CHAPTER 11

1. Let $e : \mathbb{R}^2 \to T^2$ be an embedding and let $M = T^2 - e(\text{Int } B^2)$ and $N = e(B^2)$. Then M and N are manifolds with boundary for which $\partial M = \partial N$ ($\approx S'$).

One can perform a surgery in which N is replaced by M as follows: Suppose Q^2 is any 2-manifold and $f : \mathbb{R}^2 \to Q$ is an embedding. Then $fe^{-1}|N$ enbeds N in Q. Define $\alpha : \text{Int } B^2 - \{0\} \to \mathbb{R}^2 - B^2$ by $\alpha(rx) = x/r$ where $r \in (0,1)$ and $x \in S^1$. One could say that the adjunction manifold $[Q - f(0)] \cup_{e\alpha f^{-1}} [M - \partial M]$ is obtained from Q by a surgery which replaces N by M. In this construction, T^2 could be replaced by any m-manifold and e by an embedding of $\mathbb{R}^m$ in that manifold: Q would then be an m-manifold.

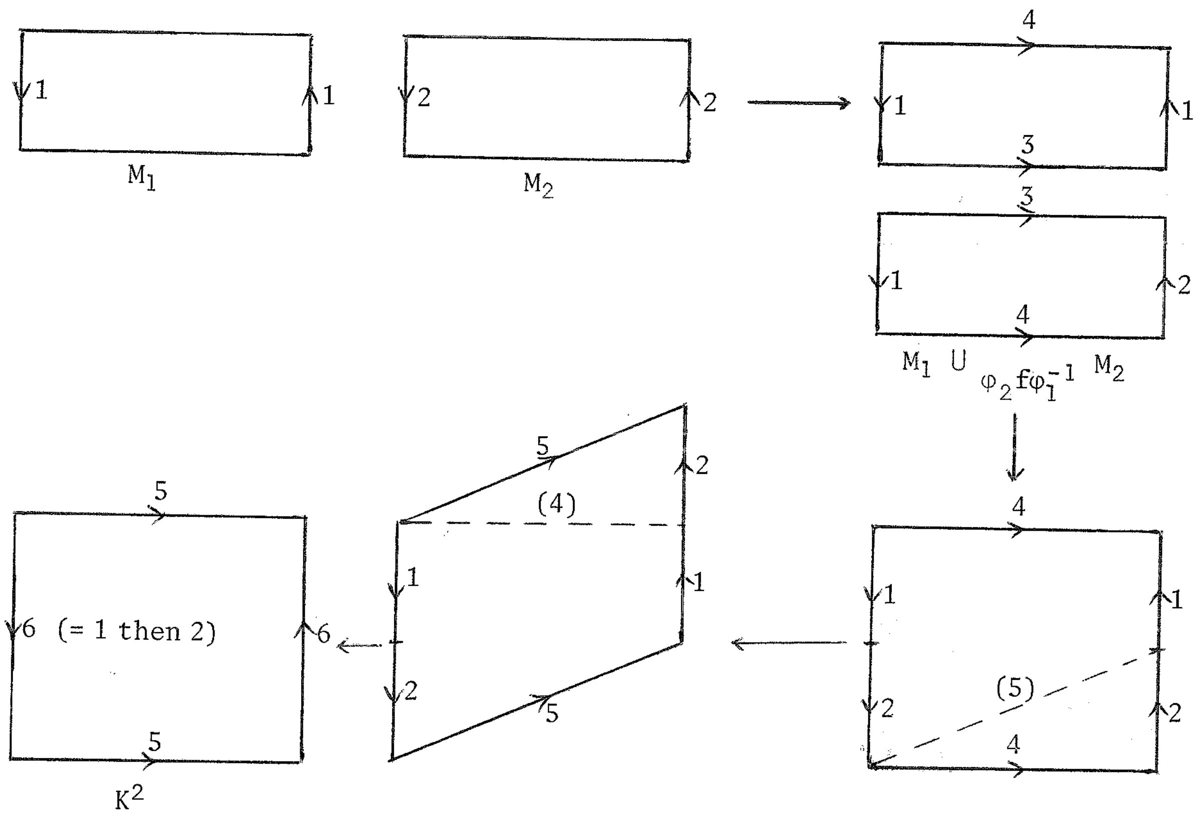
2. Suppose $N = \chi(m,e)$. Define $f : S^{n-1} \times \text{Int } B^m \to N$ by $f(x,y) = (y,x)$. Then f is an embedding. Let

