, and $\mathcal{Q}$ a property of families of subsets. Suppose $\mathcal{U}$ be an open cover of a topological space X. Then $\mathcal{V}$ is a* $\mathrm{pp}_{\mathcal{P}}^{\mathcal{Q}}$ refinement *of $\mathcal{U}$ if it is an open refinement of $\mathcal{U}$ such that if $S_V \subseteq V$, where S_V has $\mathcal{P}$, for each $V \in \mathcal{V}$, then $\{S_V\}_{V \in \mathcal{V}}$ has $\mathcal{Q}$.*

The space X is said to have property $\mathrm{pp}_{\mathcal{P}}^{\mathcal{Q}}$ *if every open cover has a $\mathrm{pp}_{\mathcal{P}}^{\mathcal{Q}}$ refinement.*

We omit the subscript, $\mathcal{P}$, or superscript, $\mathcal{Q}$, when the property is the one of the 'default' values: $\mathcal{P}$ =singleton, and $\mathcal{Q}$ =locally finite. Thus a pp refinement is an open refinement so that if a singleton point is picked from each member of the refinement, then the collection of all those points is locally finite. Clearly every pp space is T_1. In this paper all spaces are assumed to be T_1 unless otherwise stated.

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The property pp has a number of equivalents. For later use, we note (see [5, Lemma 17]) pp is equivalent to $\mathrm{pp}^{\mathrm{closed}\,\cup}$ and to $\mathrm{pp}^{\mathrm{discrete}\,\cup}$. Since pp is also equivalent to $\mathrm{pp}^{\mathrm{discrete}}$ (locally finite families of singletons are always discrete), our definition of pp coincides with that given by Matveev [7] when he introduced it as part of his study of property (a). Gauld [4], Gauld & Vamanamurthy [5] and Song [10] have continued the investigation of pp, and we answer here a number of questions raised in those papers. The central concern has been the relationship between pp and paracompactness.

Observe that $\mathrm{pp}_{\mathrm{open}}$ is trivially equivalent to paracompactness (in the form 'every open cover has a locally finite open refinement'). It is also immediate that $\mathrm{pp}_{\mathrm{open}} \implies \mathrm{pp}_{\mathrm{closed}} \implies \mathrm{pp}$. Thus paracompact spaces are pp. The converse is a more delicate matter. First countable pp spaces are paracompact [4, 10], and in section 2 we extend the equivalence of pp and paracompactness to q-spaces, T_3 LOB spaces, and (under (CH)) T_3 separable spaces. However, in section 3 we present a variety of pp spaces which are not paracompact. Indeed Hausdorff pp spaces need not be T_3, while T_3 pp spaces may fail to be T_4, and even perfectly normal, metacompact but non-paracompact pp spaces exist.

Section 4 considers the stability of property pp, again especially in relationship to the known stability characteristics of paracompactness. The most striking results here concern the behaviour of the pp property under products with a compact space. Finally in section 5 we briefly mention some results on other $\mathrm{pp}_{\mathcal{P}}^{\mathcal{Q}}$ properties which cropped up during our work on pp. The property $\mathrm{pp}_{\mathrm{closed}}$ looks quite intriguing.

2 Property pp and paracompactness

For a point x in a space X define a *local q-basis* at x to be any family, $\mathcal{Q}$, of open neighbourhoods of x such that if points x_Q distinct from x are chosen from each $Q \in \mathcal{Q}$, then $\{x_Q\}_{Q \in \mathcal{Q}}$ is not closed discrete. Note that a local basis at x is a local q-basis (the point x is not an x_Q but is in the closure of the x_Qs). Define the *q-character of* x, $q\chi(x, X)$, to be $\aleph_0$ if x is isolated and the minimal size of a local q-basis at x otherwise; and define the *q-character of* X, denoted $q\chi(X)$, to be the supremum of all $q\chi(x, X)$. In a *q-space* each point admits a sequence $\langle Q_n \rangle$ of neighbourhoods such that if $x_n \in Q_n$ then the sequence $\langle x_n \rangle$ clusters.

Lemma 2 *A space is a q-space if and only if it has countable q-character.*

Proof. First suppose the space X is a q-space. By definition every isolated point of X has countable q-character. So fix a non-isolated point x, and a sequence of open neighbourhoods of x, $\langle Q_n \rangle$, as in the definition of q-space. Now if for each n we have $x_n \in Q_n$ where $x_n \neq x$, then we know $\{x_n\}_n$ clusters. This implies $\{x_n\}_n$ is not closed discrete. So the family $\mathcal{Q} = \{Q_n\}_{n \in \omega}$ demonstrates that $q\chi(x, X) \leq \aleph_0$.

Now suppose every point of the space X has countable q-character. Take any point x in X. If x is isolated then define $\langle Q_n \rangle = \langle \{x\} \rangle$. This clearly works (the point x is a cluster point of any sequence of points chosen from the open neighbourhoods, Q_n, of x). So let us assume x is not isolated. Then there is a local q-basis $\{Q_n\}_{n \in \omega}$ at x. Let $\langle Q_n \rangle$ be the corresponding sequence. We show it satisfies the conditions in the definition of a q-space.

