Note that h takes the vertical part of L and stretches it around the upper semi-circle of 3 and stretches the horizontal part of L around the lower semi-circle of 3. Using exercise 1 - 5 one can verify that h and h^{-1} are continuous.

8. We must show that (i) $\forall\ \nu \in N$, $\beta\ \alpha\ (\nu) = \nu$ and (ii) $\forall\ \mathcal{T} \in T$, $\alpha\ \beta\ (\mathcal{T}) = \mathcal{T}$.

(i) Suppose $\nu \in N$. By definition,

$x\ \beta\ \alpha\ (\nu)\ A$ iff $\forall\ U \subset X$ satisfying $x \in U$ and $\forall\ y \in U$, $y \not\nu\ (X - U)$, we have $U \cap A \neq \phi$.

If $x \in X$, $A, U \subset X$ satisfy $x \in U$ and $\forall\ y \in U$, $y \not\nu\ (X - U)$ but $U \cap A = \phi$, then $A \subset X - U$, so by Near 4 and Near 2, $\forall\ y \in U$, $y \not\nu\ A$; in particular $x \in U$, so $x \not\nu\ A$. Thus $x\ \nu\ A \Rightarrow x\ \beta\ \alpha\ (\nu)\ A$.

If $x\ \beta\ \alpha\ (\nu)\ A$ but $x \not\nu\ A$, then $U = \{y \in X\ /\ y \not\nu\ A\}$ contains x and satisfies $\forall\ y \in U$, $y \not\nu\ (X - U)$ [for $\forall\ z \in X - U$, $z\ \nu\ A$ so by Near 4, $y\ \nu\ (X - U) \Rightarrow y\ \nu\ A$]. Thus $U \cap A \neq \phi$ which contradicts Near 2. Thus $x\ \beta\ \alpha\ (\nu)\ A \Rightarrow x\ \nu\ A$.

(ii) Suppose $\mathcal{T} \in T$. By definition,

$$\alpha\ \beta\ (\mathcal{T}) = \{V \subset X\ /\ \forall\ x \in V,\ \exists\ U \in \mathcal{T} \ni x \in U \subset V\}.$$

By Open 4, $\alpha\ \beta\ (\mathcal{T}) \subset \mathcal{T}$ and clearly $\mathcal{T} \subset \alpha\ \beta\ (\mathcal{T})$. Thus $\alpha\ \beta\ (\mathcal{T}) = \mathcal{T}$.

CHAPTER 3

1. Let $x, y \in X$ be distinct points in an infinite space X having the cofinite topology and let U and V be open neighbourhoods of x and y respectively: thus $X - U$ and $X - V$ are finite. Hence $(X - U) \cup (X - V) = X - (U \cap V)$ is finite, so cannot be all of X and hence $U \cap V \neq \phi$.

2. Let X_1 denote X with the discrete topology and X_2 denote X with any Hausdorff topology. Define $h : X_1 - X_2$ to be the identity function. Since it has a discrete domain, h is continuous. Clearly h is a bijection. Further, X_1 is compact since X is finite. Hence, since X_2 is Hausdorff, theorem 6 tells us that h is a homeomorphism, and hence by continuity of h^{-1}, every subset of X_2 is open, i.e. X_2 is discrete.

3. (a) ϕ is the only compact set. $\mathbb{R}$ is not compact because it is not bounded. $\{x \in \mathbb{Q} / 0 \leq x \leq 1\}$ and $\{x \in \mathbb{Q} / 0 < x < 1\}$ are not compact because they are not closed: for example $1/\sqrt{2}$ is near each of these sets but is not a member of either.

   (b) Only the third set is compact since it is the only one which is both closed and bounded. The first two sets are not bounded, for example they each contain the points $(n, 1/2n)$ for any positive integer n. The last set is not closed, for example $(0,1)$ is near this set but is not a member of it.

4. (i) Let A be a subset of a discrete space. Then $\{\{a\} / a \in A\}$ is an open cover of A: the only subcover is the whole family itself, which is finite iff A is finite. Thus only the finite subsets of a discrete space are compact.

   (ii) No open cover of a subset of a concrete space can have more than two members, so it is already finite. Thus every subset of a concrete space is compact.

5. Two details need checking.

   (i) $\{f^{-1}(U) / U \in \mathcal{U}\}$ is a cover of C. Indeed, if $x \in C$ then $f(x) \in f(C)$, so $f(x) \in U$ for some $U \in \mathcal{U}$. This implies that $x \in f^{-1}(U)$.

