distance ½ from 0 radially outwards. If the points $h(y)$ and $h(z)$ of the solution to exercise 3.8 are such that $\beta(h(y)) = h(z)$, then we may let the required g be β. Otherwise, we find a diffeomorphism γ of $\mathbb{R}^m$ so that $\beta\gamma h(y) = \gamma h(z)$ and let the required g be $\gamma^{-1}\beta\gamma$, which is 1 outside the bounded set $\gamma^{-1}B(0;1)$. For example, γ might be the composition $\zeta\,\varepsilon\,\delta$, where δ and ζ are translations and ε a contraction defined by

$$\delta : x \mapsto x - h(y)\ , \quad \varepsilon : x \mapsto \frac{x}{k|h(z)-h(y)|}\ , \quad \zeta : x \mapsto x + \frac{h(z)-h(y)}{2|h(z)-h(y)|}\ .$$

7. Write $V_y = f(U_x)$ when $x \in f^{-1}(y)$. Then V_y is open in M, by Invariance of Domain. Let

$$\mathcal{B} = \{(V_y, (f\,|\,U_x)^{-1}) \ / \ x \in f^{-1}(y) \text{ and } y \in M\}\ .$$

Surjectivity of f guarantees that $\mathcal{B}$ is an atlas. Suppose $(V_y, (f\,|\,U_x)^{-1}), (V_\eta, (f\,|\,U_\xi)^{-1}) \in \mathcal{B}$. Then

$$U_x \cap f^{-1}(V_\eta) \xrightarrow{\ f\ } V_y \cap V_\eta \xrightarrow{(f\,|\,U_\xi)^{-1}} \mathbb{R}^m$$

is C^r at each point of its domain. Indeed, let $z \in U_x \cap f^{-1}(V_\eta)$ and set $z' = (f\,|\,U_\xi)^{-1}f(z)$. Then the coordinate transformation when restricted to $U_x \cap f^{-1}(V_\eta) \cap U_z$ (a neighbourhood of z in $\mathbb{R}^m$) is also the restriction of $(f\,|\,U_{z'})^{-1}(f\,|\,U_z)$ which, by hypothesis, is C^r.

CHAPTER 6

1. Let M denote the upper half plane and N the plane minus the non-negative real axis. Give M and N the orientations determined by the bases $\{(M,\varphi)\}$, $\{(N,\varphi)\}$ respectively, where φ is the forgetful function given by $\varphi(x+iy) = (x,y)$. Then $\varphi f\varphi^{-1}(x,y) = \varphi f(x+iy) = \varphi(x^2 - y^2 + i\cdot 2xy)$, so $\Delta(\varphi f\varphi^{-1})(x,y) = 2(x^2+y^2) > 0$ for each $(x,y) \in \varphi(M)$. Thus f is orientation preserving.

2. Orient $\mathbb{R}^3$, $\mathbb{R}^3 - \{(x,0,z) \ / \ x \geq 0\}$ and $(0,\infty) \times (0,2\pi) \times (0,\pi)$ using the identity charts. The diffeomorphism is a composition of two, viz a standard orientation preserving diffeomorphism $\mathbb{R}^3 \to (0,\infty) \times (0,2\pi) \times (0,\pi)$ and, in the notation of the solution to exercise 4.2, the diffeomorphism

T. In that solution, it was found that $\Delta T(r,\theta,\varphi) = -r^2 \sin\varphi$, which is negative when $\varphi \in (0,\pi)$, so T, and hence also the diffeomorphism $\mathbb{R}^3 \to \mathbb{R}^3 - \{(x,0,z) / x \geq 0\}$, is orientation reversing.

3. Let M^m be a connected manifold having orientation $\mathcal{B}$. Let

$$\mathcal{B}' = \{(U,\rho\varphi) / (U,\varphi) \in \mathcal{B}\},$$

where $\rho : \mathbb{R}^m \to \mathbb{R}^m$ is the reflection of lemma 2. It is readily checked that $\mathcal{B}'$ is (a basis for) an orientation of M, and that $\mathcal{B} \neq \mathcal{B}'$, so that M has at least two orientations. On the other hand, if $\mathcal{C}$ is another orientation for M, then $\exists\ (U,\varphi) \in \mathcal{C}$ for which U is connected, so by lemma 2, either $(U,\varphi) \in \mathcal{B}$ or $(U,\varphi) \in \mathcal{B}'$: suppose the former. It is claimed that $\mathcal{B} \subset \mathcal{C}$ and hence by maximality that $\mathcal{B} = \mathcal{C}$. It is enough to show that if $(V,\psi) \in \mathcal{B}$ is such that V is connected and $U \cap V \neq \phi$ then $(V,\psi) \in \mathcal{C}$ since, by exercise 5.6, such charts form a basis for $\mathcal{B}$. If (V,ψ) is such a chart, then by lemma 2 either $(V,\psi) \in \mathcal{C}$ or $(V,\rho\psi) \in \mathcal{C}$. The latter is impossible since $(U,\varphi), (V,\psi) \in \mathcal{B} \Rightarrow \varphi\psi^{-1}$ is orientation preserving so that $\varphi(\rho\psi)^{-1}$ is orientation reversing yet $(U,\varphi) \in \mathcal{C}$. Thus $(V,\psi) \in \mathcal{C}$.