$$\beta : S^{n-1} \times [\text{Int } B^m - \{0\}] \to [\text{Int } B^n - \{0\}] \times S^{m-1},$$

be defined by $\beta(v,ru) = (rv,u)$, where $u \in S^{m-1}$, $v \in S^{n-1}$ and $r \in (0,1)$. It is claimed that M is diffeomorphic to $\chi(N,f) = [N - f(S^{n-1} \times \{0\})] \cup_{\beta f^{-1}} [\text{Int } B^n \times S^{m-1}]$. Indeed, define $g : M \to \chi(N,f)$ and $h : \chi(N,f) \to M$ by $g(x) = x$ if $x \in M - e(S^{m-1} \times \{0\})$ and if $e(u,rv) \in e(S^{m-1} \times \text{Int } B^n)$ set $g(e(u,rv)) = (rv,u)$; if $x \in M - e(S^{m-1} \times \{0\}) \subset \chi(N,f)$ then set $h(x) = x$, if $(ru,v) \in [\text{Int } B^m - \{0\}] \times S^{n-1}$ set $h(ru,v) = e(u,rv)$ and if $(rv,u) \in \text{Int } B^n \times S^{m-1}$, set $h(rv,u) = e(u,rv)$. One can check that g and h are well-defined, hence smooth, and that g and h are mutual inverses. Thus M is diffeomorphic to $\chi(N,f)$.

3. Let M_1 and M_2 be two Möbius bands, with no boundary. Let C_1 and and C_2 denote the central circles of M_1 and M_2. Then $M_i - C_i$ is diffeomorphic to $S^1 \times (-1,1)$: say $\varphi_i : S^1 \times (-1,1) \to M_i$ is an embedding with $\varphi_i(S^1 \times (-1,1)) = M_i - C_i$ $\ni$ points of C_i are near $\varphi_i(S^1 \times (-1,0))$: thus for each $u \in S^1$, as $t \to -1^+$, $\varphi_i(u,t) \to C_i$. Limerick 1.II says that $M_1 \cup_{\varphi_2 f \varphi_1^{-1}} M_2$ is diffeomorphic to the Klein

bottle, where $f : S^1 \times (-1,1) \to S^1 \times (-1,1)$ is defined by $f(u,t) = (u,-t)$. As in this module, we have glued M_1 and M_2 along a neighbourhood of the edges then omitted the edges to ensure smoothness. The following sequence of pictures illustrates that $M_1 \cup_{\varphi_2 f \varphi_1^{-1}} M_2$ is diffeomorphic to K^2. Numbered arrows indicate that the corresponding edges are the same (cf figure 13).



CHAPTER 12

1. Firstly, h is well-defined because if $(q,t) \in [M_{-1} - \underset{\sim}{e}(S^{\lambda-1} \times \{0\})] \times [-1,1]$ and $(x,y) \in P_{\lambda,m-\lambda}$ represent the same point of $\omega(M_{-1},\underset{\sim}{e})$, then by definition, $\beta(\underset{\sim}{e}^{-1}(q), t) = (x,y)$, so that if $q = \underset{\sim}{e}(u,rv)$, then by definition of β, (x,y) must be that point on the curve $\gamma_{\varphi(q)}$ satisfying $-|x|^2 + |y|^2 = t$. Since φ carries integral curves on M to integral curves on $P_{\lambda,m-\lambda}$, we must have $h(q,t) = \varphi^{-1}(x,y)$.

Secondly, h is smooth and has rank m at each point of its domain since its restriction to the separate parts (each open) of its domain are.

Thirdly, h is a bijection with continuous inverse.