   (ii) $\{U_i / i = 1, \ldots, n\}$ is a cover of $f(C)$. Indeed, if $y \in f(C)$, then $y = f(x)$ for some $x \in C$, so $\exists\, i \ni x \in f^{-1}(U_i)$. Thus $y = f(x) \in U_i$.

6. Let X be a concrete space having at least two elements and let $A \subset X$ be a non-empty proper subset. Then A is compact but not closed.

7. Suppose C is a compact subset of $\mathbb{R}$. Since $\mathbb{R}$ is Hausdorff, theorem 5 tells us that C is closed. Consider the family $\{(-n,n) / n \in \mathbb{N}\}$ of open subset of $\mathbb{R}$. This family covers $\mathbb{R}$ and hence also forms an open cover of C. Since C is compact this cover reduces to a finite subcover and since the members of the cover get larger as n increases, C is contained in the largest member of the finite subcover. Thus $C \subset (-n,n)$ for some n and hence C is bounded.

Conversely, if C is a closed and bounded subset of $\mathbb{R}$, then $C \subset [a,b]$ for some $a, b \in \mathbb{R}$; we may assume $a < b$. Now $[a,b]$ is homeomorphic to $[0,1]$, the function $h : [0,1] \to [a,b]$ defined by $h(t) = (b-a)\,t + a$ being a homeomorphism. Thus compactness of $[0,1]$ implies compactness of $[a,b]$ and hence of its closed subset C.

8. Suppose M is connected and let $x \in M$. Set

$$A = \{y \in M \;/\; \exists \text{ open neighbourhood } U \text{ of both } x \text{ and } y \ni U \text{ is homeomorphic to } \mathbb{R}^m\}.$$

(i) A is open, being the union of some open neighbourhoods of x.

(ii) A is closed. Indeed, suppose $z \smallsmile A$. Every neighbourhood of z must meet A. Since M is a manifold, z has a neighbourhood homeomorphic to $\mathbb{R}^m$; let V be one such neighbourhood with $x \notin V$ and let $y \in A \cap V$. Since $y \in A$, $\exists$ open neighbourhood U of both x and $y \ni U$ is homeomorphic to $\mathbb{R}^m$. Pick a homeomorphism $h : V \to \mathbb{R}^m$. Since $y, z \in V$, the two points $h(y)$ and $h(z)$ are points of $\mathbb{R}^m$ and there is a homeomorphism $g : \mathbb{R}^m \to \mathbb{R}^m$ which is the identity outside some compact subset of $\mathbb{R}^m \ni g(h(y)) = h(z)$. [For example, let $r \in \mathbb{R}$ be $\ni h(y), h(z) \in B(0;r)$. Set $g(w) = w$ if $w \in \mathbb{R}^m - B(0;r)$ and set $g(h(y)) = h(z)$. If $w \in C\ell\, B(0;r) - \{h(y)\}$, then the line from $h(y)$ through w meets $Fr\, B(0;r)$ at exactly one point, say $\bar{w}$. For some (unique) $t \in [0,1]$, $w = t\,\bar{w} + (1-t)\,h(y)$: set $g(w) = t\,\bar{w} + (1-t)\,h(z)$. Continuity of g at $h(y)$ follows from the fact that if $h(y) \smallsmile A$ then A contains points w for which in the expression $w = t\,\bar{w} + (1-t)\,h(y)$, t can be arbitrarily small: such points can be used to show that $h(z) \smallsmile g(A)$. It is now clear that $g \mid \mathbb{R}^m - B(0;r)$ and $g \mid C\ell\, B(0;r)$ are continuous as are their inverses, so by exercise 1.5, g and g^{-1} are continuous.] Now define the homeomorphism $f : M \to M$ by setting $f(w) = w$ if $w \in M - V$ and $f(w) = h^{-1} g\, h(w)$ if $w \in V$. Exercise 1.5 again assures continuity of f and f^{-1}. The set $f(U)$ is an open neighbourhood of $f(x) = x$ and $f(y) = z$ and is homeomorphic to $\mathbb{R}^m$. Thus $z \in A$ and hence A is closed.

By corollary 2.5, either $A = \phi$ or $A = M$, but since $x \in A$, we must have $A = M$, and so the criterion is satisfied.