It is clear that each component of an orientable manifold may be oriented in one of two ways independently of the orientation of the other components. Thus an orientable manifold with c components has 2^c orientations.

4. Since $\{U / \exists\, \varphi \ni (U,\varphi) \in \mathcal{D}\}$ covers M, $\{U \cap N / \exists\, \varphi \ni (U,\varphi) \in \mathcal{D}\}$ covers N, each of the sets being open in N. Thus $\mathcal{E}$ is an atlas on M. Note that because N is open, $U \cap N$ is open in M so $\varphi(U \cap N)$ is open in $\mathbb{R}^m$ whenever $(U,\varphi) \in \mathcal{D}$. Condition DS1 is clearly satisfied by $\mathcal{E}$ because we are just restricting differentiable functions to open subsets. Thus $\mathcal{E}$ is at least a basis for a differential structure on N. If (V,ψ) belongs to this differential structure then, since V is open in M also, we must have $(V,\psi) \in \mathcal{D}$ (by DS2 applied to $\mathcal{D}$). By definition, $(V,\psi) \in \mathcal{E}$, so $\mathcal{E}$ also satisfies DS2. Note that $\mathcal{E} \subset \mathcal{D}$.

Now suppose (N,E) is not orientable. By theorem 3, $\exists$ charts (U,φ), $(V,\psi) \in E$ for which U and V are connected but $\Delta(\varphi\psi^{-1})$ does not have constant sign. Since $E \subset \mathcal{D}$, we have (U,φ), $(V,\psi) \in E$ also, so by theorem 3, $(M,\mathcal{D})$ is not orientable.

Since P^2 contains a Möbius strip as an open subset, one can show that P^2 is not orientable by showing that the Möbius strip is not. This may be achieved by taking as U and V two open connected subsets, one of which goes about half way round and the other the rest with some overlap at each end. However φ and ψ are chosen so that (U,φ) and (V,ψ) lie in the differential structure, $\Delta(\varphi\psi^{-1})$ must change sign.

5. Let $\sigma : \mathbb{R}^n \to S^n$ be the inverse of stereographic projection $S^n - \{(0,\ldots,0,1)\} \to \mathbb{R}^n$ and $\pi : S^n \to P^n$ the standard projection. Then $\pi\sigma : \mathbb{R}^n \to P^n$ satisfies the conditions of the function f in exercise 5.7. Thus $\pi\sigma$ determines a differential structure on P^n. Because $\pi\sigma$ is an immersion with respect to the differential structure constructed on P^n in the text, the two structures are the same.

6. Because it contains a Möbius strip which is not orientable, by exercise 4 the Klein bottle is not orientable.

CHAPTER 7

1. Let $\mathcal{D}$, E and F be the respective differential structures, m, n and p the respective dimensions, and suppose $x \in M$. Let $(U,\varphi) \in E$ and $(V,\psi) \in F$ satisfy $x \in U \cap V$, $\varphi^{-1}(\mathbb{R}^m) = U \cap M$, $\psi^{-1}(\mathbb{R}^n) = V \cap N$, $(U \cap M, \varphi \mid U \cap M) \in \mathcal{D}$ and $(V \cap N, \psi \mid V \cap N) \in E$. The formula

$$\chi(y) = \left(\varphi\psi^{-1}\left(\psi_1(y),\ldots,\psi_n(y),0,\ldots,0\right), \psi_{n+1}(y),\ldots,\psi_p(y)\right)$$

defines a function $\chi : W \to \mathbb{R}^p$, for W some open neighbourhood of x. Moreover, $(W,\chi) \in F$, $\chi^{-1}(\mathbb{R}^m) = W \cap M$ and $(W \cap M, \chi \mid W \cap M) \in \mathcal{D}$.

2. Define $F : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(\alpha,\beta,r) = \left(\left(2 + (r+1)\cos\beta\right)\cos\alpha,\ \left(2 + (r+1)\cos\beta\right)\sin\alpha,\ (r+1)\sin\beta\right)$$

Then for $\alpha, \beta \in [0,2\pi)$, $F(\alpha,\beta,0)$ is that point of T^2 with latitude α and longitude β.

$$DF(\alpha,\beta,r) = \begin{bmatrix} -\big(2+(r+1)\cos\beta\big)\sin\alpha & -(r+1)\sin\beta\cos\alpha & \cos\beta\cos\alpha \\ \big(2+(r+1)\cos\beta\big)\cos\alpha & -(r+1)\sin\beta\sin\alpha & \cos\beta\sin\alpha \\ 0 & (r+1)\cos\beta & \sin\beta \end{bmatrix},$$

so $\Delta F(\alpha,\beta,r) = (r+1)\big(2+(r+1)\cos\beta\big)$, which is non-zero if $-1 < r < 1$. By the Inverse Function Theorem, F is a local diffeomorphism.

Now let $(x,y,z) \in T^2$. Then $\exists$ open set U containing (x,y,z) $\ni$ $\varphi = F^{-1}$ is defined on U and $\varphi : U \to \mathbb{R} \times \mathbb{R} \times (-1,1) \subset \mathbb{R}^3$ is an embedding. Moreover, (U,φ) is in the usual structure of $\mathbb{R}^3$. Further, $\varphi^{-1}(\mathbb{R}^2) = U \cap T^2$ and $(U \cap T^2, \varphi \mid U \cap T^2)$ is in the structure of T^2.