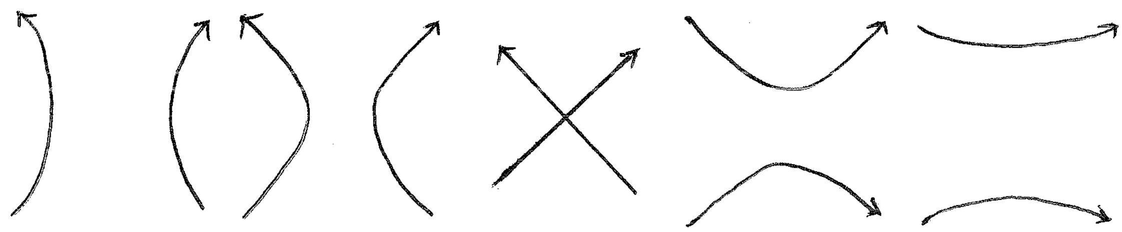
Thus h is a diffeomorphism.

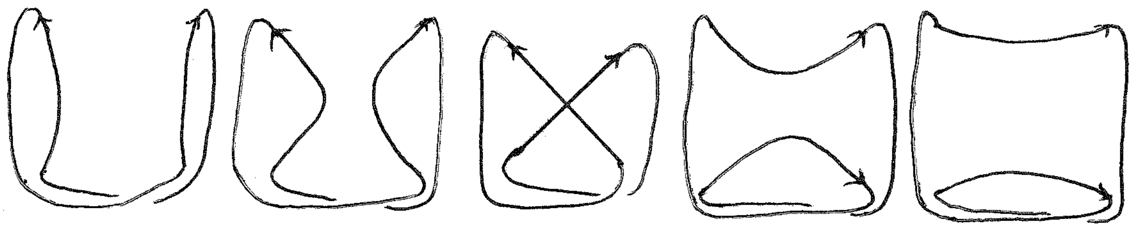
It is easily checked that $h(\partial_{\pm 1}\ \omega\ (M_{-1},\underset{\sim}{e})) = M_{\pm 1}$.

Finally, if $(q,t) \in [M_{-1} - \underset{\sim}{e}(S^{\lambda-1} \times \{0\}] \times [-1,1]$ then $fh(q,t) = t = g(q,t)$ by definition, and if $(x,y) \in P_{\lambda,m-\lambda}$, then $fh(x,y) = f\varphi^{-1}(x,y) = -|x|^2 + |y|^2 = g(x,y)$.

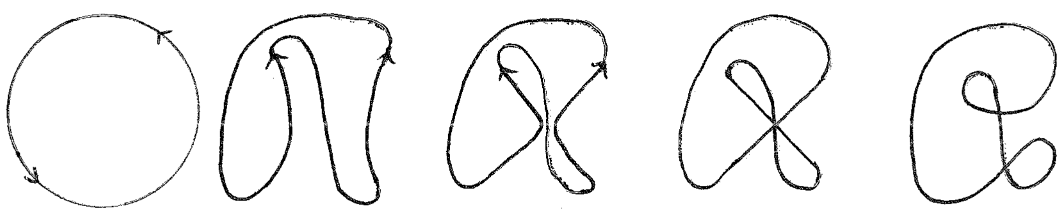
CHAPTER 13

1. The following sequences of pictures show, firstly, five different levels on $P_{1,1}$ and, secondly, five different levels of the trace of a twisting surgery based on the twisting surgery illustrated by figure 60. The arrows in the first sequence indicate an orientation of the levels of $P_{1,1}$ consistent with an orientation of the levels of the trace. Height increases to the right.

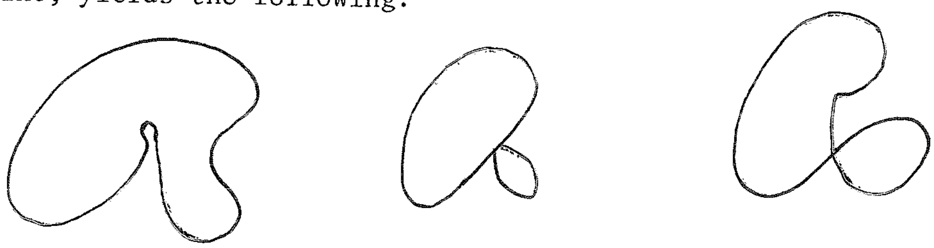




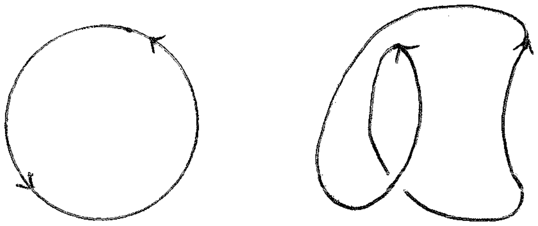
To construct a picture of the trace as in figure 61, beginning at the bottom with a circle which does not cross itself, we must somehow get two parts of the circle into a form suitable for attaching the patch. We might begin as follows:



Shrinking the loop in the middle of the last three pictures, which corresponds to concentrating much of the change about the critical point, yields the following:



which is rather like the apparent levels near the critical point in figure 61. An alternative, as illustrated by figure 67, is to separate the self-crossing away from the critical point:



etc as in the second sequence above

2. It suffices to verify the following lemma.

<u>Lemma</u>. Let $e : \mathbb{R}^m \to \mathbb{R}^m$ be an orientation preserving embedding. Then there is a diffeomorphism $g : \mathbb{R}^m \to \mathbb{R}^m$ and $\varepsilon > 0 \ni g$ has compact support (i.e. is the identity outside some compact subset of $\mathbb{R}^m$) and $g \mid \varepsilon B^m = e \mid \varepsilon B^m$.

Proof that the lemma $\Rightarrow$ the exercise: Let e and f be as in the exercise. $\exists$ a diffeomorphism $h_1 : \mathbb{R}^m \to \mathbb{R}^m$ with compact support $\ni h_1 e(0) = f(0)$. Thus $h_1 e f^{-1}$ is an embedding defined on a neighbourhood of $f(0)$. Strictly, to apply the lemma to $h_1 e f^{-1}$, we want $f(0) = 0$ and $h_1 e f^{-1}$ to have domain $\mathbb{R}^m$: this can be arranged by a normalisation process. By the lemma, $\exists$ diffeomorphism $g : \mathbb{R}^m \to \mathbb{R}^m$ with compact support $\ni g$ and $h_1 e f^{-1}$ agree in a neighbourhood of $f(0)$. Let $h = g^{-1} h_1$.

Proof of the lemma:

Case I: Assume $e(0) = 0$ and $De(0)$ is the identity. Firstly note that if U is an open convex subset of $\mathbb{R}^m$ and $f : U \to \mathbb{R}^m$ a C^1

function $\ni \forall\, i, j$, $\left|\frac{\partial f_i}{\partial x_j} - \delta_{ij}\right| < \frac{1}{m}$ throughout U (where $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$), then f is an embedding. Indeed, f has rank n throughout U , so we need only show that f is injective.

Define $f' : U \to \mathbb{R}^m$ by $f'(x) = f(x) - x$. Then $\left|\frac{\partial f'_i}{\partial x_j}\right| < \frac{1}{m}$, so by exercise 15.1, $\forall\, x, y \in U$, $|f'(x) - f'(y)| \leq |x - y|$, with equality only if $x = y$. Thus $x \neq y \Rightarrow |f'(x) - f'(y)| < |x - y|$. Now $x = f(x) - f'(x) \Rightarrow |x - y| \leq |f(x) - f(y)| + |f'(x) - f'(y)| < |f(x) - f(y)| + |x - y|$, so $|f(x) - f(y)| > 0$, i.e. $f(x) \neq f(y)$.

Now let e be as in this case of the lemma, let $h : \mathbb{R} \to \mathbb{R}$ be as in lemma 4.1 and let $\varepsilon > 0$. Define g by

$$g(x) = h(|x|/2\varepsilon)e(x) + [1 - h(|x|/2\varepsilon)]x .$$

For $|x| \geq 2\varepsilon$, $g(x) = x$ and for $|x| \leq \varepsilon$, $g(x) = e(x)$. Further, if ε is small enough, g is a diffeomorphism. Indeed,

$$\frac{\partial g_i}{\partial x_j} = h(|x|/2\varepsilon)\,\frac{\partial e_i}{\partial x_j} + [1 - h(|x|/2\varepsilon)]\delta_{ij} + \left(e_i(x) - x_i\right) \cdot h'\left(\frac{|x|}{2\varepsilon}\right) \frac{x_j}{2\varepsilon|x|} ,$$

so $$\left|\frac{\partial g_i}{\partial x_j} - \delta_{ij}\right| \leq \left|\frac{\partial e_i}{\partial x_j} - \delta_{ij}\right| + \frac{1}{2\varepsilon}\,|e_i(x) - x_i| \cdot \left|h'\left(\frac{|x|}{2\varepsilon}\right)\right| .$$

