Homogeneous and Inhomogeneous Manifolds *

Paul Gartside, Sina Greenwood and David Gauld †

April 2007

Abstract

All metaLindelöf, and most countably paracompact, homogeneous manifolds are Hausdorff. Metacompact manifolds are never rigid. Every countable group can be realized as the group of autohomeomorphisms of a Lindelöf manifold. There is a rigid foliation of the plane.

Introduction

Inspired by a recent paper, *Manifolds: Hausdorffness versus homogeneity* by Bailiff and Gabard [2], we investigate the topology of non Hausdorff manifolds and give applications to foliations. By a *manifold* we mean a connected space which is locally homeomorphic to $\mathbb{R}^n$ for some n. Manifolds are necessarily T_1, but not guaranteed to be Hausdorff. Non Hausdorff manifolds arise naturally as quotients of Hausdorff manifolds, for example as the leaf space of a foliation [6, 8, 7]; as reduced twistor spaces in relativity theory [12]; and as models of space–time in 'many-worlds' interpretations of quantum mechanics [9].

It is easy to prove that Hausdorff manifolds are always homogeneous, but this is not necessarily true of T_1 manifolds. For example, take $\mathbb{R} \times \{0, 1\}$ and identify $(x, 0)$ with $(x, 1)$ for all x except $x = 0$. The resulting space, commonly called the 'split origin' space, is a non homogeneous Lindelöf one manifold. In [2] two examples of homogeneous non Hausdorff manifolds are presented, but it is also shown that a Lindelöf *homogeneous* manifold is always Hausdorff (and hence metrizable). One is naturally led to consider under which circumstances homogeneous manifolds are Hausdorff, particularly covering properties weaker than 'Lindelöf'. We prove here that every metaLindelöf homogeneous manifold is Hausdorff (indeed metrizable) and that most countably paracompact homogeneous manifolds are Hausdorff – but *not* necessarily metrizable. Further one wants to know how 'badly' non Hausdorff manifolds can fail to be homogeneous. A space whose sole autohomeomorphism is the identity map is said to be 'rigid'. Thus rigidity and homogeneity are polar opposites. We show that metacompact

*__2000 Mathematics Subject Classification__: 54D10, 54D20, 57R30, 57N05, 57S05.

Key Words and Phrases: manifold, not Hausdorff, homogeneous, rigid, foliations.

†Supported by the Marsden Fund Council from Government funding, administered by the Royal Society of New Zealand.

manifolds are never rigid, but there is a rigid Lindelöf one manifold. Using this space we further show that every countable group can be realized as the group of autohomeomorphisms of a Lindelöf one manifold. Finally we give applications of our results to foliations of the plane. From our rigid manifold we describe an explicit construction of a rigid foliation of the plane.

Covering Properties of Non Hausdorff Manifolds

Recall that a topological space is 'Lindelöf' if every open cover has a countable subcover, and is 'hereditarily Lindelöf' if every subspace is Lindelöf. Further, a space is 'paracompact' if every open cover has a locally finite open refinement, and is 'metacompact' if every open cover has a point-finite open refinement. Finally, a space is 'metaLindelöf' if every open cover has a point-countable open refinement. Note also that a space is 'second countable' if it has a countable base for the open sets.

It is easy to see, because manifolds are locally second countable, that 'second countable', 'Lindelöf' and 'hereditarily Lindelöf' coincide in manifolds.

Paracompact vs Metacompact vs MetaLindelöf in Manifolds

Lemma 1 *Lindelöf and metaLindelöf are equivalent in manifolds.*

Proof. We need only show that a metaLindelöf manifold is Lindelöf. This follows from two well known facts. First every point-countable open cover of a locally separable space has a star-countable open refinement (each element of the cover meets only countably many other members of the cover). Second any star-countable collection $\mathcal{V}$ can be written $\mathcal{V} = \bigcup_{\lambda\in\Lambda} \mathcal{V}_\lambda$ where each $\mathcal{V}_\lambda$ is countable and $(\bigcup \mathcal{V}_\lambda) \cap (\bigcup \mathcal{V}_{\lambda'}) = \emptyset$ whenever $\lambda \neq \lambda'$.

Thus given any open cover $\mathcal{U}$ of a metaLindelöf manifold M, let $\mathcal{V}$ be a point-countable open refinement and $\mathcal{W}$ a star-countable open refinement of $\mathcal{V}$. Partition $\mathcal{W}$ as above. Since M is connected there can only be one element of Λ, and so $\mathcal{W}$ is countable and hence $\mathcal{U}$ has a countable subcover. ∎

Since 'paracompact' implies 'metacompact' which implies 'metaLindelöf':

Corollary 2 *Metacompact (and hence paracompact) manifolds are second countable.*

Example 3 *There is a second countable manifold which is not metacompact.*

Proof. Let $M = \mathbb{R} \cup (\mathbb{Q} \times \mathbb{N} \times [0, \infty))$ with the topology in which points of $\mathbb{R}$ and $\{(q, n)\} \times (0, \infty)$ have their standard neighborhoods (here $q \in \mathbb{Q}$ and $n \in \mathbb{N}$), and a basic neighborhood of $(q, n, 0)$ is $B(q, n, \epsilon) = (q - \epsilon, q) \cup \{(q, n)\} \times [0, \epsilon)$, where $\epsilon > 0$. Then M is a one dimensional second countable manifold.