3. The case $m = n$ is trivial, so assume $m < n$, Let $S = S^n - \{(0,\dots,0,1)\}$ and $\sigma : S \to \mathbb{R}^n$ be stereographic projection from $(0,\dots,0,1)$. Note that $\sigma \mid S^m$ is the inclusion. Suppose $x \in S^m$. Since S^m is a submanifold of $\mathbb{R}^n$, $\exists$ chart (V,ψ) in the usual structure of $\mathbb{R}^n$ $\ni$ $x \in V$, $\psi^{-1}(\mathbb{R}^m) = V \cap S^m$ and $(V \cap S^m, \psi \mid V \cap S^m)$ is in the structure of S^m. Let $U = \sigma^{-1}(V)$ and $\varphi : U \to \mathbb{R}^n$ be $\psi(\sigma \mid U)$. Then (U,φ) is in the structure of S^n, $x \in U$, $\varphi^{-1}(\mathbb{R}^m) = U \cap S^m$ and $(U \cap S^m, \varphi \mid U \cap S^m)$ is in the structure of S^m.

4. Again assume $m < n$. Let $x \in P^m$, and pick $y \in S^m$ $\ni$ $\pi(y) = x$, where $\pi : S^n \to P^n$ is the standard projection. By problem 3, $\exists$ chart (V,ψ) in the structure of S^n $\ni$ $y \in V$, $\varphi^{-1}(\mathbb{R}^m) = V \cap S^m$ and $(V \cap S^m, \psi \mid V \cap S^m)$ is in the structure of S^m. We may assume that $z \in V \Rightarrow -z \notin V$. Let $U = \pi(V)$ and $\varphi : U \to \mathbb{R}^n$ be $\psi(\pi \mid V)^{-1}$. Then (U,φ) is in the structure of P^n, $x \in U$, $\varphi^{-1}(\mathbb{R}^m) = U \cap P^m$ and $(U \cap P^m, \varphi \mid U \cap P^m)$ is in the structure of P^m.

5. Replacing e by f, choose (V,ψ) and (W,χ) as in the proof but further so that $f \mid V$ is an embedding. In this case the chosen (U,φ), together with (V,ψ), will satisfy the requirements.

6. By exercise 2-6(c), $\varphi \times \psi : U \times V \to \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ is an embedding so $M \times N$ is a topological manifold [it is easy to show that $M \times N$ is Hausdorff]. Clearly $\{(U \times V, \varphi \times \psi) \;/\; (U,\varphi) \in \mathcal{D}, (V,\psi) \in \mathcal{E}\}$ is an atlas. Suppose $(U_i,\varphi_i) \in \mathcal{D}$ and $(V_i,\psi_i) \in \mathcal{E}$ $(i = 1,2)$. Then $(\varphi_2 \times \psi_2)(\varphi_1 \times \psi_1)^{-1} = (\varphi_2\varphi_1^{-1}) \times (\psi_2\psi_1^{-1})$, which is differentiable, so the atlas satisfies DS1 and hence is a basis for a differential structure.

The function $\gamma : M \to \Gamma(f)$ is a homeomorphism, so $\gamma(\mathcal{D})$ is a differential structure on $\Gamma(f)$.

Suppose $\Gamma(f)$ is a submanifold of $M \times N$. Then $\gamma : M \to M \times N$ is differentiable since if $(W,\chi) \in \mathcal{D} \times \mathcal{E}$ satisfies $\chi^{-1}(\mathbb{R}^m) = W \cap \Gamma(f)$ and $\big(W \cap \Gamma(f), \chi \,|\, W \cap \Gamma(f)\big) \in \gamma(\mathcal{D})$, then $\big(W \cap \Gamma(f), \chi \,|\, W \cap \Gamma(f)\big) = \big(\gamma(U), \varphi\gamma^{-1}\big)$ for some $(U,\varphi) \in \mathcal{D} : \chi\gamma\varphi^{-1} = \chi\big(\varphi\gamma^{-1}\big)^{-1}$ is just the inclusion so is differentiable. Further, $\pi : M \times N \to N$ defined by $\pi(x,y) = y$ is also differentiable: if $(x,y) \in M \times N$, pick a chart $(U \times V, \varphi \times \psi)$ from the basis for $\mathcal{D} \times \mathcal{E} \ni (x,y) \in U \times V$; then (V,ψ) is a chart about y and $\psi\pi(\varphi \times \psi)^{-1} : \mathbb{R}^{m+n} \to \mathbb{R}^n$ is projection on the last n coordinates so is differentiable. Thus $f = \pi\gamma$ is differentiable.

Conversely, if f is differentiable and $\big(x,f(x)\big) \in \Gamma(f)$, pick charts $(U,\varphi) \in \mathcal{D}$, $(V,\psi) \in \mathcal{E}$ with $x \in U$ and $f(U) \subset V$; $\psi f \varphi^{-1}$ is differentiable. The formula $\chi(\xi,\eta) = \big(\varphi(\xi), \psi(\eta) - \psi f(\xi)\big)$ defines a function $\chi : W \to \mathbb{R}^{m+n}$ for W some open neighbourhood of $\gamma(x)$ in $M \times N$. Further, $(W,\chi) \in \mathcal{D} \times \mathcal{E}$, $\chi^{-1}(\mathbb{R}^m) = W \cap \Gamma(f)$ and $\big(W \cap \Gamma(f), \chi \,|\, W \cap \Gamma(f)\big) \in \gamma(\mathcal{D})$. Thus $\Gamma(f)$ is a submanifold of $M \times N$.