Conversely, suppose that $\forall\ x, y \in M$, $\exists$ open neighbourhood U of both x and $y \ni U$ is homeomorphic to $\mathbb{R}^m$. Pick any $x \in M$, and $\forall\ y \in M$, let U_y be an open neighbourhood of both x and y with U_y homeomorphic to $\mathbb{R}^m$. Each U_y is connected, so by theorem 1-4 the union of all of the sets U_y is connected. This union is M.

9. Let $U \subset M$ be an open neighbourhood and $h : U \longrightarrow \mathbb{R}^m$ a homeomorphism $\ni h(x) = 0$. Then $h(U \cap N)$ is a neighbourhood of 0 so $\exists\ r > 0$ for which $B(0;r) \subset h(U \cap N)$. The set $h^{-1}(B(0;r))$ is a connected neighbourhood of x contained in N.

10. Set $\mathcal{F} = \{U \subset X \ /\ \exists\ \alpha \in A \ni U \subset U_\alpha$ and $\varphi_\alpha(U)$ is open in $\mathbb{R}^m\}$. To show that $\mathcal{F}$ is a basis for a topology on X, we verify the criterion of proposition 2-2. Since each $U_\alpha \in \mathcal{F}$ and $\bigcup_{\alpha \in A} U_\alpha = X$, we have $\bigcup \mathcal{F} = X$. Suppose $U, V \in \mathcal{F}$, say $\alpha, \beta \in A$ are such that $U \subset U_\alpha$, $V \subset U_\beta$, and $\varphi_\alpha(U)$ and $\varphi_\beta(V)$ are open in $\mathbb{R}^m$. Now $\varphi_\alpha(U \cap U_\beta) = \varphi_\alpha(U_\alpha \cap U_\beta) \cap \varphi_\alpha(U)$, an open set in $\mathbb{R}^m$, so by Invariance of Domain, $\varphi_\beta(U \cap U_\beta)$ is also open in $\mathbb{R}^m$. Thus $\varphi_\beta(U \cap V) = \varphi_\beta(U \cap U_\beta) \cap \varphi_\beta(V)$ is also open in $\mathbb{R}^m$, and hence $U \cap V \in \mathcal{F}$. Therefore, $\mathcal{F}$ is a basis for a topology on X.

Suppose $x, y \in X$ with $x \neq y$. If $\exists\ \alpha \in A \ni x, y \in U_\alpha$, then the Hausdorff condition for x and y is easily verified. More generally, we may have $x \in U_\alpha$ and $y \in U_\beta$ for some $\alpha, \beta \in A$. For the Hausdorff condition to be satisfied, we need to be able to find $U, V \in \mathcal{F} \ni x \in U$, $y \in V$ and $U \cap V = \phi$. This will be possible provided the following condition is satisfied:

$\forall\ \alpha, \beta \in A$, $\forall\ x \in U_\alpha$, $\forall\ y \in U_\beta$, if $x \neq y$, then $\exists\ U \subset U_\alpha$, $V \subset U_\beta$ $\ni x \in U$, $y \in V$, $U \cap V = \phi$ and $\varphi_\alpha(U)$ and $\varphi_\beta(V)$ are open.

Finally, if the condition for the topology to be Hausdorff is satisfied then X is an m-manifold. To verify this we must exhibit, for each $x \in X$, an open neighbourhood U of x and a homeomorphism $\varphi : U \longrightarrow \mathbb{R}^m$. Let $x \in X$, and choose $\alpha \in A$ with $x \in U_\alpha$. Since $\varphi_\alpha(U_\alpha)$ is an open neighbourhood of $\varphi_\alpha(x)$ in $\mathbb{R}^m$, $\exists\ r > 0 \ni B(x;r) \subset \varphi_\alpha(U_\alpha)$. Let $U = \varphi_\alpha^{-1}(B(x;r))$. By definition, U is an open neighbourhood of x. Moreover, $\varphi_\alpha | U$ is a homeomorphism from U to $B(x;r)$, for if V is any open subset of $B(x;r)$ then $\varphi_\alpha^{-1}(V)$

is, by definition, an open subset of X, hence of U so $\varphi_\alpha | U$ is continuous, and if V is any member of the basis for the topology on X, so that $U \cap V$ is a typical member of the basis for the topology on U, then $\exists\, \beta \in A \ni V \subset U_\beta$ and $\varphi_\beta(V)$ is open in $\mathbb{R}^m$. As in our discussion above showing that $\mathcal{F}$ is a basis for a topology, $\varphi_\alpha(U \cap V)$ is open in $\mathbb{R}^m$ and hence in $B(x;r)$, so $(\varphi_\alpha | U)^{-1}$ is continuous. Follow the homeomorphism $\varphi_\alpha | U$ by a homeomorphism $B(x;r) \longrightarrow \mathbb{R}^m$ (cf exercise 1-9) to get the required homeomorphism φ.