We show that M is not metacompact. To this end let $\mathcal{U} = \{\mathbb{R}\} \cup \{\{(q, n)\} \times (0, \infty) : q \in \mathbb{Q}, n \in \mathbb{N}\} \cup \{B(q, n, 1/n) : q \in \mathbb{Q}, n \in \mathbb{N}\}$. This is an open cover of

M. Suppose $\mathcal{V}$ is any open refinement. For each (q, n) pick $\epsilon_{q,n} > 0$ such that $B(q, n, \epsilon_{q,n})$ is contained in some element of $\mathcal{V}$. Note that $\epsilon_{q,n} < 1/n$.

Let $W_n = \bigcup_{q \in \mathbb{Q}} (q - \epsilon_{q,n}, q)$. Then, for each n, W_n is an open dense subset of the reals, and hence there is an $x \in \bigcap_n W_n$. So for each n there is $q_n \in \mathbb{Q}$ with $x \in (q_n - \epsilon_{q_n,n}, q_n)$.

If infinitely many of the elements q_n are different, then we see that the point x of M is in infinitely many (distinct) elements of $\mathcal{V}$. On the other hand, if infinitely many of the elements q_n are all equal (say to q), since $\epsilon_{q,n} \to 0$, we see that x is again in infinitely many elements of $\mathcal{V}$. In either case $\mathcal{V}$ is not point finite. ■

Example 4 *There is a metacompact manifold which is not paracompact.*

Proof. Let $X = \mathbb{R} \times \mathbb{N}$. For each $x < 0$ in the real line, identify (x, n) with (x, m) for all $n, m \in \mathbb{N}$, and call it x. Let M be the resulting quotient space. Then M is a one dimensional second countable manifold which can be described as 'the line with a countable infinity of branches at the origin'.

Take any open cover $\mathcal{U}$ of M. Since $(-\infty, 0) \cup \bigcup_n (0, \infty) \times \{n\}$ is metacompact, there is a partial point finite open refinement $\mathcal{V}'$ of $\mathcal{U}$ covering all points except the $(0, n)$s. For each n pick $m_n \geq n$ so that $V_n = (-1/m_n, 0) \cup [0, 1/m_n) \times \{n\}$ is contained in some element of $\mathcal{U}$. Now $\mathcal{V} = \mathcal{V}' \cup \{V_n : n \in \mathbb{N}\}$ is a point finite open refinement of $\mathcal{U}$, as required for M to be metacompact.

Since any two neighborhoods of an (x, m) and an (x, n) meet, it is clear M is not paracompact. ■

Hereditarily Lindelöf vs Hereditarily Separable in Manifolds A space is said to be an 'S-space' if it is hereditarily separable (every subspace has a countable dense subset) but not hereditarily Lindelöf. It is a beautiful result of Todorcevic that it is consistent with the axioms of set theory that T_3 S-spaces do not exist. Thus it is consistent that Hausdorff hereditarily separable manifolds are (hereditarily) Lindelöf and so metrizable.

Question 5 *Is there, in ZFC, an S-manifold?*

We can show there is a *consistent* non Hausdorff answer to part (a). (Consistent Hausdorff examples have also been constructed – but note that there *are* non Hausdorff S-spaces in ZFC, although none are locally compact.) The example relies on a general construction of unusual manifolds. Let X be a subset of the reals with a topology τ refining the subspace topology such that (X, τ) is locally compact and zero-dimensional.

Define $M(X) = (\mathbb{R} \times \{0\}) \cup (X \times \{1\})$. Topologise this set so that $\mathbb{R} \times \{0\}$ has its usual topology and for any $x \in X$, compact τ-open neighborhood K of x and Euclidean open U containing K let $((U \setminus K) \times \{0\}) \cup (K \times \{1\})$ be a basic neighborhood of $(x, 1)$.

Then $M(X)$ is a manifold (which is only Hausdorff if X is empty) containing a closed copy of (X, τ).

Example 6 *It is consistent [11] that there is an X obtained by refining the topology on a subset of the reals which is locally compact, zero-dimensional, hereditarily separable but not Lindelöf.*

Then $M(X)$ is a non Hausdorff one dimensional manifold which is hereditarily separable but not Lindelöf.

It is a very interesting question of [2] as to whether every *homogeneous* hereditarily separable manifold is necessarily Hausdorff, and hence consistently (hereditarily) Lindelöf.

When Homogeneous Manifolds are Hausdorff

We start with a general discussion of 'points of Hausdorffness' and 'points of regularity' in manifolds. This will be applied below and in the next section.

If a manifold is Hausdorff, then of course it is regular; and conversely. Define for any subset A of any space X, $R(A) =$ all points of A with a neighborhood base in X of closed sets. Obviously X is regular if and only if $R(X) = X$. Further define $H(A) =$ all points of X which can be separated by disjoint open sets from every other point of A, and $NH(A) = X \setminus H(A)$. Write $NH(x)$ for $NH(\{x\})$, so $NH(x)$ consists of all points that can not be Hausdorff separated from x.

In the case of a manifold we clearly have:

Lemma 7 *For an m-manifold M, $R(M) =$ all points with a closed neighborhood homeomorphic to a closed ball in $\mathbb{R}^m$. Hence $R(M)$ is open.*

Lemma 8 *In a manifold M, the set $NH(x)$ is closed for every point x of M.*

Lemma 9 *Let M be a manifold. Then $R(M) = \emptyset$ if and only if $NH(M)$ is dense.*

Also, $R(M)$ is not dense if and only if $NH(M)$ is somewhere dense (dense in some non empty open set).

Proof. If $R(M)$ is non empty then it contains a (non empty) open set. This set clearly can not contain any non Hausdorff points, so $NH(M)$ is not dense in this case.