CHAPTER 8

1. It is clear that $\mathcal{F}(X,V)$ is closed under addition and scalar multiplication. Associativity and commutativity of addition follow from properties of $\mathbb{R}$. The additive identity is the function $0 : X \to V$ defined by $0(x) = 0 \;\forall\, x \in X$. For $f : X \to V$ define $-f : X \to V$ by $(-f)(x) = -f(x)$: $-f$ is the additive inverse of f. The distributive laws follow from those for $\mathbb{R}$, for example, $(r+s)f = rf + sf$ because if $x \in X$ then $\big((r+s)f\big)(x) = (r+s) \cdot f(x) = r \cdot f(x) + s \cdot f(x) = (rf + sf)(x)$. Associativity of scalar multiplication follows from multiplicative associativity in $\mathbb{R}$, and $1f = f$ because $(1f)(x) = 1 \cdot f(x) = f(x)$.

2. We must show that $u + v$ satisfies Tang 2 and Tang 3 and that rv satisfies Tang 1, Tang 2 and Tang 3.

$u + v$ satisfies Tang 2 because if $f, g \in C^\infty(M, \mathbb{R})$, then

$$\begin{aligned}(u+v)(f\times g) &= u(f\times g) + v(f\times g) && \text{(definition of vector addition)}\\ &= u(f)g(p) + f(p)u(g) + v(f)g(p) + f(p)v(g) && \text{(u and v satisfy Tang 1)}\\ &= \big(u(f)+v(f)\big)g(p) + f(p)\big(u(g)+v(g)\big) && \text{(axioms for } \mathbb{R})\\ &= (u+v)(f)g(p) + f(p)(u+v)(g) && \text{(definition of vector addition).}\end{aligned}$$

$u + v$ satisfies Tang 3 because if $f, g \in C^\infty(M, \mathbb{R})$ and $f|U = g|U$ for some neighbourhood U of p, then

$$\begin{aligned}(u+v)(f) &= u(f) + v(f) && \text{(definition)}\\ &= u(g) + v(g) && \text{(Tang 3)}\\ &= (u+v)(g) && \text{(definition).}\end{aligned}$$

rv satisfies Tang 1 because if $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(M, \mathbb{R})$, then

$$\begin{aligned}(rv)(\alpha f + \beta g) &= r\cdot v(\alpha f + \beta g) && \text{(definition)}\\ &= r[\alpha v(f) + \beta v(g)] && \text{(Tang 1)}\\ &= \alpha r v(f) + \beta r v(g) && \text{(axioms for } \mathbb{R})\\ &= \alpha(rv)(f) + \beta(rv)(g) && \text{(definition)}\end{aligned}$$

rv satisfies Tang 2 because if $f, g \in C^\infty(M, \mathbb{R})$, then

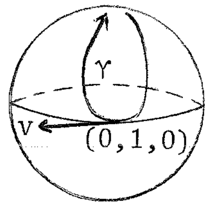
$$\begin{aligned}(rv)(f\times g) &= r\cdot v(f\times g) && \text{(definition)}\\ &= r[v(f)g(p) + f(p)v(g)] && \text{(Tang 2)}\\ &= rv(f)g(p) + f(p)rv(g) && \text{(axioms for } \mathbb{R})\\ &= (rv)(f)g(p) + f(p)(rv)(g) && \text{(definition)}\end{aligned}$$

rv satisfies Tang 3 because if $f, g \in C^\infty(M, \mathbb{R})$ and $f|U = g|U$ for some neighbourhood U of p, then

$$(rv)(f) = rv(f) = rv(g) = (rv)(g)\,.$$

3. Since $\varphi(0,1,0) = (0,1)$ and from the calculation in chapter 6, $D(\psi\varphi^{-1})(0,1) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, the required components are given by

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$ i.e. the components of v with respect to (V,ψ) are $(-1,0)$. Since $\{(t,1) \,/\, t \in \mathbb{R}\}$ is a curve in $\mathbb{R}^2$ with direction ratios $(1,0)$, the curve $\varphi^{-1}\{(t,1) \,/\, t \in \mathbb{R}\}$, more precisely $\gamma : \mathbb{R} \to S^2$ defined by $\gamma(t) = \varphi^{-1}(t,1)$, has velocity vector v at $(0,1,0)$.



4. $df_p(u+v) = df_p(u) + df_p(v)$, for if $g \in C^\infty(N,\mathbb{R})$, then

$$\begin{aligned} df_p(u+v)(g) &= (u+v)(gf) && \text{(definition of } df_p\text{)} \\ &= u(gf) + v(gf) && \text{(definition of addition)} \\ &= df_p(u)(g) + df_p(v)(g) && \text{(definition of } df_p\text{)} \\ &= \big(df_p(u) + df_p(v)\big)(g) \end{aligned}$$