Since $De(0)$ is the identity, $\lim\limits_{x \to 0} \frac{|f(x) - x|}{|x|} = 0$, so if ε is small enough, then $\frac{1}{2\varepsilon}\,|e_i(x) - x_i| \cdot \left|h'\left(\frac{|x|}{2\varepsilon}\right)\right| < \frac{1}{2m}$ when $|x| \leq 3\varepsilon$, and by continuity, again if ε is small enough, $\left|\frac{\partial e_i}{\partial x_j} - \delta_{ij}\right| < \frac{1}{2m}$ when $|x| \leq 3\varepsilon$. Thus $\left|\frac{\partial g_i}{\partial x_j} - \delta_{ij}\right| < \frac{1}{m}$ on $3\varepsilon B^m$ and by the previous paragraph, g is an embedding on $3\varepsilon B^m$. On the other hand, if $x, y \in \mathbb{R}^m$ with $|x| \leq 2\varepsilon$ and $|y| \geq 3\varepsilon$, then $g(y) = y$, so if $g(x) = g(y)$, then $h(|x|/2\varepsilon)\ e(x) + [1 - h(|x|/2\varepsilon)]\ x = y$, i.e. $h(|x|/2\varepsilon)(e(x) - x) = y - x$ and hence $|e(x) - x| \geq |y - x| \geq \varepsilon$. On the other hand, since $|x| \leq 3\varepsilon$, $\frac{1}{2\varepsilon}\,|e_i(x) - x_i| \cdot |h'(|x|/2\varepsilon)| < 1/2m$ from which $|e(x) - x| < \varepsilon$, a contradiction. Thus $g(x) \neq g(y)$. In all other cases where $x, y \in \mathbb{R}^m$ with $x \neq y$ it is clear that $g(x) \neq g(y)$. Thus g is a diffeomorphism.

Case II: Assume that $e(0) = 0$. Since e is orientation preserving, $\exists$ a diffeomorphism with compact support $h : \mathbb{R}^m \to \mathbb{R}^m \ni Dh(0) = De(0)$: h might restrict on B^m to the linear transformation whose matrix representation is $De(0)$. If also $h(0) = 0$ then $h^{-1}e$ satisfies the conditions of case I, so given $g' : \mathbb{R}^m \to \mathbb{R}^m \ni g' = h^{-1}e$ on some ball around 0 , we may let $g = hg'$.

Case III: General case. As in the solution to exercise 5.6, we may find a diffeomorphism $h : \mathbb{R}^m \to \mathbb{R}^m$ with compact support $\ni he(0) = 0$. Now proceed as in case III.

3. Since S is not orientable, by theorem 6.3, $\exists$ charts (U,φ), (V,ψ) in the structure of $S \ni U$ and V are connected but $\Delta(\psi\varphi^{-1})$ does not have a constant sign throughout $\varphi(U \cap V)$. Inspection of the proof of theorem 6.3 reveals that it may be assumed that $\varphi(U) = \psi(V) = \mathbb{R}^m$. Moreover, given $x \in S$, we may assume that $x \in U \cap V$, for exercise 5.6 provides us with a chart with domain containing x and meeting $U \cap V$ and image $\mathbb{R}^2$. Using a function of the type of g in the solution to exercise 5.6 and the chart about x we are able to modify the charts (U,φ) and (V,ψ) so that $x \in U \cap V$. Thus we have shown that $\forall\ x \in S$, $\exists$ charts (U,φ) and $(V,\psi) \ni x \in U \cap V$, $\varphi(U) = \psi(V) = \mathbb{R}^2$ and $\Delta(\psi\varphi^{-1})$ does not have constant sign: it may be assumed that $\Delta(\psi\varphi^{-1})(\varphi(x)) > 0$.