CHAPTER 4

1. We have $Df(x,y) = \begin{bmatrix} 2x+2y & 2x+2y \\ y^2+2xy & 2xy+x^2 \end{bmatrix}$, so $\Delta f(x,y) = 2(x+y)^2(x-y)$. Now $\text{rank}(f) = 2$ iff $\Delta f(x,y) \neq 0$ iff $x \neq \pm y$, so f has rank 2 at those points off the lines $y = x$ and $y = -x$. Next, $\text{rank}(f) = 0$ iff all entries in $Df(x,y)$ are 0 iff $x = y = 0$, so f has rank 0 at the origin. Finally f has rank 1 at those points where it does not have rank 0 or 2, i. e. at all points on either, but not both, of the lines $y = x$ and $y = -x$. Since f does not have rank 2 everywhere, f cannot be a diffeomorphism.

2. Denote the transformation by $T : [0,\infty) \times [0,2\pi) \times [0,\pi] \longrightarrow \mathbb{R}^3$. Then

$$DT(r,\theta,\varphi) = \begin{bmatrix} \cos\varphi \sin\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\varphi & 0 & -r\sin\varphi \end{bmatrix},$$ and hence

$\Delta T(r,\theta,\varphi) = -r^2\sin\varphi$. Thus T has rank 3 unless $r = 0$ or $\varphi = 0$ or π, i.e. T has rank 3 on $(0,\infty) \times [0,2\pi) \times (0,\pi)$. By inspection, when $r = 0$ the matrix $DT(0,\theta,\varphi)$ has rank 1, so there T has rank 1. When $\varphi = 0$ or π but $r \neq 0$, $DT(r,\theta,\varphi)$ has rank 2, so T has rank 2 on $(0,\infty) \times [0,2\pi) \times \{0,\pi\}$.

Since T has rank 3 on $(0,\infty) \times (0,2\pi) \times (0,\pi)$, and T is clearly a homeomorphism when restricted to this set, T restricts to a diffeomorphism $(0,\infty) \times (0,2\pi) \times (0,\pi) \longrightarrow T\big((0,\infty) \times (0,2\pi) \times (0,\pi)\big)$. The domain of this diffeomorphism is diffeomorphic to $\mathbb{R}^3$ and the range consists of all points of $\mathbb{R}^3$ except those which are expressible in spherical polar coordinates in which $r = 0$, $\theta = 0$, $\varphi = 0$ or $\varphi = \pi$, i.e. $\mathbb{R}^3 - \{(x,y,z) \in \mathbb{R}^3 \,/\, y = 0 \text{ and } x \geq 0\}$. Thus this set is diffeomorphic to $\mathbb{R}^3$.

3. Since $Df(x,y) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ y+2 & x+3 \end{bmatrix}$ has its first two rows linearly independent, the proof of corollary 3 suggests defining $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^3$ by $F(x,y,z) = f(x,y) + (0,0,z) = (x+2y,\ x+y,\ xy+2x+3y+z)$. To find an inverse of F around 0 , set $F(x,y,z) = (\xi,\eta,\zeta)$ and solve the equations for x, y, z in terms of ξ, η, ζ. We get $x + 2y = \xi$, $x + y = \eta$ and $xy + 2x + 3y + z = \zeta$, having solutions $(x,y,z) = (2\eta - \xi,\ \xi - \eta,\ \zeta - \xi - \eta + \xi^2 + 2\eta^2 - 3\xi\eta)$. Thus we should define the diffeomorphism g by $g(x,y,z) = (2y - x,\ x - y,\ z - x - y + x^2 + 2y^2 - 3xy)$.