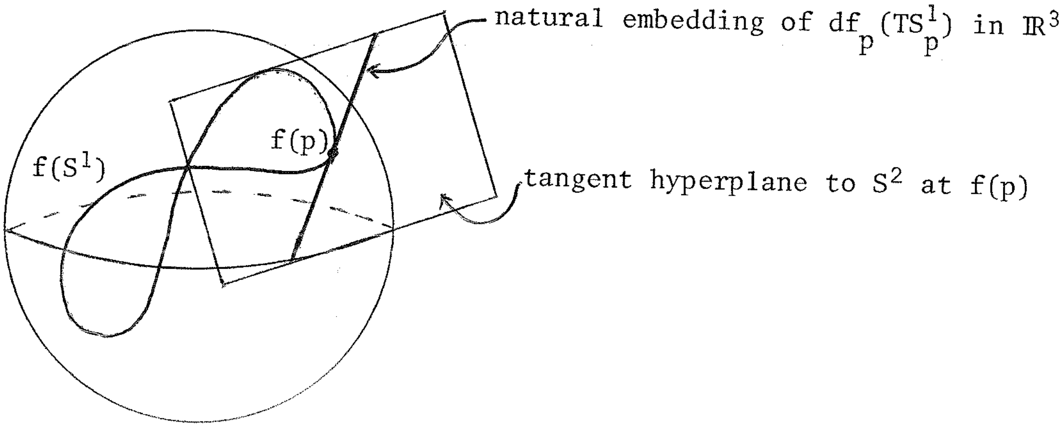
Similarly $df_p(rv) = r\,df_p(v)$, so df_p is a linear transformation.

Let (U,φ) and (V,ψ) be any two charts about p and $f(p)$ respectively with $f(U) \subset V$. The rank of f at p is the rank of the Jacobian matrix $D(\psi f\varphi^{-1})(\varphi(p))$. On the other hand the rank of df_p is the rank of the matrix representation of df_p chosen with respect to any bases on TM_p and $TN_{f(p)}$. The proof of theorem 1 provides bases: $\left\{\frac{\partial}{\partial\varphi_i}\Big|_p \Big/ i = 1,\dots,m\right\}$ and $\left\{\frac{\partial}{\partial\psi_i}\Big|_{f(p)} \Big/ i = 1,\dots,n\right\}$ respectively. The matrix representation of df_p with respect to these bases is (a_{ij}), where $df_p\left(\frac{\partial}{\partial\varphi_j}\Big|_p\right) = \Sigma_{k=1}^n a_{kj} \frac{\partial}{\partial\psi_k}\Big|_{f(p)}$. As in the proof of theorem 1, we may extend the restriction of ψ_i to some neighbourhood of $f(p)$ over all of N, calling such an extension of ψ_i also. Then the previous equation, when applied to ψ_i, yields $\frac{\partial}{\partial\varphi_j}(\psi_i f)\Big|_p = a_{ij}$, since $\frac{\partial\psi_i}{\partial\psi_k}$ is 0 if $k \neq i$ and 1 if $k = i$.

Now $\frac{\partial}{\partial\varphi_j}(\psi_i f)\Big|_p = \frac{\partial(\psi_i f\varphi^{-1})}{\partial x_j}\Big|_{\varphi(p)}$, which is the (i,j) entry of $D(\psi f\varphi^{-1})(\varphi(p))$. Thus the matrix representation of df_p is just the Jacobian matrix $D(\psi f\varphi^{-1})(\varphi(p))$, and so the rank of df_p is the rank of f at p.

To transfer the Jacobian to a manifold, we must think of it as representing a linear transformation on the tangent space [but $T\mathbb{R}^m_p$ is naturally identifiable with $\mathbb{R}^m$]. This linear transformation carries over to manifolds.

5.



If $\gamma : I \to S^1$ is any curve with $\gamma(0) = p$, then the natural embedding of $df_p(TS^1_p)$ in $\mathbb{R}^3$ contains the tangent line to the curve f_γ at 0. More generally, for arbitrary immersion $f : M \to N$, the natural embedding of $df_p(TM_p)$ in $\mathbb{R}^q$ contains the tangent line at $f(p)$ to any curve $f\gamma$, where γ is a curve in M.

CHAPTER 9

1. Set $S = \{p \in M \,/\, f \text{ is regular at } p\}$ and let $p \in S$. Let (U,φ) and (V,ψ) be charts as in theorem 1. Clearly $\psi f\varphi^{-1}$ has rank n throughout $\varphi(U)$, i.e. f has rank n throughout U. Thus $U \subset S$, so S is open.

2. We consider the critical point (1,0,0) and modify the latitude/longitude chart of chapter 5. As in the solution to exercise 7.2, the inverse of the function $(\alpha,\beta) \mapsto \big((2+\cos\beta)\cos\alpha,\ (2+\cos\beta)\sin\alpha,\ \sin\beta\big)$

defines a chart about $(1,0,0)$, with (α,β) near $(0,\pi)$: denote this chart by (U,φ). Then $f\varphi^{-1}(\alpha,\beta) = (2+\cos\beta)\cos\alpha$. Since $f(1,0,0) = 1$, we require a diffeomorphism ψ from a neighbourhood of $(0,\pi)$ onto a neighbourhood of $(0,0)$ so that $f\varphi^{-1}\psi^{-1}(\xi,\eta) = 1-\xi^2+\eta^2$. Writing $\psi(\alpha,\beta) = (\xi,\eta)$, we want $\psi(0,\pi) = (0,0)$ and $1-\xi^2+\eta^2 = (2+\cos\beta)\cos\alpha$.