Now let $A = \{x\} \cup \{y \in S \ / \ \exists$ charts (U,φ), $(V,\psi) \ni x, y \in U \cap V$,

$$\varphi(U) = \psi(V) = \mathbb{R}^2,\ \Delta(\psi\varphi^{-1})(\varphi(x)) > 0,\ \Delta(\psi\varphi^{-1})(\varphi(y)) < 0\}.$$

Clearly $A - \{x\}$ is open; A is a neighbourhood of x , for if (U,φ), (V,ψ) are charts about x as in the previous paragraph then $U \subset A$ because given $y \in U - \{x\}$, then as in the previous paragraph we can modify (V,ψ) so that a point at which $\Delta(\psi\varphi^{-1})$ is negative moves to y but ψ is unchanged at x . On the other hand, as in the solution to exercise 5.6, A is closed. Thus $A = S$.

Suppose given $e, f : \text{Int } B^2 \to S$ as in the exercise and suppose $e(\text{Int } B^2) \cap f(\text{Int } B^2) = \phi$. Then $\exists$ charts (U,φ) and $(V,\psi) \ni e(0)$, $f(0) \in U \cap V$, $\varphi(U) = \psi(V) = \mathbb{R}^2$, $\Delta(\psi\varphi^{-1})(\varphi(x)) > 0$ and

$\Delta(\psi\varphi^{-1})(\varphi(y)) < 0$. If $\Delta(\varphi e)(0) < 0$, precede φ and ψ by a reflection: the only change will be to ensure that $\Delta(\varphi e)(0) > 0$. Either $\Delta(\varphi f)(0) > 0$ or $\Delta(\varphi f)(0) < 0$ and in the latter case, $\Delta(\psi f)(0) > 0$. Thus for one of the charts (U,φ) and (V,ψ), assume the former, we have $\Delta(\varphi e)(0) > 0$ and $\Delta(\varphi f)(0) > 0$, i.e. φe and φf are both orientation preserving at 0 and hence in a neighbourhood of 0 . If the neighbourhood is not $\frac{3}{4} B^2$ then, in a now standard way, we can enlarge U within $e(\text{Int } B^2) \cup f(\text{Int } B^2)$ so that it does contain $e(\frac{3}{4} B^2) \cup f(\frac{3}{4} B^2)$.

Finally since $\varphi e, \varphi f : \frac{3}{4} B^2 \to \mathbb{R}^2$ are orientation preserving and may be extended to orientation preserving embeddings as in exercise 2, $\exists$ a diffeomorphism $g : \mathbb{R}^2 \to \mathbb{R}^2 \ni g\varphi e / \varepsilon B^2 = \varphi f$ for some $\varepsilon > 0$ and g is the identity outside some compact subset of $\mathbb{R}^2$. Let h be 1 outside the image under φ^{-1} of this compact subset and $\varphi^{-1} g \varphi$ inside U .

If e and f are as in the exercise, then we can introduce a third embedding $e' : \text{Int } B^2 \to S$ whose image is disjoint from those of e and f , and using the procedure of the previous paragraph, construct two diffeomorphisms $h_1, h_2 : S \to S$ so that $h_1 e \mid \varepsilon B^2 = e' \mid \varepsilon B^2$ and $h_2 e' \mid \varepsilon B^2 = f \mid \varepsilon B^2$. Then $h = h_2 h_1 : S \to S$ is a diffeomorphism and $he \mid \varepsilon B^2 = f \mid \varepsilon B^2$.

4. Let $h_1 : S \to S$ be a diffeomorphism as given by exercise 3 $\ni \forall x \in \varepsilon B^2$, $h_1 e(1,x) = f(1,x)$. Consider the two embeddings $\text{Int } B^2 \to S$ given by $x \mapsto h_1 e(-1,x)$ and $x \mapsto h_1 f(-1,x)$. Again by exercise 3 $\exists$ a diffeomorphism $h_2 : S \to S \ni \forall x \in \varepsilon B^2$, $h_2 h_1 e(-1,x) = f(-1,x)$. Moreover as constructed in exercise 3, h_2 is the identity outside some chart: it may be assumed that this chart is disjoint from $h_1 e(\{1\} \times \varepsilon B^2)$. Let $h = h_2 h_1$.