4. Case I. Suppose that the $n \times n$ matrix with (i,j) entry $\frac{\partial f_i}{\partial x_j}$ is non-singular at 0 , i.e. the first n columns of $Df(0)$ are linearly independent. Define $F : U \to \mathbb{R}^n \times \mathbb{R}^{m-n}$ by $F(x_1,\dots,x_m) = \big(f(x_1,\dots,x_m), x_{n+1},\dots,x_m\big)$. Then $DF(x)$ has the form

$$\left[\begin{array}{c|c} \multicolumn{2}{c}{Df(x)} \\ \hline 0 & \text{Identity} \end{array}\right]$$

so is non-singular at 0 . Let h be a local inverse of F as given by the Inverse Function Theorem. Then when $(x_1,\dots,x_m)$ lies in the domain of h , we have $Fh(x_1,\dots,x_m) = (x_1,\dots,x_m)$, so, looking at the first n coordinates only we have $fh(x_1,\dots,x_m) = (x_1,\dots,x_n)$ as required.

Case II. General case. Some n columns of $Df(0)$ are linearly independent, say the columns $i_1, \dots, i_n$ (in ascending order). Let $i_{n+1}, \dots, i_m$ denote the remaining integers between 1 and m and define the diffeomorphism $p : \mathbb{R}^m \to \mathbb{R}^m$ by $p(x_1,\dots,x_m) = (x_{i_1},\dots,x_{i_m})$. Then $fp^{-1} : p(U) \to \mathbb{R}^n$ satisfies the conditions of Case I, so $\exists$ a diffeomorphism g of a neighbourhood of 0 in $\mathbb{R}^m$ onto another such neighbourhood $\ni g(0) = 0$ and $fp^{-1}g(x_1,\dots,x_m) = (x_1,\dots,x_n)$. We may set $h = p^{-1}g$.

5. $Df(x,y) = [10x - 2y \quad 4y - 2x]$, so $Df(0) = [0 \quad 0]$ and hence 0 is a critical point. $Hf(x,y) = \begin{bmatrix} 10 & -2 \\ -2 & 4 \end{bmatrix}$, which is non-singular, so 0 is a non-degenerate critical point. Finally, $\det(\lambda 1 - Hf(0)) = \lambda^2 - 14\lambda + 36$, which has no negative roots, so the critical point has index 0.

$Dg(x,y) = [5y - 6x \quad 5x]$, so $Dg(0) = [0 \quad 0]$ and hence 0 is a critical point. $Hg(x,y) = \begin{bmatrix} -6 & 5 \\ 5 & 0 \end{bmatrix}$, which is non-singular, so 0 is a non-degenerate critical point. Finally, $\det(\lambda 1 - Hg(0)) = \lambda^2 + 6\lambda - 25$, which has 1 negative root, so the critical point has index 1.

$Dh(x,y) = [y \cos xy \cdot \cos(\cos xy) \quad x \cos xy \cdot \cos(\cos xy)]$, so $Dh(0) = [0 \quad 0]$ and hence 0 is a critical point. $Hh(0) = \begin{bmatrix} 0 & \cos 1 \\ \cos 1 & 0 \end{bmatrix}$, which is non-singular, so 0 is a non-degenerate critical point. Finally, $\det(\lambda 1 - Hh(0)) = \lambda^2 - (\cos 1)^2$, which has 1 negative root, so the critical point has index 1.

$Di(x,y,z) = [y + z \quad x + z \quad y + x]$, so $Di(0) = [0 \quad 0 \quad 0]$ and hence 0 is a critical point. $Hi(x,y,z) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, which is non-singular, so the critical point is non-degenerate. Finally, $\det(\lambda 1 - Hi(0)) = (\lambda + 1)^2(\lambda - 2)$, which has 2 negative roots, so the critical point has index 2.

$Dj(x,y,z) = [2x + y + z \quad 2y + z + x \quad 2z + x + y]$, so $Dj(0) = [0 \quad 0 \quad 0]$ and hence 0 is a critical point. $Hj(x,y,z) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, which is non-singular, so the critical point is non-degenerate. Finally, $\det(\lambda 1 - Hj(0)) = (\lambda - 1)^2(\lambda - 4)$, which has no negative roots, so the critical point has index 0.