When $\beta = \pi$, η will have to be 0, so the equation reduces to $1-\xi^2 = \cos\alpha$, so $\xi = \pm\sqrt{1-\cos\alpha}$. Trying $1-\xi^2 = \cos\alpha$ even when $\beta \neq \pi$, we get $\cos\alpha + \eta^2 = (2+\cos\beta)\cos\alpha$, so $\eta^2 = (1+\cos\beta)\cos\alpha$, and $\eta = \pm\sqrt{(1+\cos\beta)\cos\alpha}$.

Now $\dfrac{\partial\xi}{\partial\alpha} = \pm\dfrac{\sin\alpha}{2\sqrt{1-\cos\alpha}}$, $\dfrac{\partial\xi}{\partial\beta} = 0$, $\dfrac{\partial\eta}{\partial\alpha} = \mp\dfrac{(1+\cos\beta)\sin\alpha}{2\sqrt{(1+\cos\beta)\cos\alpha}}$ and

$\dfrac{\partial\eta}{\partial\beta} = \mp\dfrac{\sin\beta\cos\alpha}{2\sqrt{(1+\cos\beta)\cos\alpha}}$. Defining ψ by

$$\psi(\alpha,\beta) = \left((\operatorname{sign}\alpha)\sqrt{1-\cos\alpha},\ (\operatorname{sign}(\beta-\pi))\sqrt{(1+\cos\beta)\cos\alpha}\right),$$

one checks that ψ is a diffeomorphism from a neighbourhood of $(0,\pi)$ onto a neighbourhood of $(0,0)$ $\Big[$e.g. $\left.\dfrac{\partial\xi}{\partial\alpha}\right|_0 = \lim\limits_{h\to 0}\dfrac{(\operatorname{sign} h)\sqrt{1-\cos h}}{h} = \dfrac{1}{\sqrt{2}}$ $= \lim\limits_{\alpha\to 0}\dfrac{\partial\xi}{\partial\alpha}$, so $\dfrac{\partial\xi}{\partial\alpha}$ is continuous at 0; similarly the other partial derivatives are continuous in a neighbourhood of $(0,\pi)$, and hence ψ is differentiable near $(0,\pi)$. On the other hand, $D(\psi)(0,\pi) = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}$,

which is non-singular, so ψ is a local diffeomorphism by the Inverse Function Theorem.$\Big]$ Making U smaller if necessary, we obtain a chart $(U,\psi\varphi)$ about $(1,0,0)$. Further, $f\varphi^{-1}\psi^{-1}(\xi,\eta) = f\varphi^{-1}\left((\operatorname{sign}\xi)\cos^{-1}(1-\xi^2),\ \cos^{-1}\left(\dfrac{\eta^2}{1-\xi^2}-1\right)\right) = 1-\xi^2+\eta^2$, where $\cos^{-1}\left(\dfrac{\eta^2}{1-\xi^2}-1\right)$ is understood to be in $(\pi,3\pi/2)$ if $\eta > 0$ and in $(\pi/2,\pi)$ if $\eta < 0$.

3. Let $h : \mathbb{R} \to \mathbb{R}$ be as in lemma 4.1. Then $f \cdot h : \mathbb{R} \to \mathbb{R}$ is a C^2 function, being a product of such functions. Since $f \cdot h$ agrees with f on $[-\frac{1}{2},\frac{1}{2}]$, $f \cdot h$ has infinitely many non-degenerate critical points. Further, since $f \cdot h$ is identically zero outside $[-1,1]$, it may be used to define a function $g : S^1 \to \mathbb{R}$ of the required type. For example, set $g(x,y) = f(2x)h(2x)$ if $y > 0$ and $g(x,y) = 0$ if $y < \frac{1}{2}$.

4. The statement is false, for if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^4$, then $C = \{0\}$, with the critical point being degenerate. However, $C - \{0\} = \phi$, so $0 \not\in \bar{C} - \{0\}$.

5. Let $g : M \to \mathbb{R}$ be a Morse function. By corollary 5, g has only finitely many critical points, all isolated. Suppose p is a critical point of g. Let (U,φ) be a chart as given by Morse's theorem. Defining $h : M \to \mathbb{R}$ by $h(q) = g(q) - g(p)$, we have $h\varphi^{-1}(x) = -\Sigma_{i=1}^{\lambda} x_i^2 + \Sigma_{i=\lambda+1}^{m} x_i^2 \quad \forall\, x \in \varphi(U)$. Since $\varphi(U)$ is a neighbourhood of 0, it contains $B(0;r)$ for some $r > 0$. As in lemma 4.6, we can find a C^∞ function $\alpha : B(0;r) \to \mathbb{R}$ which agrees with $h\varphi^{-1}$ on $B(0;r) - B(0;r/2)$, has a single critical point which is at 0, is non-degenerate and has index λ, and satisfies $\alpha(0) \neq h(q)$ $\forall$ critical point q of h (hence g). Define $f : M \to \mathbb{R}$ by $f(q) = h(q)$ if $q \in M - \varphi^{-1}(B(0;r/2))$ and $f(q) = \alpha\varphi(q)$ if $q \in \varphi^{-1}(B(0;r))$. Then f is a Morse function and the critical point p has a different value from those of all of the other critical points of f. Continuing finitely many times we obtain the result.