To this end take any $x_n \in Q_n$. If infinitely many of the x_ns are x, then the x_ns cluster at x, and we are done. Otherwise only finitely many of the x_ns are equal to x. Relabelling if necessary we may assume that $x_1, \ldots, x_N$ equal x but all $x_n \neq x$ for $n > N$. As x is not isolated, for $n = 1, \ldots, N$ we can pick $y_n \in Q_n \setminus \{x\}$, and set $y_n = x_n$ for $n > N$. Now all the y_ns are distinct from x and chosen from the corresponding Q_n. By the definition of q-basis, the set $\{y_n\}_n$ is not closed discrete — say every neighbourhood of some point y meets this set in at least two points. Clearly y can not be isolated. Further, since X is T_1, every neighbourhood of y must meet $\{y_n\}_n$ in an infinite set. But this means every neighbourhood of y must meet $\{x_n\}_n \supseteq \{y_n : n > N\}$ in an infinite set, which is equivalent to saying the x_ns cluster at y, as required. ■

Theorem 3 *Fix a space X. Let $\mathcal{V}$ be a pp refinement of an open cover $\mathcal{U}$ of X, and let x be a non-isolated point. Then there is an open neighbourhood N_x of x meeting $< q\chi(x, X)$ many members of $\mathcal{V}$.*

Define $C = X \setminus \bigcup\{N_x : x \text{ is non-isolated}\}$, and $\mathcal{V}^* = \{V \setminus C : V \in \mathcal{V}\} \cup \{\{c\} : c \in C\}$. Then $\mathcal{V}^*$ is a pp refinement of $\mathcal{U}$ which is locally $< q\chi(X)$.

Call a pp refinement $\mathcal{V}^*$ of an open cover $\mathcal{U}$ with the above property a tidy pp refinement.

Proof. We only need prove the first part. The remainder follows immediately. Fix, then, a non-isolated point x with local q-basis $\{Q_\alpha : \alpha < \kappa\}$, where $\kappa = q\chi(x, X)$.

Suppose, for a contradiction, $\mathcal{V}$ is not locally $< \kappa$ at x. By transfinite recursion pick points x_α and sets V_α, for all $\alpha < \kappa$, such that $x_\alpha \in V_\alpha \cap Q_\alpha$ and $V_\beta \neq V_\alpha$ when $\alpha \neq \beta$. This is possible because if x_β and V_β have been selected for $\beta < \alpha$, then $\mathcal{V}_\alpha = \{V \in \mathcal{V} : V \cap Q_\alpha \neq \emptyset\}$ has size at least κ, and $\mathcal{V}_\alpha \setminus \{V_\beta : \beta < \alpha\}$ is non-empty – so pick V_α from this set, and $x_\alpha \in V_\alpha \cap Q_\alpha$.

Now the x_αs above come from different members of the pp-refinement $\mathcal{V}$, so they form a closed discrete set. But the Q_αs form a local q-basis, so the x_αs are not closed discrete – contradiction. ■

The following corollary of Theorem 3 extends the result of [4] and [10], that a first countable pp space is paracompact.

Corollary 4 *Every q-space with property pp is paracompact.*

Compare the following result with the result that every T_3 separable paracompact space is Lindelöf.

Proposition 5 (CH) *Suppose that X is a T_3, separable space with property pp. Then X is Lindelöf, hence paracompact.*

Proof. Fix a countable dense subset $D \subset X$. Note that by regularity the collection of regular open sets forms a basis for the topology on X and, by separability, there are no more than $2^{\aleph_0}$ such sets. Thus, by (CH), the q-character of X is $\leq \aleph_1$.

Let $\mathcal{U}$ be an open cover of X and let $\mathcal{V}^*$ be a tidy pp refinement. Then $\mathcal{V}^*$ is a locally $< \aleph_1$ open refinement of $\mathcal{U}$. Since locally countable families of open sets are countable in separable (or even calibre ω_1) spaces, we are done. ■

Question 6 *Is there a consistent example of a T_3 separable pp space which is not paracompact?*

Can 'separability' in the preceding proposition be weakened to 'calibre ω_1' or 'countable chain condition'?

Proposition 7 *If X is a T_3 space having property pp and every point of X has a totally ordered local base then X is paracompact.*

Proof. We use the fact that every T_3 space in which every open cover has a closure-preserving open refinement is paracompact.

Let $\mathcal{U}$ be an open cover of X and let $\mathcal{V}$ be a tidy pp refinement. It is claimed that $\mathcal{V}$ is closure-preserving. Suppose instead that $\mathcal{V}$ is not closure-preserving: then there is a subfamily $\mathcal{V}' \subseteq \mathcal{V}$ and $x \in X$ with $x \in \overline{\bigcup\{V : V \in \mathcal{V}'\}}$ but $x \notin \bigcup\{\overline{V} : V \in \mathcal{V}'\}$. Note that x cannot be an isolated point.

Let $\mathcal{B}_x$ be a local base at x totally ordered by $\subseteq$. Assume that $|\mathcal{B}_x| = \chi(x, X) = \kappa$; note that κ is regular. By Theorem 3 $\mathcal{V}$ is locally $< \kappa$ at x.

Suppose that $O \subseteq X$ is an open set containing x such that $\mathcal{V}_O = \{V \in \mathcal{V}' : O \cap V \neq \varnothing\}$ has $|\mathcal{V}_O| < \kappa$.

As $x \notin \cup\{\overline{V} : V \in \mathcal{V}'\}$, for each $V \in \mathcal{V}_O$ we may choose $B_V \in \mathcal{B}_x$ with $B_V \cap V = \varnothing$. Because $|\{B_V : V \in \mathcal{V}_O\}| < \kappa$ it follows that there is $\widehat{B} \in \mathcal{B}_x$ such that $\widehat{B} \subseteq B_V$ for each $V \in \mathcal{V}_O$. We may assume that $\widehat{B} \subseteq O$.