6. Special case: $p = 0$ and $f(x) = \Sigma_{i=1}^{m} \pm x_i^2$. We have

$$D(f)(0) = [\pm 2x_1 \quad \pm 2x_2 \quad \dots \quad \pm 2x_m] \text{ and hence } H(f)(0) = \begin{bmatrix} \pm 2 & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ 0 & & \ddots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & \pm 2 \end{bmatrix}.$$

By the chain rule, $D(fh)(q) = D(f)(0) \cdot D(h)(q) = [0 \quad \dots \quad 0]$, so q is a critical point. Writing $h = (h_1, \dots, h_m)$, the (i,j) entry of $D(h)(q)$ is $\left.\frac{\partial h_i}{\partial x_j}\right|_q$, and hence

$$D(h)(q)^* \cdot H(f)(0) \cdot D(h)(q) = \left[\left.\frac{\partial h_j}{\partial x_i}\right|_q\right] \begin{bmatrix} \pm 2 & 0 & \cdots & 0 \\ & & & \vdots \\ 0 & & & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & \pm 2 \end{bmatrix} \left[\left.\frac{\partial h_i}{\partial x_j}\right|_q\right]$$

$$= \left[\left.\frac{\partial h_j}{\partial x_i}\right|_q\right] \left[\pm 2 \left.\frac{\partial h_i}{\partial x_j}\right|_q\right]$$

$$= \left[\left.\Sigma_{k=1}^{m} \pm 2 \frac{\partial h_k}{\partial x_i} \frac{\partial h_k}{\partial x_j}\right|_q\right].$$

On the other hand,

$$\begin{aligned} D(fh)(x) &= D(f)\big(h(x)\big) \cdot D(h)(x) \\ &= [\pm 2h_1(x) \quad \dots \quad \pm 2h_m(x)] \left[\left.\frac{\partial h_i}{\partial x_j}\right|_x\right] \\ &= \left[\left.\Sigma_{k=1}^{m} \pm 2h_k(x) \cdot \frac{\partial h_k}{\partial x_j}\right|_x\right], \end{aligned}$$

so

$$\begin{aligned} H(fh)(q) &= \left[\left.\Sigma_{k=1}^{m} \pm 2 \frac{\partial}{\partial x_i}\left(h_k(x) \cdot \frac{\partial h_k}{\partial x_j}\right)\right|_q\right] \\ &= \left[\left.\Sigma_{k=1}^{m} \pm 2 \left(\frac{\partial h_k}{\partial x_i}\frac{\partial h_k}{\partial x_j} + h_k \frac{\partial^2 h_k}{\partial x_i \partial x_j}\right)\right|_q\right] \\ &= \left[\left.\Sigma_{k=1}^{m} \pm 2 \frac{\partial h_k}{\partial x_i} \cdot \frac{\partial h_k}{\partial x_j}\right|_q\right] \qquad \text{since } h_k(q) = 0 \\ &= D(h)(q)^* \cdot H(f)(0) \cdot D(h)(q). \end{aligned}$$

Since the three matrices $D(h)(q)^*$, $H(f)(0)$ and $D(h)(q)$ are all non-singular, so is $H(fh)(q)$ and hence q is a non-degenerate critical point of fh.

General case: Letting θ be as in the hints, we have $q = (\theta^{-1}h)^{-1}(0)$ and $f\theta$ is expressible as a sum of squares as was f in the special case. Thus $fh = (f\theta)(\theta^{-1}h)$ has a non-degenerate critical point at q.

CHAPTER 5

1. Let $E = \{(x,y,z) \in \mathbb{R}^3 \,/\, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$. Then the function $h : E \to S^2$ defined by $h(x,y,z) = (x,y,z) \,/\, \sqrt{x^2+y^2+z^2}$ is a homeomorphism. Let $\mathcal{D}$ be the usual C^∞ differential structure on S^2. Define a differential structure $\mathcal{E}$ on E by setting

$$\mathcal{E} = \{(h^{-1}(U), \varphi h) \,/\, (U,\varphi) \in \mathcal{D}\}.$$

Then h is a diffeomorphism from $(E,\mathcal{E})$ to $(S^2,\mathcal{D})$.

2. Let M^m, N^n and P^p be differentiable manifolds of class at least C^r and $f : M \to N$ and $g : N \to P$ be C^r functions. We use the criterion of lemma 2 to show that gf is C^r. Let $x \in M$, and let (U,φ) and (W,χ) be any charts in the structures of M and P respectively for which $x \in U$ and $gf(x) \in W$. Let (V,ψ) be a chart in the structure of N for which $f(x) \in V$. In a neighbourhood of $\varphi(x)$, we have $\chi g f \varphi^{-1} = (\chi g \psi^{-1})(\psi f \varphi^{-1})$. Now $\chi g \psi^{-1}$ is C^r at $\psi f(x)$ and $\psi f \varphi^{-1}$ is C^r at $\varphi(x)$. Thus by the chain rule, $\chi g f \varphi^{-1}$ is C^r at $\varphi(x)$.