If $NH(M)$ is not dense then $H(M)$ contains a point with a compact neighborhood homeomorphic to a Euclidean closed ball, K. By a standard argument every point of M outside K can be separated from K by an open set. Hence K is closed in M, and Lemma 7 applies.

The second claim follows similarly. ∎

Lemma 10 *If $R(M) \neq \emptyset$ where M is a homogeneous manifold then M is regular ($R(M) = M$) (and Hausdorff, of course).*

Corollary 11 (to Lemma 1) *MetaLindelöf homogeneous manifolds are Hausdorff (indeed metrizable).*

A space is said to be 'countably compact' if every countable cover has a finite subcover, and is 'countably paracompact' if every countable cover has a locally finite open refinement. Clearly countably compact spaces are countably paracompact. The long line is a Hausdorff manifold which is countably compact but not Lindelöf. We now show that in most circumstances, countably paracompact homogeneous manifolds are Hausdorff.

Let X be a space, and A, B any subsets of X. Define $\rho(A, B)$ (the 'reflection' of A in B) to be $\rho(A, B) = \{x \in B : NH(x) \cap A \neq \emptyset\}$.

Lemma 12 *Let M be a manifold. If U and V are Euclidean open sets in M, then $\rho(V, U)$ is nowhere dense in U.*

Proof. Observe that $V \cap \rho(V, U) = \emptyset$ since V is Euclidean open. In the subspace U, let I be the interior of the closure of $\rho(V, U)$. Suppose, for a contradiction, $I \neq \emptyset$. Note that $\rho(V, U)$ is dense in I. Take any $p \in \rho(V, U) \cap I$. Then there is a $p' \in NH(p) \cap V$. Every neighborhood of p' meets every neighborhood of p, hence $V \cap I \neq \emptyset$. Now $V \cap I$ is an open subset of I and $\rho(V, U)$ is dense in I, hence $\rho(V, U) \cap V \cap I \neq \emptyset$, a contradiction. ■

Theorem 13 *Let M be a countably paracompact homogeneous manifold. If for some $x \in M$ either (i) $NH(x)$ is not countably compact or (ii) $NH(x)$ is Lindelöf, then M is Hausdorff.*

Proof. Let M be a homogeneous non Hausdorff manifold. We show that if either (i) or (ii) hold, then the manifold is not countably paracompact.

(i): First suppose x is in M and $NH(x)$ is not countably compact. This is equivalent to saying that $NH(x)$ contains an infinite closed discrete subspace, $C = \{x_n\}_{n \in \mathbb{N}}$. Since $NH(x)$ is closed in M, C is closed discrete in M. For each x_n pick an open Euclidean neighborhood, T_n, witnessing closed discreteness. Now $\mathcal{U} = \{M \setminus C\} \cup \{T_n : n \in \mathbb{N}\}$ is a countable open cover of M, no open refinement of which can be locally finite at x.

(ii): Now suppose that x is in M and $NH(x)$ is Lindelöf. By homogeneity and non Hausdorffness of M, every $NH(y)$ is (Lindelöf and) non empty. Fix Euclidean open U containing x. We **claim** there are sequences $(x_n)_n$ and $(x'_n)_n$ where: each x_n is in U and $x_n \to x$; and $x'_n \in NH(x_n)$, for each n; but $C = \{x'_n : n \in \mathbb{N}\}$ is closed discrete. Let T_n witness closed discreteness for x_n. Then $\mathcal{U} = \{M \setminus C\} \cup \{T_n : n \in \mathbb{N}\}$ is a countable open cover of M no open refinement of which is locally finite at x.

To establish the claim, start by observing if (x_n) is a sequence on U converging to x, and x'_n is in $NH(x_n)$ for each n, then $\langle x'_n \rangle$ can converge only to elements of $NH(x)$.

For each $y \in NH(x)$ pick a Euclidean open V_y containing y. As $NH(x)$ is Lindelöf, we can find a countable subcollection $V_{y_1}, V_{y_2}, \ldots$ covering $NH(x)$. By Lemma 12, $\rho(V_{y_i}, U)$ is nowhere dense in U. So $U \setminus \bigcup_{i=1}^{\infty} \rho(V_{y_i}, U)$ is dense in U. Thus we can pick a sequence $(x_n)_n$ from this set converging to x, and pick x'_n in $NH(x_n)$ for each n. No x'_n is in any V_{y_i} otherwise x_n is in $\rho(V_{y_i}, U)$. Hence they can not converge to anything in $NH(x)$ (or, as observed above, anything else). ■

Rigidity and Non–Rigidity

We start this section by giving conditions under which a manifold is not rigid. Then we construct a variety of rigid manifolds, and manifolds with specified autohomeomorphism or isotopy group.

Conditions for Non–Rigidity

Lemma 14 *If M is a manifold and $R(M) \neq \emptyset$ then M is not rigid. Indeed, in this case the autohomeomorphism group of M contains many uncountable free subgroups.*

Proof. To see this fix an open set U in M homeomorphic to some $\mathbb{R}^m$ with closure homeomorphic to the closed ball in $\mathbb{R}^m$. Then any non-identity homeomorphism of the closure of U which is the identity on the boundary of U can be extended to a non-trivial homeomorphism of M which is the identity everywhere outside U.

A minor modification of the argument in [3] shows that in fact the group of all autohomeomorphisms of the closed ball in $\mathbb{R}^m$ fixing the boundary is 'almost free' (almost all, in the sense of Baire category, n-tuples of autohomeomorphisms freely generate a free subgroup). Hence, see [5], almost all (again in the sense of Baire category) uncountable compact subsets of the autohomeomorphism group freely generate a free subgroup. ■

Theorem 15 *Every metacompact manifold M has (open) dense $R(M)$, and hence is far from being rigid.*

Proof. The 'and hence' part follows because as $R(M)$ is open and dense it is non empty and Lemma 14 applies.