Note that $\widehat{B}$ is a neighbourhood of x which meets no member of $\mathcal{V}_O$ and hence that $\widehat{B} \cap (\bigcup\{V : V \in \mathcal{V}'\}) = \varnothing$, which implies that $x \notin \overline{\bigcup\{V : V \in \mathcal{V}'\}}$. ■

As noted in [4], locally compact pp spaces are paracompact. Recall that a space is countably compact if and only if all closed discrete subsets are finite. So it is immediate that:

Lemma 8 *Locally countably compact pp spaces are paracompact.*

Question 9 *Are (Tychonoff) pseudocompact pp spaces (para)compact?*

In a similar vein, Lindelöf degree and extent agree in pp spaces. Recall that the *Lindelöf degree*, $L(X)$, of a space X is $\leq \kappa$ if and only if each open cover has a subcover of cardinality at most κ, while the *extent*, $e(X)$, is $\leq \kappa$ if and only if each closed discrete subset of X has cardinality at most κ.

Proposition 10 *If X has property pp then $L(X) = e(X)$.*

Proof. As it is always the case that $e(X) \leq L(X)$ we need only show that $L(X) \leq e(X)$. More specifically we show that if κ is any cardinal and $L(X) > \kappa$ then $e(X) > \kappa$.

Suppose that $\mathcal{U}$ is an open cover of X having no subcover of cardinality $\leq \kappa$. Let $\mathcal{V}$ be given by applying property pp to $\mathcal{U}$. For any ordinal $\alpha < \kappa^+$ pick recursively x_α and V_α with $x_\alpha \in V_\alpha \in \mathcal{V}$ and $x_\beta \neq x_\alpha$ and $V_\beta \neq V_\alpha$ whenever $\beta < \alpha$. If x_β and V_β have been picked for all $\beta < \alpha$ then $\{V_\beta : \beta < \alpha\}$ cannot cover X so there is $V_\alpha \in \mathcal{V} \setminus \{V_\beta : \beta < \alpha\}$ with $V_\alpha \setminus (\bigcup\{V_\beta : \beta < \alpha\}) \neq \varnothing$; pick x_α from within this set.

Set $E = \{x_\alpha : \alpha < \kappa^+\}$. Then by the pp choice of $\mathcal{V}$, the set E is closed and discrete. Moreover $|E| > \kappa$. Hence $e(X) > \kappa$. ■

Since stationary subsets of uncountable regular cardinals are easily seen to not be pp, it follows from the Rudin-Balogh theorem [1] that:

Lemma 11 *Monotonically normal pp spaces are paracompact.*

This provides a nice companion to the results of the next section.

3 PP is not paracompact

Example 12 *Let X be ω_1 with the co-countable topology. This space is T_1, pp but not T_2 or paracompact.*

Example 13 *Let X be $\mathbb{R}$ with topology σ obtained from the standard topology, τ, by declaring all sets of the form $U \setminus C$ to be open, where U is a standard open set and C is countable.*

This space is T_2, pp but not T_3 (and hence not paracompact).

Proof. Since σ refines τ, certainly X is T_2.

Now observe that (1) every countable subset of X is closed and discrete, and (2) X is Lindelöf. It follows immediately that X is pp. For if $\mathcal{U}$ is an open cover of X then there is a countable subcover, say $\mathcal{V}$. Now, for any selection of points $x_V \in V$, for $V \in \mathcal{V}$, the set $\{x_V : V \in \mathcal{V}\}$ is countable, hence closed and discrete in X.

Since the closure of an open set, $U \setminus C$, coincides with the Euclidean closure, it is evident no neighbourhood of 0 has closure contained in $(-1,1) \setminus \{q + \pi : q \in \mathbb{Q}\}$. Hence this space is not regular. ∎

A whole class of T_3, pp not paracompact examples can be created via the following two constructions.

Construction 1 *Let (X', σ') be a space with subspace M whose complement contains only isolated points. Define (X, σ) to have underlying set X' and topology σ consisting of all sets of the form $U \setminus S$ where U is σ'-open and $S \subseteq X \setminus M$ with $|S| \leq w(M)$.*

If M is pp, then (X, σ) is pp.

If (X', σ') has property $\mathcal{P}$ then (X, σ) has $\mathcal{P}$ for each of the following properties: 'zero-dimensional', '(collectionwise) T_2', '(collectionwise) normal', '(countably) metacompact', '(countably) paracompact', and 'perfect'.

If

> *$(*)$ for all σ'-open sets U and V both meeting M such that $U \cap V \neq \emptyset$, we have $|U \cap V| > w(M)$,*

then (X', σ') has property $\mathcal{P}$ when (X, σ) has $\mathcal{P}$ for each of the following properties: 'zero-dimensional', '(collectionwise) T_2', '(collectionwise) normal', and 'perfect'.

If $(*)$ is strengthened to

> *$(*_\omega)$ for all countable families of σ'-open sets $(U_n)_n$ each meeting M such that $\bigcap_n U_n \neq \emptyset$, we have $|\bigcap_n U_n| > w(M)$,*

then if (X, σ) is (countably) metacompact or paracompact, then so is (X', σ').

Proof. Note first that (X, σ) really is a topological space and M has the same subspace topology in both σ' and σ.