CHAPTER 10

1. The family $\{\hat{\varphi}^{-1}(V \times W) \,/\, (U,\varphi) \in \mathcal{D},$ V and W are open in $\mathbb{R}^m$ and $V \subset \varphi(U)\}$ forms a basis for the topology on TM. The sets $\hat{U}$ lie in this basis so are open. Further the function $\hat{\varphi}$ are embeddings. Thus $\{(\hat{U},\hat{\varphi}) \,/\, (U,\varphi) \in \mathcal{D}\}$ forms an atlas.

Suppose (U,φ), $(V,\psi) \in \mathcal{D}$. Then $\hat{\psi}\hat{\varphi}^{-1}(x,y) = \left(\psi\varphi^{-1}(x), D(\psi\varphi^{-1})(x)^*(y)\right) \quad \forall (x,y) \in \hat{\varphi}(\hat{U} \cap \hat{V})$, where by $D(\psi\varphi^{-1})(x)^*$ we mean the transpose of the Jacobian $D(\psi\varphi^{-1})(x)$: it converts components of a vector with respect to (U,φ) into components with respect to (V,ψ). Since $\psi\varphi^{-1}$ is C^r, $D(\psi\varphi^{-1})(\ )^*$ is C^{r-1}. Thus $\hat{\psi}\hat{\varphi}^{-1}$ is C^{r-1}, so we have a basis for a C^{r-1} structure on TM.

2. The equations $\frac{dx_i}{dt} = -x_i$ and $\frac{dy_j}{dt} = y_j$ have solutions $x_i = A_i e^{-t}$ and $y_j = B_j e^t$, so that $(x,y) = (Ae^{-t}, Be^t)$, where $A = (A_1,\dots,A_\lambda)$ and $B = (B_1,\dots,B_{m-\lambda})$ are constants. Thus $t \mapsto \varphi^{-1}(Ae^{-t},Be^t)$, are integral curves for ξ within U. When $0 < \lambda < m$, we have $|x| \cdot |y| = |A|e^{-t} \cdot |B|e^t = |A| \cdot |B|$, which is constant. When $\lambda = 0$, there are no x-coordinates and $t \mapsto \varphi^{-1}(Be^t)$ are integral curves for ξ within U; in this case as $t \to \infty$, $|y| \to \infty$ also and the curves emanate radially (with respect to (U,φ)) from p. When $\lambda = m$, there are no y-coordinates and $t \mapsto \varphi^{-1}(Ae^{-t})$ are integral curves for ξ within U; as $t \to \infty$, $|x| \to 0$ and the curves converge radially towards p.

3. Let (U,φ) and (V,ψ) be charts about $(0,0,-1)$ and $(0,0,1)$ $\ni$ $f\varphi^{-1}(x,y) = -1 + x^2 + y^2$ and $f\psi^{-1}(x,y) = 1 - x^2 - y^2$ $\forall\ (x,y) \in \varphi(U)$ and $\forall\ (x,y) \in \psi(V)$ respectively. We may assume that $\varphi(U) = \psi(V) = B(0;r)$ for some $r \in (0,1)$: thus $(x,y,z) \in U$ iff $z < -1 + r^2$ and $(x,y,z) \in V$ iff $z > 1 - r^2$. Let (W,χ) and (X,ω) be charts $\ni$ $W \cup X = S^2 - \{(0,0,-1),(0,0,1)\}$ and $\chi(x,y,z)$ and $\omega(x,y,z)$ are each $(\theta,\tilde{\varphi})$, where (x,y,z) has spherical polar coordinates $(1,\theta,\tilde{\varphi})$.

(a) Let $\xi(p)$ have components $\varphi(p)$ with respect to (U,φ) for $p \in \varphi^{-1}(B(0;r/2))$;
Let $\xi(p)$ have components $-\psi(p)$ with respect to (V,ψ) for $p \in \psi^{-1}(B(0;r/2))$;
Let $\xi(p)$ have components $(0,1)$ with respect to (W,χ) or (X,ω) for $p \in \{(x,y,z) \in S^2 \,/\, -1 + 3r^2/4 \le z \le 1 - 3r^2/4\}$. Note that the integral curves of ξ as defined so far are parts of lines of longitude. Moreover the definition of ξ on part of $W \cup X$ may be extended over all of $W \cup X$: thus $\exists$ a smooth function $k : (-1,-1+r^2) \to (0,\infty)$ $\ni$ $\forall\ p \in U - \{(0,0,-1)\}$, the vector field with components at p equal to $(0,1)$ with respect to (W,χ) or (X,ω) has components $kf(p) \cdot \varphi(p)$ with respect to (U,φ). Letting $h : \mathbb{R} \to \mathbb{R}$ be the function of lemma 4.1, define $\ell : (-1,-1+r^2) \to (0,\infty)$ by

$$\ell(t) = h\left(\frac{t+1}{r^2} + \tfrac{1}{4}\right) + k(t)\left[1 - h\left(\frac{t+1}{r^2} + \tfrac{1}{4}\right)\right].$$

For $t \le -1 + r^2/4$, $\ell(t) = 1$ and for $t > -1 + 3r^2/4$, $\ell(t) = k(t)$. Thus we may extend ξ over $\{(x,y,z) \,/\, -1 + r^2/4 \le z \le -1 + 3r^2/4\}$ by letting $\xi(p)$ have components $\ell f(p) \cdot \varphi(p)$ with respect to (U,φ). Similarly we may extend ξ over the corresponding annulus in the northern hemisphere. Such extension is smooth, and satisfies the requirements.