For the first part fix a metacompact manifold M. Suppose, for a contradiction, that $R(M)$ is not dense, so by Lemma 9 for some open subspace U of M homeomorphic to $\mathbb{R}^m$ we have $NH(M)$ dense in U.

Let $\mathcal{U}$ be an open cover of M by sets homeomorphic to some $\mathbb{R}^m$. Let $\mathcal{V}$ be any open refinement of $\mathcal{U}$. We show $\mathcal{V}$ is not point-finite.

Let $D = \{x_n\}_{n\in\mathbb{N}}$ be a dense set of non Hausdorff points in U. Let $D' = \{x'_n\}_{n\in\mathbb{N}}$ be such that x'_n can not be Hausdorff separated from x_n (a twin). Note that each x'_n is not in U, but is in $\overline{U}$. For each n, pick $V_n \in \mathcal{V}$ so that $x'_n \in V_n$. Let $W_n = V_n \cap U$. Then $\{W_n\}_n$ is a collection of open sets in $\mathbb{R}^m$ such that $x_n \in \overline{W_n} \setminus W_n$ for each n.

We show this latter collection is not point-finite: set $n_1 = 1$, select non empty open S_1 whose closure in U lies in W_{n_1}, and inductively pick n_i so that $x_{n_i} \in S_{i-1}$, select non empty open S_i whose closure in U lies in $S_{i-1} \cap W_{n_i} \subseteq$

$\bigcap_{j\le i} W_{n_j}$. Then there is a point $z \in \bigcap_{i\in\mathbb{N}} S_i$, and this point is in infinitely many of the sets W_n (namely W_{n_i} for each i). ∎

If every Lindelöf manifold had $R(M)$ non empty, this would have given an alternative proof that Lindelöf homogeneous manifolds are Hausdorff. But this is not the case.

Example 16 *There is a second countable manifold with empty $R(M)$. The given example is not rigid.*

Proof. The manifold is called 'La plume composee' in [7]. We start by describing the 'plume': the underlying set is $P = (0, \infty) \cup (\mathbb{Q} \times [0, \infty))$, points in the 'shaft' $(0, \infty)$ have their usual neighborhoods, a basic neighborhood of $(q, 0)$ is $B(q, 0, \epsilon) = \{q\} \times [0, \epsilon) \cup (q, q + \epsilon)$, and points on the 'barb' $\{q\} \times (0, \infty)$ have their usual neighborhoods (in the 'barb').

Now we can get the 'plume double' by replacing each 'barb' by a 'plume' – and iterate countably many times. This gives a second countable one dimensional manifold M, the 'plume composee'. Note that this manifold is far from being rigid.

The key property of this example is that the set of non Hausdorff points is dense. Hence Lemma 9 applies. ∎

Rigid Manifolds

Example 17 *There is a second countable rigid manifold.*

Proof. For each $n \ge 1$ let $M(n) = \big((-1, 0) \cup ([0, 1) \times \{1, \ldots, n\})\big) \times \{n\}$ with topology where points with real part non-zero have their usual Euclidean neighborhoods, and a basic neighborhood of $(0, i)$ is $B(0, i, \epsilon) = (-\epsilon, 0) \cup ([0, \epsilon) \times \{i\})$. Then $M(n)$ is a second countable one dimensional manifold with an order inherited from the order on the real parts. Call points with real part < 0 the 'left part' of $M(n)$, call points with real part > 0 the 'right part' of $M(n)$, and call the points $(0, i)$ the 'branch points'. Note that the branch points are the non Hausdorff points of $M(n)$.

Now let $X = (0, 1) \cup \bigcup_{n\ge1} M(n)$. Fix a countable dense set D of $(0, 1)$ and the right parts of the $M(n)$. Enumerate $D = \{x_m\}_{m\ge1}$. (If we arrange that $x_m \notin M(n)$ for $n \ge m$, then the resulting manifold will have no 'loops', and so is orientable.) For each $m \ge 1$ fix an open neighborhood U_m of x_m and a homeomorphism $h_m : (-1, +1) \to U_m$ so that $h_m(0) = x_m$, $x_i \notin U_m$ for $i < m$ and U_m contains no branch points. We may assume that h preserves the order. Identify the left part of $M(m)$ with $h_m((0, 1))$ via h_m for each m. This gives a second countable one dimensional manifold M.

We prove that M is rigid. Note first that the non Hausdorff points, $NH(M)$, are precisely the branch points and the points of D, and are dense. Take any autohomeomorphism h of M. Then h carries (non) Hausdorff points to (non) Hausdorff points, respectively. Thus if h is the identity on $NH(M)$ (which is dense), by Hausdorffness of all other points of M, h is forced to be the identity on the whole of M.

It remains to show that h is indeed the identity on $NH(M)$. To do this we associate with each point of $NH(M)$ an object which is invariant under homeomorphisms and so that distinct points of $NH(M)$ receive different objects.

The invariant object depends on an analysis of the non Hausdorff points of M. The relation $\sim$ on $NH(M)$ given by $x \sim x'$ if and only if x and x' can not be Hausdorff separated is, *for this particular manifold*, an equivalence relation. So $NH(M)$ is partitioned into equivalence classes N_m, for $m \geq 1$, where N_m consists of x_m and the branch points of $M(m)$. Note that the definition of N_m is purely topological: it is the unique equivalence class of $\sim$ with precisely $m+1$ elements.