Take any open cover $\mathcal{U}$ of (X, σ). Then $\mathcal{U}|M = \{U \cap M : U \in \mathcal{U}\}$ is an open cover of M. By hypothesis, it has a pp-refinement, $\mathcal{V}|M$. Since M has weight $w(M)$, we can assume $\mathcal{V}|M$ has cardinality $\leq w(M)$. For each element of $\mathcal{V}|M$ pick a σ-open set contained in some member of $\mathcal{U}$. Gather these together in $\mathcal{V}_0$. Let $\mathcal{V}_1 = \{\{x\} : x \notin \bigcup\{V : V \in \mathcal{V}_0\}\}$. Then $\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1$ is an open refinement of $\mathcal{U}$. We check it is pp-refinement.

Well, take a point x_V from each $V \in \mathcal{V}$. Three cases arise. The first is when x_V is in $R = X \setminus \bigcup\{V : V \in \mathcal{V}_0\}$. These cause no trouble because R is closed discrete. The second is when x_V is in M. But in this case x_V is in element of the pp-refinement $\mathcal{V}|M$, and since M is closed these also cause no trouble. Finally x_V might be in some V in $\mathcal{V}_0$ but not in M. However, there are $\leq w(M)$ many elements of $\mathcal{V}_0$, and so $\leq w(M)$ many such x_V. Now if y is any point in X, then either $\{y\}$, if y is isolated, or $U \setminus S$ where $U = X$ and $S = \{x_V : V \in \mathcal{V}_0, x_V \notin M\}$, if y is in M, is an open neighbourhood meeting these x_Vs in at most one point. Thus the x_Vs form a closed discrete set, as required.

It is easy to check that (X, σ) is zero-dimensional if and only if (X', σ') is zero-dimensional.

To see that (collectionwise) Hausdorff and (collectionwise) normal transfer from (X', σ') to (X, σ), it really suffices to observe that we need only separate points or closed sets in M, and that σ refines σ'.

To move these separation axioms from (X, σ) to (X', σ'), assuming $(*)$, the key is that if U, V are σ'-open sets which meet M and each other, then for any $S, T \subseteq X \setminus M$ with $|S|, |T| \leq w(M)$ the σ-open sets $U \setminus S$ and $V \setminus T$ must also meet (indeed, $|(U \setminus S) \cap (V \setminus T)| \geq |(U \cap V) \setminus (S \cup T)| > w(M)$). Now if (X', σ') is not Hausdorff (for example), then there are two points of M which can not be separated by σ'-open sets, and we now see they can not be separated by σ-open sets either.

Perfectness can be treated similarly.

Now suppose (X', σ') is metacompact. (The cases of countable metacompactness and (countable) paracompactness follow similarly.) Take any open cover $\mathcal{U}$ of (X, σ). Then each element of $\mathcal{U}$ is of the form $U \setminus S$ as in the definition of σ. Collect all these Us together to form $\mathcal{U}'$. This is a σ'-open cover of X'. Take a point-finite open refinement $\mathcal{V}'$. For each $V \in \mathcal{V}'$ pick a $U_V \setminus S_V$ from $\mathcal{U}$ so that $V \subseteq U_V$. Set $\mathcal{V}$ to be all $V \subseteq S_V$ such that $V \in \mathcal{V}'$ along with all singletons not covered so far. This is easily checked to be a point-finite open refinement of $\mathcal{U}$.

Our last task is to show that if $(*_\omega)$ holds, and (X,σ) is metacompact then so is (X',σ') (the other cases are similar). Take any σ'-open cover, $\mathcal{U}'$ of X'. This is also a σ-open cover of X, so it has a point-finite σ-open refinement $\mathcal{V}$. For each $x \in M$ pick $V_x \setminus S_x$ from $\mathcal{V}$ containing x. Since $\mathcal{V}$ refines $\mathcal{U}''$ there is an U_x in $\mathcal{U}'$ such that $V_x \setminus S_x \subseteq U_x$. Let $\mathcal{V}_0' = \{U_x \cap V_x : x \in M\}$, $\mathcal{V}_1' = \{\{x\} : x \notin \bigcup \mathcal{V}_0'\}$ and $\mathcal{V}' = \mathcal{V}_0' \cup \mathcal{V}_1'$. Then $\mathcal{V}'$ is an open refinement of $\mathcal{U}'$. We check it is point-finite. If not, then there must be $V_1 \cap U_1, V_2 \cap U_2, \ldots V_n \cap U_n, \ldots$ in $\mathcal{V}_0'$ each meeting M and with non-empty intersection. Condition $(*_\omega)$ then tells us that $\bigcap V_n$ has cardinality $> w(M)$. But now this means that the intersection of the elements of $\mathcal{V}$ corresponding to the $V_n \cap U_n$s, say $V_1 \setminus S_1, V_2 \setminus S_2, \ldots$, have non-empty intersection (indeed, $|\bigcap_n (V_n \setminus S_n)| \geq |\bigcap_n V_n - \bigcup_n S_n| > w(M)$). ■

Construction 2 *Let (X'',σ'') have a subspace M whose complement contains only isolated points. Let λ be a cardinal. Define (X',σ') to have underlying set $X' = M \cup (X'' \times \lambda)$ and topology σ' where all points of $X'' \times \lambda$ are isolated, and other basic open sets are of the form $(U \cap M) \cup (U \times (\lambda \setminus F))$ for any σ''-open set and finite $F \subseteq \lambda$.*

Then (X'',σ'') is T_4 (respectively, metacompact) if and only if (X',σ') is T_4 (resp., metacompact).

Further, if M as a subspace of (X'',σ'') is λ-perfect (every open subset is the union of $\leq \lambda$-many closed sets), then (X',σ') is λ-perfect.