(b) Let ξ be as in (a) on $U \cup V$, but introduce a spiral on the equatorial annulus $\{(x,y,z) \in S^2 \,/\, -1 + r^2 \le z \le -1 + r^2\}$. For example, with $h : \mathbb{R} \to \mathbb{R}$ as in (a), let $\xi(\chi^{-1}(\theta,\tilde{\varphi}))$ or $\xi(\omega^{-1}(\theta,\tilde{\varphi}))$ have components $\{ch(\tilde{\varphi}/(1 - r^2)), 1\}$ with respect to (W,χ) or (X,ω), where c is any constant: the value of c determines the number of times the integral curves spiral around S^2.

4. Define $h : S^1 \times [0,1] \to \{x \in \mathbb{R}^2 \,/\, 1 \le |x| \le 2\}$ by $h(x,t) = (t+1)x$. Then h is a homeomorphism. Suppose $(\bar{x},\bar{t}) \in S^1 \times [0,1]$ with $\bar{x} = (\bar{x}_1,\bar{x}_2)$ where $\bar{x}_2 > 0$. Then $(x_1,x_2) \mapsto x_1$ determines a chart on S^1 about $\bar{x}$. Since $[0,1] \subset \mathbb{R}$, we may take $([0,1], 1)$ as a chart on $[0,1]$ about $\bar{t}$ and the identity function also gives a chart on the range. Differentiability of h at $(\bar{x},\bar{t})$ transfers to differentiability of the function $(x_1,t) \mapsto \left((t+1)x_1, (t+1)\sqrt{1-x_1^2}\right)$ at $(\bar{x}_1,\bar{t})$, and this is clearly differentiable since $x_1^2 < 1$. Moreover, its Jacobian matrix is $\begin{bmatrix} t+1 & x_1 \\ \dfrac{-x_1(t+1)}{\sqrt{1-x_1^2}} & \sqrt{1-x_1^2} \end{bmatrix}$, which is non-singular. Similar reasoning applies to other points of $S^1 \times [0,1]$. Thus h is a diffeomorphism.

5. By the Inverse Function Theorem, it suffices to show that Γ has rank m at each point of its domain. The theory of differential equations tells us that in $\mathbb{R}^m$, in the absence of singularities, the integral curves determine a family of diffeomorphisms as follows: let β_x denote the

integral curve through $x \ni \beta_x(0) = x$ and define $B_t : \mathbb{R}^m \to \mathbb{R}^m$ by $B_t(x) = \beta_x(t)$. Each B_t is a diffeomorphism. Transferring this to M enables us to slide a chart in M up the integral curves and obtain a new chart.

Let $(p,t) \in M_c \times [c,d]$ and choose a chart (U,φ) about p in M for which $\varphi(q) = (\varphi\gamma_q(c), f(q) - c)$. Defining $g : U \to M$ by $g(q) = \gamma_q(t + f(q) - c)$, we obtain a chart $(g(U), \varphi g^{-1})$ in M about $g(p) = \gamma_p(t)$. Now $\varphi g^{-1}\Gamma((\varphi \mid U \cap M_c) \times 1)^{-1}(x,s) = (x, s-t)$, whose Jacobian is the identity: thus Γ has rank m at (P,t).

6. Let the sequence (x_n) be as in the hint, and let (x_{n_k}) be a subsequence converging, say, to x. By compactness, $x \in M - U$. On the other hand, by continuity of f, $f(x) = \lim_{k\to\infty} f(x_{n_k}) = b$, so that $x \in M_b \subset U$.

7. Let $f : M \to \mathbb{R}$ be a Morse function having exactly two critical points. Since M is compact, so is $f(M)$ which, therefore, must contain its maximum and minimum. Thus f must have a maximum and a minimum, which must be the only critical points of f: call them p and q respectively, and suppose $f(p) = 1$, $f(q) = -1$. Let (U,φ) and (V,ψ) be charts about p and q given by Morse's theorem. Assume that $\varphi(U) = \psi(V) = B(0;2r)$, where $0 < r < \frac{1}{2}$. By theorem 2, $f^{-1}([-1+r^2, 0]$ is diffeomorphic to $f^{-1}(-1+r^2) \times [-1+r^2, 0]$. But $f^{-1}(-1+r^2) = \psi^{-1}\{(x_1,\dots,x_m) \in \mathbb{R}^m \ / \ \Sigma x_i^2 = r^2\}$, which is an $(m-1)$-sphere. Thus, using the diffeomorphism given by theorem 2, we may extend ψ to a diffeomorphism $\psi : f^{-1}([-1,0]) \to B^m$. Similarly, φ may be extended to a diffeomorphism $\varphi : f^{-1}([0,1]) \to B^m$. It may be that φ and ψ disagree on $f^{-1}(0)$: let $\alpha : B^m \to B^m$ be a homeomorphism for which $\alpha \mid S^{m-1} = \varphi(\psi \mid S^{m-1})^{-1}$. Define the required homeomorphism $h : S^m \to M$ by $h(x) = \psi^{-1}(x_1,\dots,x_m)$ if $x_{m+1} \le 0$ and $h(x) = \varphi^{-1}\alpha(x_1,\dots,x_m)$ if $x_{m+1} \ge 0$. Note that h might not be a diffeomorphism since we cannot be sure that α is a diffeomorphism.