Define for $x \in NH(M)$:

$$\sigma(x) = \{\{|N_m| : N_m \cap U \neq \emptyset\} : U \text{ is an open neighborhood of } x\}.$$

Then it is easy to check that $\sigma(x) = \sigma(h(x))$ (so σ is invariant) and $\sigma(x) \neq \sigma(x')$ for any distinct $x, x' \in NH(M)$ — as required. ∎

Example 18 *There are rigid manifolds of arbitrarily large cardinality.*

Proof. It is known that for every cardinal κ, there is a cardinal $\lambda \geq \kappa$ so that $\aleph_\lambda = \lambda$. Here $\aleph_\alpha$ denotes the αth infinite cardinal.

Repeat the argument for the 'small' rigid manifold above (Example 17) with the following amendments. For each $\beta \leq \lambda$ let $M(\aleph_\beta) = \big((-1,0) \cup ([0,1) \times \{\aleph_\beta : \beta \leq \lambda\})\big) \times \{\beta\}$ with obvious topology. Let $X = (0,1) \cup \bigcup_{\beta \leq \lambda} M(\aleph_\beta)$. Let D be a dense subset of $(0,1)$ and the right parts of the $M(\aleph_\beta)$ such that $|D| = \aleph_\lambda$. Now proceed as before. The equivalence classes of $\sim$ will all have different cardinalities and so the invariant σ works as before. ∎

Realizing Groups as Autohomeomorphism Groups of Manifolds Let G be a group. The directed graph with vertices G and an edge from g to g' of color h if and only if $g.h = g'$ is called the *Cayley graph* of G, denoted $CG(G)$. An *automorphism* of $CG(G)$ is a permutation α of the vertices such that (g, g') is an edge of $CG(G)$ if and only if $(\alpha(g), \alpha(g'))$ is an edge of $CG(G)$. Further, α is said to be *color preserving* if for every edge (g, g'), the edges (g, g') and $(\alpha(g), \alpha(g'))$ have the same color. The key property of the Cayley graph is that its group of color preserving automorphisms, $Aut(CG(G))$, is isomorphic to G.

Theorem 19 *For every countable group G there is a second countable one dimensional manifold $M(G)$ such that the group of autohomeomorphisms of $M(G)$ is isomorphic to G.*

Proof. We will show that given a countable group $G = \{g_n\}_{n \in \mathbb{N}}$ there is a second countable one dimensional manifold $M(G)$ such that the group of autohomeomorphisms of $M(G)$, $Homeo(M(G))$, is isomorphic to $Aut(CG(G))$ (and hence G). The rigid manifold M of Example 17 is the main building block in the construction, playing the role of both vertices and edges of the Cayley graph of G. The terminology and notation of that example is used freely here.

Fix two disjoint countable subsets of $(0,1)$ in M, both disjoint from D, and label them $g_1, g_2, \ldots$ and $g_1', g_2', \ldots$. Think of the points g_i and g_i' as having 'color' g_i. An edge attached to a g_i and a g_i' will be considered to 'start' at g_i and 'end' at g_i' and will inherit their color, namely g_i. Extend $(0,1)$ in M left and right to $(-1,2)$ in the natural way to get manifold M' (so all points of $(-1,2)$ have their standard Euclidean open neighborhoods).

Let $\mathbf{V} = M \times G$. Think of $M \times \{g\}$ as being the vertex g. Let $\mathbf{E} = M' \times (G \times G)$. Think of $M' \times \{g\} \times \{h\}$ as being an edge (as yet unattached) — once attached (using the 'flaps' $(-1,0)$ and $(1,2)$) — it will start at vertex g and end at vertex $g.h$. Let $X(G) = \mathbf{V} \cup \mathbf{E}$.

For each $g \in G$ and $g_i \in G$ identify the left part of a small neighborhood of the point g_i in 'vertex' $M \times \{g\}$ with the 'left flap' $(-1,0)$ in the 'edge' $M' \times \{g\} \times \{g_i\}$, and identify the right part of a small neighborhood of the point g_i' in 'vertex' $M \times \{g.g_i\}$ with the 'right flap' $(1,2)$ in the 'edge' $M' \times \{g\} \times \{g_i\}$. This gives a second countable one dimensional manifold $M(G)$.

Then $M(G)$ is as required: $Homeo(M(G)) = Aut(CG(G))$, see the proof of Theorem 2 from [4] for a similar argument. ■

Using Example 18 and the technique of the preceding theorem it is clear that:

Theorem 20 *For every group G there is a manifold M whose autohomeomorphism group is isomorphic to G.*

Isotopy Let M be a manifold. Let I be all $h \in H(M)$ which are isotopic to the identity through autohomeomorphisms. Then I is a normal subgroup of $H(M)$. The quotient $H(M)/I$ is the 'isotopy group' of M.

Take any countable group G, and consider the manifold $M(G)$ of Theorem 19 whose group of autohomeomorphisms coincides with G. If $M(G)$ were Hausdorff then an isotopy between any two distinct autohomeomorphisms as in the definition of I would give a non-trivial path in $Homeo(M(G))$ – which is impossible as non-trivial paths are uncountable and $Homeo(M(G))$ is countable. Since $M(G)$ is not Hausdorff this argument can not be applied directly. Nevertheless, given $h \neq g$ in $Homeo(M(G))$ and fix a point x of $M(G)$ moved to different places by the h and g. An isotopy between h and g through autohomeomorphisms gives a non-trivial path in $M(G)$ which should only have countably many values – this is impossible (even in a non Hausdorff manifold). Thus:

Theorem 21 *Every countable group can be realized as the isotopy group of a second countable manifold.*

By way of contrast note that this is certainly not true for Hausdorff second countable manifolds. Indeed no metrizable manifold has trivial isotopy group. The non–metrizable manifold known as the long ray does have trivial isotopy group. (See [1] for more on isotopy groups of non-metrizable manifolds.)