Now suppose $f : M \to N$ and $g : N \to P$ are C^r diffeomorphisms. Then gf is a C^r homeomorphism and gf has rank m at each point of its domain. Indeed, let $x \in M$, and choose charts (U,φ), (V,ψ) and (W,χ) in the structures of M, N and P $\ni$ $x \in U$, $f(x) \in V$ and $gf(x) \in W$. Then $\psi f \varphi^{-1}$ and $\chi g \psi^{-1}$ have rank m at $\varphi(x)$ and $\psi f(x)$ respectively and hence $D(\psi f \varphi^{-1})(\varphi(x))$ and $D(\chi g \psi^{-1})(\psi f(x))$ are non-singular. Thus $D(\chi g f \varphi^{-1})(\varphi(x))$ is non-singular, so gf has rank m at x.

3. A C^{r+1} structure on M is an atlas on M and it satisfies DS1 because every C^{r+1} function is C^r.

We take $M = \mathbb{R}$ and $r = 1$. Let $\mathcal{D}$ be the usual C^2 structure on $\mathbb{R}$ and let $\mathcal{E}$ be the C^2 structure on $\mathbb{R}$ having as basis $\{(\mathbb{R},h)\}$, where $h : \mathbb{R} \to \mathbb{R}$ is the homeomorphism defined by $h(x) = x + x|x|$. Since h is not C^2, $(\mathbb{R},h) \notin \mathcal{D}$ and so $\mathcal{D} \neq \mathcal{E}$. However, h is a C^1 diffeomorphism, so $\{(\mathbb{R},1), (\mathbb{R},h)\}$ form a basis for a C^1 structure on $\mathbb{R}$, and hence $\mathcal{D}$ and $\mathcal{E}$ are bases for the same C^1 structure.

4. S^1 is a compact manifold. For each $r > 0$, let $h_r : \mathbb{R} \to \mathbb{R}$ be defined by $h_r(t) = t$ if $t \le 0$ and $h_r(t) = rt$ if $t \ge 0$. Define $i : (-\pi,2\pi) \to S^1$ by $i(t) = (\cos t, \sin t)$. Then $\{(i(-\pi,\pi),\ h_r(i|(-\pi,\pi))^{-1}),\ (i(0,2\pi),\ (i|(0,2\pi))^{-1})\}$ is a basis for a differential structure $\mathcal{D}_r$ on S^1. Furthermore, $\mathcal{D}_r = \mathcal{D}_s$ iff $r = s$.

5. By lemma 1, $\exists\ (V,\psi), (W,\chi) \in \mathcal{D} \ni \psi(x) = \chi(y) = 0$. Since $x \neq y$, we may assume $V \cap W = \phi$. Let $\alpha, \beta : \mathbb{R}^m \to \mathbb{R}^m$ be the translations defined by $\alpha(z) = z + u$ and $\beta(z) = z + v$. Now $(V,\alpha\psi), (W,\beta\chi) \in \mathcal{D}$ with $\alpha\psi(x) = u$ and $\beta\chi(y) = v$. Since $u \neq v$, we may assume that V and W are small enough so that $\alpha\psi(V) \cap \beta\chi(W) = \phi$. Set $U = V \cup W$ and define $\varphi : U \to \mathbb{R}^m$ by $\varphi | V = \alpha\psi$, $\varphi | W = \beta\chi$. It is readily checked that (U,φ) satisfies the requirements.

6. As in the solution to exercise 3.8, if M is connected and $x \in M$, let $A = \{y \in M \ / \ \exists\ (U,\varphi) \in \mathcal{D}$ with $x, y \in U$ and $\varphi(U) = \mathbb{R}^m\}$.