And if $\lambda > w(M)$, or if (X'',σ'') satisfies $(*_\omega)$, then (X',σ') satisfies $(*_\omega)$.

Proof. First note that σ' is indeed a topology on X', and M has the same subspace topology in each topology. It is entirely routine checking that normality and metacompactness are preserved between (X'',σ'') and (X',σ').

Now suppose M is λ-perfect, and U is a σ'-open subset of X'. We need to show that U is the union of $\leq \lambda$ many σ'-closed sets. As M is λ-perfect, and σ'-closed, $U \cap M$ is the union of $\leq \lambda$ many closed (in either topology) sets. Moreover, each $X'' \times \{\alpha\}$ is closed discrete in (X',σ'), so each $(X'' \times \{\alpha\}) \cap U$ is σ'-closed, and their union, $(X' \setminus M) \cap U$ is the union of $\leq \lambda$ many σ'-closed sets. Together these last two statements prove that U is the union of $\leq \lambda$ many σ'-closed sets — as required.

If (X'',σ'') satisfies $(*_\omega)$ then it is clear that so does (X',σ'). While if $\lambda > w(M)$, then given a countable family $(U_n)_n$ of σ'-open sets each meeting M and with non-empty intersection, we can assume the U_ns are all basic, say $U_n = (V_n \cap M) \cup (V_n \times (\lambda \setminus F_n))$, where each U_n is open in (X'',σ'') and

F_n is finite, and we can fix some $x \in M \cap \bigcap_n U_n$. Since λ is uncountable, there are λ many αs in $\lambda \setminus \bigcup_n F_n$, and λ many corresponding points (x, α) in the intersection of the U_ns. Since $\lambda > w(M)$, $(*_\omega)$ holds. ∎

Example 14 *There is a T_3, pp space which is not T_4.*

Proof. Let (X', σ') be the space obtained by starting with $X' = 2^{\mathfrak{c}}$, setting $M = \{\mathbf{x}_\alpha : \alpha \in \mathfrak{c}\}$ where $\mathbf{x}_\alpha(\beta) = 1$ if and only if $\beta = \alpha$, and refining the Tychonoff product topology (call it τ) on X' by declaring the points in the complement of M to be isolated.

Let (X, σ) be the space given by Construction 1. This space is pp.

Note that non-empty τ-open sets have cardinality $2^{\mathfrak{c}} > \mathfrak{c} = w(M)$. Hence (X', σ') satisfies $(*)$. Since (X', σ') is T_2, zero-dimensional, so to is (X, σ).

Note that M is closed discrete in (X', σ'), but (X', τ) is separable, so, by a standard Jones' Lemma argument, (X', σ') is not normal. Hence (X, σ) is also not normal. ∎

Example 15 *There is a T_4, pp but not collectionwise Hausdorff or metacompact space.*

There is a T_4, metacompact, pp space which is not collectionwise T_2.

Proof. Let (X', σ') be Bing's G ([2], or see [6] pg 748). Then (X', σ') is created in a similar way to the preceding example. Hence, as there, (X', σ') satisfies $(*)$. Indeed, non-empty G_δ subsets of a Tychonoff cube 2^κ, have cardinality 2^κ. Hence (X', σ') satisfies the stronger property $(*_\omega)$. Let (X, σ) be the space created from Bing's G by Construction 1.

Then (X, σ) is pp, T_4 but not collectionwise normal or metacompact, because (X', σ') has the same properties.

To get metacompactness, start instead with Michael's metacompact variation ([8] or see [6] pg 707) of Bing's G, call it (X'', σ''). Here both M and its complement have cardinality ω_1. Hence this can not be plugged directly into Construction 1. Instead first apply Construction 2 with $\lambda = \omega_2$ giving (X', σ'). This latter space is normal, metacompact, not collectionwise Hausdorff, and satisfies $(*_\omega)$. Hence Construction 1 applied to this gives a normal, metacompact non-collectionwise T_2 pp space. ∎

Example 16 *There is a perfectly normal pp space which is not collectionwise T_2 or metacompact.*

There is a space which is perfectly normal, metacompact and pp but not collectionwise T_2.

Proof. To get perfect normality, proceed as above but additionally apply Construction 2 with $\lambda = \omega$. (Since M is perfect, this yields perfectness of the final space. Also note that Construction 2 applied to Bing's G is precisely Bing's H.) ■

Example 17 *There is a normal, metacompact, paraLindelöf, collectionwise Hausdorff pp space which is not collectionwise normal.*

Proof. Apply Construction 1 to Navy's space ([9] or see [6] pp 750–752). This has the relevant properties. Further it is of the form $M \cup Q$ where M is metrisable of weight ω_1 and points of Q are isolated points. The trace of an open neighbourhood of a point in M onto Q is a union of traces of a Bing's G open set onto the isolated points (of Bing's G). Hence, as for Bing's G, Navy's space has $(*_\omega)$, so the desired properties are transferred to (X, σ). ■

The most complete failure of paracompactness in a pp space is given by:

Example 18 *There is a T_3 pp space which is not countably metacompact.*

Proof. We use Chaber's almost Dowker space ([3] or see [6] pg 775). This has $\mathfrak{c}$ many isolated points (the cardinality of the irrationals) while the set of non-isolated points, M, is closed discrete and has cardinality $\leq \mathfrak{c}$ (the cardinality of $|\mathbb{Q}|$ many collections of maximal almost disjoint families on $\mathbb{Q}$).