Example 22 *There is a non Hausdorff second countable manifold whose isotopy group is trivial.*

Proof. Take two copies of the second countable rigid manifold Example 17, and connect any two different points of the two copies by an open interval in the now standard way. Then no autohomeomorphism can 'flip' the copies, but every orientation preserving autohomeomorphism of the open interval can be extended over the entire manifold. These are all isotopic. ■

Applications to Foliations

A partition $\mathcal{F} = \{L_\alpha : \alpha \in A\}$ of an n-manifold M by arcwise connected subsets is a k*-dimensional foliation* of M, where $0 \leq k \leq n$, provided:

- for each $p \in M$ there is a chart (U, φ) on M about p such that for each $\alpha \in A$ if $L_\alpha \cap U \neq \emptyset$ then each arcwise component of $\varphi(L_\alpha \cap U)$ is of the form $\{(x_1, \ldots, x_n) \in \varphi(U) : x_{k+1} = c_{k+1}, \ldots, x_n = c_n\}$, where $c_{k+1}, \ldots, c_n$ are constants determined by the component.

Each set L_α is called a *leaf* of the foliation. The set of leaves forms a space called the *leaf space*; it carries the obvious quotient topology from M. Each leaf may be given an orientation. By an *oriented* foliation we mean that each leaf has been given an orientation in such a way that neighboring leaves have compatible orientations. See [10] for an introduction to foliations.

We are interested here only in foliations where $n = 2$, usually $M = \mathbb{R}^2$, and $k = 1$. In this case every foliation is orientable, and the leaf space is a simply connected, orientable, Lindelöf one manifold (not necessarily Hausdorff). Haefliger and Reeb showed that plane foliations are nearly classified by their leaf spaces. A complete classification requires some additional structure on the non Hausdorff points of the leaf space.

Let M be a simply connected, orientable, Lindelöf one manifold, and let $\mathcal{A}$ be an oriented atlas for M. Define two equivalence relations $\sim^+$ and $\sim^-$ on $NH(M)$ as follows. Let x_1 and x_2 be two points of M which can not be Hausdorff separated. Pick (U_1, ϕ_1) and (U_2, ϕ_2) in $\mathcal{A}$ such that $x_i \in U_i$ and ϕ_i is an orientation preserving homeomorphism of $(-1, 1)$ with U_i, for $i = 1, 2$. Set $x_1 \sim^- x_2$ if 'U_1 and U_2 meet on the left': $\phi_1[(-1, 0)] \cap \phi_2[(-1, 0)] \neq \emptyset$. Set $x_1 \sim^+ x_2$ if 'U_1 and U_2 meet on the right': $\phi_1[(0, 1)] \cap \phi_2[(0, 1)] \neq \emptyset$. These two relations are equivalence relations. Their equivalence classes are countable.

An 'order structure' on a simply connected, orientable, Lindelöf one manifold is an explicit enumeration of the $\sim^+$ and $\sim^-$ equivalence classes.

Theorem 23 (Haefliger & Reeb, [7]) *There is a bijective correspondence between oriented foliations of the plane and Lindelöf, one dimensional, orientable, simply connected manifolds with an order structure.*

It is shown in [7] that a one manifold is simply connected if and only if every point is a cut point (remove that point and the space is disconnected). This makes identifying simply connected one manifolds very straightforward.

Thus, Examples 3 and 4 are both easily checked to be second countable, orientable, simply connected 1-manifolds, so there are foliations of the plane whose leaf spaces are (1) not metacompact and (2) metacompact but not paracompact, respectively.

An *automorphism* of an oriented foliation $\mathcal{F}$ is an orientation preserving autohomeomorphism of the plane taking leaves to leaves. Denote the group of all automorphisms of $\mathcal{F}$ by $Aut(\mathcal{F})$. It corresponds to an autohomeomorphism of the leaf space preserving the order structure. Call an automorphism 'inner' if it carries each leaf to itself. Let $Inn(\mathcal{F})$ be the collection of all inner automorphisms of $\mathcal{F}$. Let $Out(\mathcal{F}) = Aut(\mathcal{F})/Inn(\mathcal{F})$. Every automorphism of a foliation $\mathcal{F}$ induces an autohomeomorphism of the leaf space. Two automorphisms of a foliation yield the same autohomeomorphism of the leaf space if and only they differ only by an inner automorphism. The correspondence theorem implies that any autohomeomorphism of a leaf space can be lifted to an automorphism of the foliation provided it respects the order structure.

Lemma 24 *Let $\mathcal{F}$ be a foliation of the plane. Then $Inn(\mathcal{F})$ contains an uncountable free subgroup. In particular, the full automorphism group is never trivial.*

Proof. The group of all orientation preserving autohomeomorphisms of $(-1, 1)$ contains many uncountable free subgroups because it is almost free [5]. We show it embeds inside $Inn(\mathcal{F})$.