It is clear that A is open and to follow the previous solution, we need only prove that A is closed. The previous solution carries over provided differentiability is respected. Thus (V,h) should belong to $\mathcal{D}$. Unfortunately the $g : \mathbb{R}^m \to \mathbb{R}^m$ constructed there is not differentiable at either Fr $B(0;r)$ or $h(y)$, so that construction must be modified. Let $h : \mathbb{R} \to \mathbb{R}$ be as in lemma 4.1. On $[-1,1]$, hence on $\mathbb{R}$, h' is bounded, say $x \in \mathbb{R} \Rightarrow |h'(x)| < k$. Define $\alpha : \mathbb{R} \to \mathbb{R}$ by $\alpha(x) = x + h(4(x-\frac{1}{2}))/4k$. Then $\forall\ x \in \mathbb{R}$, $\alpha'(x) > 0$, and α is surjective, so α is a diffeomorphism: $\alpha(x) = x$ unless $|x-\frac{1}{2}| \le \frac{1}{4}$, but $\alpha(\frac{1}{2}) \neq \frac{1}{2}$. Define $\beta : \mathbb{R}^m \to \mathbb{R}^m$ by $\beta(x) = \alpha(|x|)x/|x|$ if $x \neq 0$, and $\beta(0) = 0$. Note that if $|x| \le \frac{1}{4}$ then $\alpha(|x|) = |x|$ so that $\alpha(|x|)x/|x|$ reduces to x and hence β is a diffeomorphism of $\mathbb{R}^m$ which is the identity outside $B(0;1)$ but β moves points distance ½ from 0 radially outwards. If the points $h(y)$ and $h(z)$ of the solution to exercise 3.8 are such that $\beta(h(y)) = h(z)$, then we may let the required g be β. Otherwise, we find a diffeomorphism γ of $\mathbb{R}^m$ so that $\beta\gamma h(y) = \gamma h(z)$ and let the required g be $\gamma^{-1}\beta\gamma$, which is 1 outside the bounded set $\gamma^{-1}B(0;1)$. For example, γ might be the composition $\zeta\,\varepsilon\,\delta$, where δ and ζ are translations and ε a contraction defined by

$$\delta : x \mapsto x - h(y)\ , \quad \varepsilon : x \mapsto \frac{x}{k|h(z)-h(y)|}\ , \quad \zeta : x \mapsto x + \frac{h(z)-h(y)}{2|h(z)-h(y)|}\ .$$

7. Write $V_y = f(U_x)$ when $x \in f^{-1}(y)$. Then V_y is open in M, by Invariance of Domain. Let

$$\mathcal{B} = \{(V_y, (f\,|\,U_x)^{-1}) \;/\; x \in f^{-1}(y) \text{ and } y \in M\}\ .$$

Surjectivity of f guarantees that $\mathcal{B}$ is an atlas. Suppose $(V_y, (f\,|\,U_x)^{-1}), (V_\eta, (f\,|\,U_\xi)^{-1}) \in \mathcal{B}$. Then

$$U_x \cap f^{-1}(V_\eta) \xrightarrow{\ f\ } V_y \cap V_\eta \xrightarrow{(f\,|\,U_\xi)^{-1}} \mathbb{R}^m$$

is C^r at each point of its domain. Indeed, let $z \in U_x \cap f^{-1}(V_\eta)$ and set $z' = (f\,|\,U_\xi)^{-1}f(z)$. Then the coordinate transformation when restricted to $U_x \cap f^{-1}(V_\eta) \cap U_z$ (a neighbourhood of z in $\mathbb{R}^m$) is also the restriction of $(f\,|\,U_{z'})^{-1}(f\,|\,U_z)$ which, by hypothesis, is C^r.

CHAPTER 6

1. Let M denote the upper half plane and N the plane minus the non-negative real axis. Give M and N the orientations determined by the bases $\{(M,\varphi)\}$, $\{(N,\varphi)\}$ respectively, where φ is the forgetful function given by $\varphi(x+iy) = (x,y)$. Then $\varphi f\varphi^{-1}(x,y) = \varphi f(x+iy) = \varphi(x^2-y^2+i\cdot 2xy)$, so $\Delta(\varphi f\varphi^{-1})(x,y) = 2(x^2+y^2) > 0$ for each $(x,y) \in \varphi(M)$. Thus f is orientation preserving.

2. Orient $\mathbb{R}^3$, $\mathbb{R}^3 - \{(x,0,z) \;/\; x \geq 0\}$ and $(0,\infty) \times (0,2\pi) \times (0,\pi)$ using the identity charts. The diffeomorphism is a composition of two, viz a standard orientation preserving diffeomorphism $\mathbb{R}^3 \to (0,\infty) \times (0,2\pi) \times (0,\pi)$ and, in the notation of the solution to exercise 4.2, the diffeomorphism