So apply Construction 2 to Chaber's space with $\lambda = 2^{\mathfrak{c}}$. This gives a non-countably metacompact T_3 space with $(*_\omega)$. Applying Construction 1 gives the desired pp example. ■

Question 19 *Is there a normal pp space which is not countably paracompact? In other words, is there a pp Dowker space?*

4 Stability of property pp

The stability of property pp has a lot in common with that of paracompactness. But at each step there are some interesting differences.

First consider subspaces. It is clear that every closed subspace of a space with property pp has property pp. However, ω_1 is an open subspace of its one-point compactification, $\omega_1 + 1$, and while $\omega_1 + 1$ is certainly pp (it is compact!), ω_1 is not pp (it is first countable, but not paracompact).

So far pp follows paracompactness. But let us turn to inheritance to (regular) F_σ subspaces. Recall that both paracompactness and metacompactness are hereditary to F_σ subsets. Thus perfectly normal paracompact (respectively, metacompact) spaces are hereditarily paracompact (resp., metacompact). It seems likely to the authors that pp is not F_σ hereditary, although we don't have a counterexample. However, pp is hereditary to regular F_σ subsets (a subset A of a space is a *regular* F_σ if there are open sets $U_1, U_2, \ldots$ so that $A = \bigcup_n U_n = \bigcup_n \overline{U_n}$). It is easy to check that pp is hereditary (to all subspaces) provided it is hereditary to all open subspaces. Hence perfectly normal pp spaces are hereditarily pp.

Theorem 20 *The pp property is hereditary to regular* F_σ *subsets*

Proof. Let X be a space and Y a regular F_σ subset. Fix open sets $U_1, U_2, \ldots$ so that $Y = \bigcup_n U_n = \bigcup_n \overline{U_n}$. We can suppose that $\overline{U_n} \subset U_{n+1}$ for all n. We shall also declare $U_0 = U_{-1} = \varnothing$. Let $\mathcal{U}$ be a cover of Y by open subsets of Y, hence open subsets of X. For each $n \in \mathbb{N}$, let $\mathcal{U}_n = \{U \cap U_n \ : \ U \in \mathcal{U}\}$. The collection $\{X \setminus \overline{U_{n-1}}\} \cup \mathcal{U}_n$ is an open cover of X and must have a pp refinement, say $\mathcal{V}_n$. Let $\mathcal{W}_n = \{V \cap (Y \setminus \overline{U_{n-2}}) \ : \ V \in \mathcal{V}_n \text{ and } V \cap U_{n-1} \neq \varnothing\}$. We claim that the collection $\mathcal{W} = \bigcup_{n=1}^{\infty} \mathcal{W}_n$ is a pp refinement for $\mathcal{U}$.

We show that $\mathcal{W}_n$ refines $\mathcal{U}_n$, and hence $\mathcal{W}$ refines $\mathcal{U}$. Take any $W \in \mathcal{W}_n$, say $W = V \cap (Y \setminus \overline{U_{n-2}})$ with $V \in \mathcal{V}_n$ and $V \cap U_{n-1} \neq \varnothing$. As $\mathcal{V}_n$ refines $\{X \setminus \overline{U_{n-1}}\} \cup \mathcal{U}_n$, there is a U in this latter collection such that $V \subseteq U$. Since V meets U_{n-1} we must have U also meeting U_{n-1} and hence $U \in \mathcal{U}_n$. Then $W \subseteq U \in \mathcal{U}_n$.

Now we show that $\mathcal{W}$ covers Y. Take any $y \in Y$ and choose the least n with $y \in \overline{U_n}$. As $\mathcal{V}_{n+1}$ covers X there is $V \in \mathcal{V}_{n+1}$ with $y \in V$. Since $y \in \overline{U_n}$ it follows that $V \cap U_n \neq \varnothing$ so $W = V \cap (Y \setminus \overline{U_{n-1}}) \in \mathcal{W}_{n+1}$. Also as $y \notin \overline{U_{n-1}}$ it follows that $y \in W$.

Finally we show that $\mathcal{W}$ is a pp refinement. Suppose for each $W \in \mathcal{W}$ there is given a point $y_W \in W$. We will show that $S = \{y_W \ : \ W \in \mathcal{W}\}$ is closed and discrete by breaking it into a locally finite collection of closed discrete pieces. Let $S_n = \{y_W \ : \ W \in \mathcal{W}_n\}$. We see that S_n is closed and discrete because $\mathcal{W}_n$ is a precise refinement of $\mathcal{V}_n$ so, like $\mathcal{V}_n$, must be pp. For $m \geq n-1$, let $T_{n,m} = S_n \cap (\overline{U_m} \setminus U_{m-1})$, also closed and discrete because $\overline{U_m} \setminus U_{m-1}$ is closed. As $S_n \subseteq Y \setminus \overline{U_{n-2}}$ it follows that $\bigcup_{m \geq n-1} T_{n,m} = S_n$ and hence that $S = \bigcup_n S_n = \bigcup_n \bigcup_{m \geq n-1} T_{n,m}$. The collection of all sets $T_{n,m}$ is locally finite because every $y \in Y$ is in some U_p and this open neighbourhood of y can meet $T_{n,m}$ only for $n - 1 \leq m \leq p$. ■

Corollary 21 *Perfectly normal pp spaces are hereditarily pp.*

Now we turn to the behaviour of pp under products.