Fix any element U in a chart for $\mathcal{F}$. Then U is homeomorphic to $(-2, 2)^2$ and the leaves correspond to horizontal lines. Take any orientation preserving autohomeomorphism h of $(-1, 1)$. Then h can be extended to an inner automorphism of $\mathcal{F}$ as follows. First make it the identity outside U. Now identify U with $(-2, 2)^2$, and declare the extension to be the identity outside $V = \{(x, y) \in U : |x| + |y| < 1\}$, and to be $\big(h(x/(1 - |y|)), y\big)$ for (x, y) in V. ■

A foliation is said to be 'rigid' if all its automorphisms are internal, in other words Out is trivial, or equivalently its leaf space is a rigid manifold.

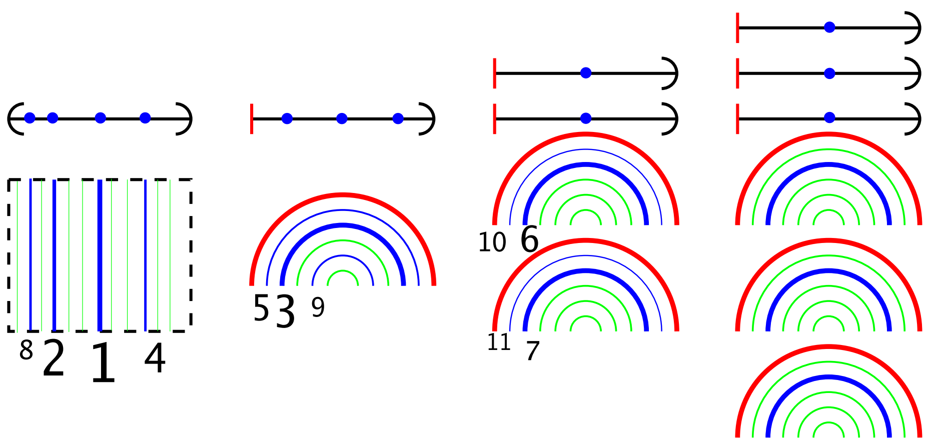
Example 25 *There is a foliation of the plane which is rigid.*

Proof. The rigid manifold Example 17 is a second countable one manifold, it can be made to be orientable, as explained in the construction, and in this case it is clear every point is a cut point. So the claim follows as explained above.

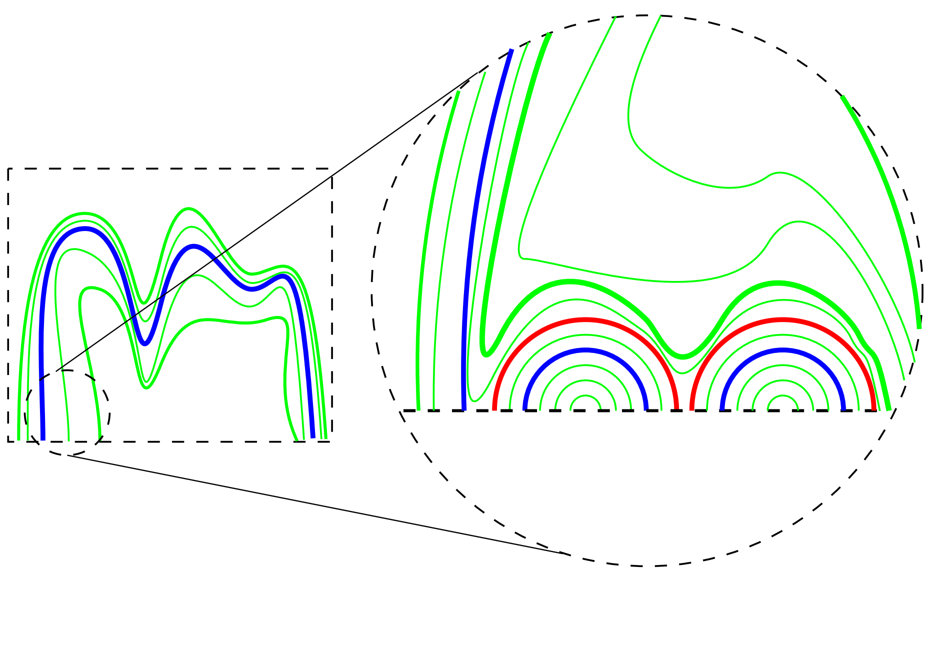
It seems interesting to provide an explicit construction of the rigid foliation whose leaf space is homeomorphic to the rigid manifold of Example 17, particularly because the procedure for generating a foliation from an oriented, simply connected, second countable one manifold is not detailed in [7].

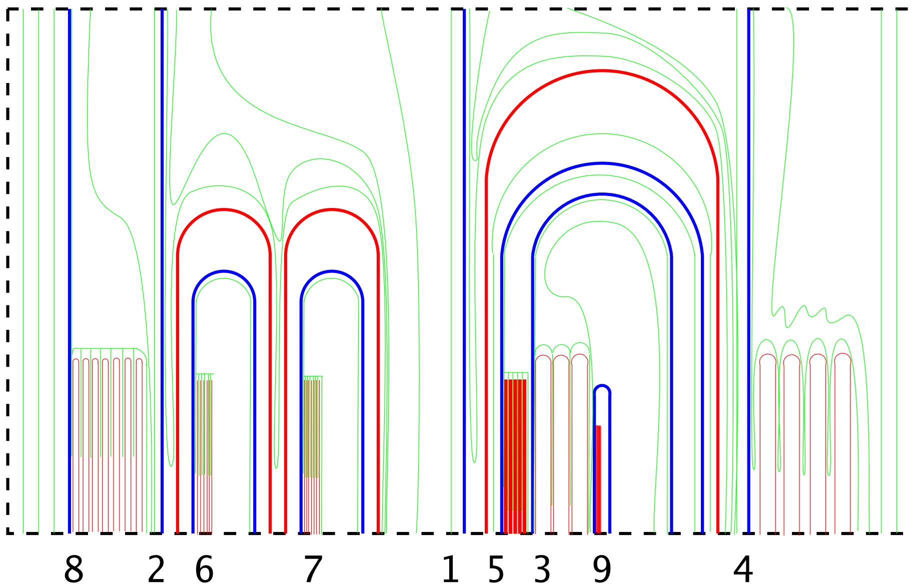
The 'ingredients' going in to the rigid manifold are: a copy of $(0, 1)$; for each $n \geq 1$, n copies of $[0, 1)$; and a 'nice' enumeration of a countable dense set of all the open intervals $(0, 1)$. The construction proceeds by gluing in the 'n copies of $[0, 1)$' to the nth element of the dense set. Denote by M_n the manifold obtained at the nth step of this process.