Example 22 *Let* X *be the real line with the right half-open topology (the Sorgenfrey line). Then* X *has property pp but* $X \times X$ *does not.*

Proof. X has property pp, being T$_1$ and paracompact. Also X, and hence $X \times X$, is first countable. Then by Corollary 4 $X \times X$ does not have property pp as it is not paracompact. ∎

So the squaring behaviour of pp mirrors that of paracompactness. The situation with products of pp spaces and compact spaces is more interesting. Recall that the product of a paracompact space and a compact space is always paracompact. Also a product of metacompact and compact spaces is metacompact. The situation with pp is different: the product of a pp space and a compact space may, or may not, be pp. Which case occurs seems to depend on metacompactness and the strengthening $\mathrm{pp}_{\text{finite}}$ of pp in which finite subsets can be chosen from the elements of the open refinement, not just singletons.

Theorem 23 *If every open cover of a space X has a point-finite $\mathrm{pp}_{\mathit{finite}}$ refinement, then $X \times K$ is pp for every compact K.*

Proof. Let $\mathcal{U}_{X\times K}$ be an open cover of $X \times K$. We can assume all members of the cover are open rectangles. For each $x \in X$, pick $S_{x,i} \times T_{x,i}$, for $i = 1 \dots n_x$, from $\mathcal{U}_{X\times K}$, covering $\{x\} \times K$. Let $U_x = \bigcap_i S_{x,i}$ and $\mathcal{U}_X = \{U_x : x \in X\}$.

By hypothesis, $\mathcal{U}_X$ has a point-finite $\mathrm{pp}_{<\omega}$ refinement $\mathcal{V}_X$. For each $V \in \mathcal{V}_X$, fix $U_{x_V} \in \mathcal{U}_{X\times K}$ so that $V \subseteq U_{x_V}$. Let $\mathcal{V}_{X\times K} = \{V \times T_{x_V,i} : V \in \mathcal{V}_X, \text{ and } i = 1 \dots n_{x_V}\}$. Then $\mathcal{V}_{X\times K}$ is an open cover of $X \times K$ refining $\mathcal{U}_{X\times K}$. We show it is a pp refinement.

Suppose, for each $V \in \mathcal{V}$ and $1 \leq i \leq n_{x_V}$, we select a point (p_i, q_i) in $V \times T_{x_V,i}$. The finite set $F_V = \{p_i : i = 1 \dots n_{x_V}\}$ is a subset of V. Hence, by the $\mathrm{pp}_{<\omega}$ property, their union $P = \bigcup_{V\in\mathcal{V}} F_V$ is closed discrete. Take any point (p, q) in the product. If $p \notin P$, then p has a neighbourhood N_p disjoint from P. In this case $N_p \times K$ is a neighbourhood of (p, q) disjoint from all the (p_i, q_i)s — as required for discreteness. If $p \in P$, then p has a neighbourhood N_p meeting P only at p. So the neighbourhood $N_p \times K$ of (p, q) can only contain a (p_i, q_i) if $p_i = p$. Since $\mathcal{V}_X$ is point-finite at p, and for each V from $\mathcal{V}_X$ containing p there are only finitely many corresponding $T_{x,i}$s, there are only finitely many selected points of the form (p, q_i). Hence by the T_1 property those (p, q_i) distinct from (p, q) can be deleted from $N_p \times K$ to get a neighbourhood of (p, q) meeting all (p_i, q_i)s only in (p, q) — as required for discreteness. ∎

Theorem 24 *If for some infinite compact space, K, the product $X \times K$ is pp, then every open cover of X has a point-finite pp refinement. In particular, X is metacompact.*

Proof. Fix K an infinite compact space so that $X \times K$ is pp. Fix a non-isolated point x_0 of K. For each n, we can find (as K is infinite) an increasing open cover $\mathcal{T}_n$ of K containing at least n elements. Let $\mathcal{U}_X$ be an open cover of X, and N be a function from $\mathcal{U}$ to $\mathbb{N}$. Let $\mathcal{U}_{X\times K} = \{U \times T_i : U \in \mathcal{U}$, and $T_i \in \mathcal{T}_{N(U)}$ for $1 \leq i \leq N(U)\}$. This open cover of $X \times K$ has a pp refinement $\mathcal{V}_{X\times K}$. Let $\mathcal{V}_X = \{V \cap (X \times \{x_0\}) : V \in \mathcal{V}_{X\times K}\}$. Identifying X with $X \times \{x_0\}$, this gives a pp refinement of $\mathcal{U}$. It remains to show $\mathcal{V}_X$ is point-finite.

Suppose not, say (x, x_0) is in distinct $V_n \in \mathcal{V}_{X\times K}$ $(n \in \mathbb{N})$. Then by the definition of the product topology, and recalling that x_0 is non-isolated, a simple counting argument can be applied to pick $(x, y_n) \in V_n$ where the y_ns are distinct. But these points selected from elements of the pp refinement $\mathcal{V}_{X\times K}$ form an infinite closed discrete set in the (countably) compact space $\{x\} \times K$ — impossible. ∎

Example 25 *Let X be the perfectly normal, non-metacompact pp space of Example 16. Since X is not metacompact, $X \times (\omega + 1)$ is not pp.*

Example 26 *Let X be the perfectly normal, metacompact but not paracompact space of Example 16. Then for any compact space K, the product $X \times K$ is pp.*