Analogously, the rigid foliation is made from: a copy of $(0, 1)^2$ foliated by the vertical lines (0); and for each $n \geq 1$, n copies of $[0, 1) \times (0, 1)$ foliated by vertical lines (n); and a nice enumeration of a dense countable family of leaves. We think of each $[0,1) \times (0,1)$ as being 'buckled over', see the diagram. The 'nice' enumeration uses the natural enumeration of the dyadic rationals between 0 and 1, namely: $1/2, 1/4, 3/4, 1/8, 3/8, 5/8, 7/8, 1/16, \ldots$. In fact the dense leaves are enumerated: $(1/2, 0)$; $(1/4, 0)$, $(1/2, 1, 1)$; $(3/4, 0)$, $(1/4, 1, 1)$, $(1/2, 2, 1)$, $(1/2, 2, 2)$; $(1/8, 0)$, $(3/4, 1, 1)$, $(1/4, 2, 1)$, $(1/4, 2, 2)$, $(1/2, 3, 1)$, $(1/2, 3, 2)$, $(1/2, 3, 3)$; $\ldots$. Here $(x, 0)$ stands for the leaf $\{x\} \times (0,1)$ in $(0,1)^2$, and (x, n, m) stands for the leaf $\{x\} \times (0,1)$ in the mth copy of $[0,1) \times (0,1)$ of the nth group.



The construction starts with $\mathcal{F}_0$ which is just the copy of $(0,1)^2$ foliated by the vertical lines. At stage n of the construction the n 'buckles' are inserted into a tubular neighborhood of the nth dense leaf, as in the diagram for two buckles.





This gives foliation $\mathcal{F}_n$ whose leaf space is M_n. (The diagram above shows the case $n = 9$.) In the limit we get our rigid foliation. ∎

Theorem 26 *Every finite group can be realized as the outer automorphisms of a foliation of an open (metrizable) surface.*

Proof. Given a finite group G, construct the manifold $M(G)$ of Theorem 19. This has the property that its autohomeomorphism group is isomorphic to G. Since $M(G)$ is constructed from finitely many copies of the small rigid manifold (Example 17) it is easy to see that $M(G)$ is a non Hausdorff one manifold with finite fundamental group. Further, because of the property of the rigid manifold, M, that the invariant $\sigma(x)$ is different for every point of M, any autohomeomorphism of $M(G)$ *must* respect any order structure placed upon it.

Haefliger ([6], page 8) has shown: *for every orientable non Hausdorff one manifold with finitely generated fundamental group, M, there is an orientable plane foliation $\mathcal{F}$ on a (metrizable) surface S, such that the leaf space of $\mathcal{F}$ is M.*

Now the surface S and foliation $\mathcal{F}$ for $M(G)$ given by the above result are as required. ∎

Question 27 *Which finite groups can be represented as the group of outer automorphisms of a foliation of the plane? What about the cyclic group of order 3?*

References

[1] M. Baillif, S. Deo & D. Gauld, *The mapping class group of powers of the long ray and other non-metrisable surfaces*, (to appear).

[2] M. Baillif & A. Gabard, *Manifolds: Hausdorffness versus Homeogeneity*, to appear in Proc. AMS.

[3] A. Blass & J. Kister, *Free subgroups of the homeomorphism group of the reals*, Topology and its Applications Volume 24, Issues 1-3, December 1986, Pages 243-252.

[4] P. Gartside & A. Glyn, *Autohomeomorphism groups*, Topology Appl. 129 (2003), no. 2, 103–110.

[5] P. Gartside & R. Knight, *Ubiquity of free subgroups*, Bull. Lond. Math. Soc. 35, No. 5, 624-634 (2003).

[6] A. Haefliger, *Sur les feuilletages des varietes de dimension n par des feuilles fermees de dimension $n-1$*, Colloque de Topologie de Strasbourg, 1955.

[7] A. Haefliger & G. Reeb, *Varietes (non separees) a une dimension et structures feuilletees du plan*, Enseignement Math. (2) 3 (1957), 107–125.

[8] S. Harris & R. Law, *Causal monotonicity, omniscient foliations and the shape of space*, Class. Quantum Grav. 18 (2001), 27–43.

[9] S. Hawking & G. Ellis, *The Large Scale Structure of Space-Time*, New York: Cambridge University Press, 1975.

[10] I. Tamura, *Topology of Foliations: An Introduction*, Translations of Math. Monographs v. 97, Amer. Math. Soc., (Providence).

[11] S. Todorcevic, *Forcing positive partition relations*, Trans. Amer. Math. Soc. 280 (1983), no. 2, 703–720.

[12] N. Woodhouse & L. Mason, *The Geroch group and non-Hausdorff twistor spaces*, 1988 Nonlinearity 1, 73-114