Proof. We need to check this space satisfies the conditions of Theorem 23. Take any open cover $\mathcal{U}$ of X. For this particular space, the set M is (closed) discrete, so we can assume (taking a refinement if necessary) that for each $m \in M$ there is a V_m in $\mathcal{U}$ so that $V_m \cap M = \{m\}$, but that all other elements of $\mathcal{U}$ are singletons from $X \setminus M$. Take a point-finite open refinement, $\mathcal{V}$, of $\mathcal{U}$. We check $\mathcal{V}$ is a $\text{pp}_{\text{finite}}$ refinement. So for each $V \in \mathcal{V}$ fix a finite $F_V \subseteq V$. We will show that $F = \bigcup_V F_V$ is the union of three closed discrete sets, and hence is closed discrete. Let $R = X \setminus \bigcup\{V \in \mathcal{V} : V \cap M = \emptyset\}$. Then R is closed discrete, and so is $F \cap R$. Also M is closed discrete, and thus $F \cap M$ is closed discrete. Finally, the set $F' = F \setminus (R \cup M)$ consists of (the union of) finite subsets of $(X \setminus M)$ each selected from an element of $\mathcal{V}$ meeting M. But there are only $|M|$ many such members of $\mathcal{V}$. Hence F' is disjoint from M with $|F'| \leq w(M)$, and the definition of the topology σ on X ensures that such sets are closed discrete. ∎

To complete this examination of the stability of pp we consider maps. Since the projection map onto a compact space is a perfect map, Example 25 above demonstrates that the perfect pre-image of a pp space need not be pp. Again, this is in contrast to the behaviour of paracompactness. On the other hand for forward images pp follows paracompactness. Since every first countable space is the open image of a metrisable space, it is easy to see that the open image of a pp (indeed metrisable) space need not be pp. For closed images we have preservation:

Lemma 27 *The closed image of a pp space is pp.*

Proof. Let f be a closed map from the pp space X onto Y. Recall that pp is equivalent to $\text{pp}^{\text{closed } \cup}$, so it is sufficient to show that Y is $\text{pp}^{\text{closed } \cup}$. In other words, for every open cover of Y we need to find an open refinement $\mathcal{V}_Y$ so that if we choose $y_V \in V$ for each $V \in \mathcal{V}_Y$, then $\{\{y_V\} : V \in \mathcal{V}_Y\}$ has closed union – i.e. $\{y_v : V \in \mathcal{V}_Y\}$ is closed.

To this end, take any $\mathcal{U}_Y$ an open cover of Y. Set $\mathcal{U}_X = \{f^{-1}(U) : U \in \mathcal{U}_Y\}$. Let $\mathcal{V}_X$ be a pp refinement of $\mathcal{U}_X$. Now set $\mathcal{V}_Y = \{f^{\#}(V) : V \in \mathcal{V}_X\}$ where $f^{\#}(U) = \{y \in Y : f^{-1}\{y\} \subseteq U\}$ (the 'small image' of U).

Then, since f is closed, $\mathcal{V}_Y$ is an open refinement of $\mathcal{U}_Y$. Suppose $y_V \in V$ for each $V \in \mathcal{V}_Y$. For each $V \in \mathcal{V}_Y$, pick $V' \in \mathcal{V}_X$ so that $f^{\#}(V') = V$ and $x_V \in V'$ such that $f(x_V) = y_V$. Now, as $\mathcal{V}_X$ is a pp refinement, the set $C = \{x_V : V \in \mathcal{V}_Y\}$ is closed, and since f is a closed map, the set $\{y_V : V \in \mathcal{V}_Y\} = f(C)$ is also closed — as desired. ■

5 Strengthening property pp

Considering the gap between property pp and paracompactness, we are led to examine strengthenings of pp which might close the difference. Two different approaches suggest themselves: either say something stronger about the closed discrete set given by pp; or select sets larger than single points. (In this section we defer all proofs to another paper.)

In the first direction, recall that paracompact T_2 spaces are strongly collectionwise T_2 — every closed discrete set has a discrete open expansion. Thus we introduce *stronger pp*=$\text{pp}^{\text{discrete open expansion}}$ and *strong pp*=$\text{pp}^{\text{disjoint open expansion}}$, and we see that paracompact T_2 implies stronger pp, which implies strong pp. Since pp is equivalent to $\text{pp}^{\text{discrete } \cup}$, strong pp implies pp.

In fact, strong pp is equivalent to 'pp and collectionwise T_2', and stronger pp is equivalent to 'pp and strongly collectionwise T_2'. Examples 14, 15 and 16 all give T_3 pp spaces which are not collectionwise Hausdorff, and hence not strong pp. Example 17 (built from Navy's space) is T_3, pp, collectionwise Hausdorff and hence strong pp, but is not strongly collectionwise T_2, and so is not stronger pp. XXXX a 'stronger pp not paracompact' space???

In the second direction attention is immediately drawn to $\mathrm{pp}_{\mathrm{closed}}$. (In the following we restrict our attention to T_3 spaces.) It is immediate that $\mathrm{pp}_{\mathrm{closed}}$ combined with the shrinking property of open covers (for every open cover, $\mathcal{U}$, there is closed cover $\mathcal{C} = \{C_U : U \in \mathcal{U}\}$, where $C_U \subseteq U$) is equivalent to paracompactness. The shrinking property implies collectionwise normality and countable paracompactness. We can show that $\mathrm{pp}_{\mathrm{closed}}$ implies collectionwise Hausdorff. But after that we are left with numerous intriguing open questions:

Question 28 *Are T_3 pp_{closed} spaces paracompact?*

What if we add collectionwise normality? Or metacompactness?

At the other extreme, is there a pp_{closed} space which is not countably metacompact? Or a pp_{closed} Dowker space?

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