5. Let $h : S \to S$ be the diffeomorphism given by exercise 4. Define $g : \chi(S,e) \to \chi(S,f)$ by letting $g(x) = h(x)$ if $x \in S - e(S^0 \times \{0\})$ and $g(ru,v) = (ru,v)$ if $(ru,v) \in \varepsilon \text{ Int } B^1 \times S^1$. This g is well-defined for in $\chi(S,e)$, x and (ru,v) represent the same element iff $\alpha e^{-1}(x) = (ru,v)$ iff $x = e(u,rv)$, in which case
$g(x) = h(x) = he(u,rv) = f(u,rv) = f\alpha^{-1}(ru,v) = f\alpha^{-1}g(ru,v)$: thus $g(x)$ and $g(ru,v)$ represent the same element of $\chi(S,f)$. One can check that g is a diffeomorphism.

CHAPTER 14

1. Choose a chart (U,φ) on $S \ni \varphi(x) = 0$, $[-1,1] \times [0,1] \subset \varphi(U)$ and the components of ξ with respect to (U,φ) are $(0,1)$. Assume that the second integral curve emanating from s does not meet U. Let $h : \mathbb{R} \to \mathbb{R}$ be the function of lemma 4.1. Since $h'(t) = 0$ for $|t| \geq 1$, by compactness h' is bounded, say $k \in \mathbb{R}$ satisfies $\forall\, t \in \mathbb{R}$, $|h'(t)| < k$. Define $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ by $\alpha(t,u) = \big(t + h(2t) \cdot h(u)/2k,\ u\big)$. Then α is the identity outside $[-1,1] \times [-1,1]$ and within that square, displaces part of the u-axis horizontally. The components of $\xi\big(\varphi^{-1}(t,u)\big)$ with respect to $(U,\alpha\varphi)$ are $\big(h(2t) \cdot h'(u)/2k,\ 1\big)$. Let $\xi'\big(\varphi^{-1}(t,u)\big)$ have components $\big(h(2t) \cdot h'(u)/2k,\ 1\big)$ with respect to (U,φ) when $(t,u) \in (-1,1) \times (0,1)$ and let ξ' agree with ξ elsewhere.

2. As in exercise 5.6, we may find a chart (U,φ) from the orientation for $S \ni e(1,0),\ f(1,0) \in U$ and $\varphi(U) = \mathbb{R}^2$. Using this chart as in the solution to exercise 13.3, we may construct a diffeomorphism $h_1 : S \to S$ so that $h_1 e \mid \{1\} \times \varepsilon B^2 = f \mid \{1\} \times \varepsilon B^2$. Similarly we may construct a diffeomorphism $h_2 : S \to S$ so that $h_2 h_1 e \mid \{-1\} \times \varepsilon B^2 = f \mid \{-1\} \times \varepsilon B^2$, and as in the solution to exercise 13.4, we may assume that h_2 is the identity on $h_1 e(\{1\} \times \varepsilon B^2)$. Let $h = h_2 h_1$.

3. Let T have m handles and suppose $m \geq n$. Then T may be obtained from S by adding $m - n$ handles, i.e. by performing $m - n$ surgeries of type $(1,2)$. Reversing these surgeries, S may be obtained from T by performing $m - n$ surgeries of type $(2,1)$, and hence from S by performing $m + 1 - n$ surgeries of type $(2,1)$. Each such surgery increases the genus by at least 1, as illustrated by figure 78. Repeating the cycle of surgeries ℓ times, we deduce that S has infinite genus.

CHAPTER 15

1. Since $f_i(x) - f_i(y) = Df_i(z_i) \cdot (x - y)$, by the Cauchy-Schwarz inequality,

$$|f_i(x) - f_i(y)| \leq |Df_i(z_i)| \cdot |x - y|$$
$$\leq b\sqrt{n} \cdot |x - y|$$ since each entry in $Df_i(z_i)$ lies between $-b$ and b. Thus

$$|f(x) - f(y)|^2 \leq \Sigma_{i=1}^{n} (b\sqrt{n}\ |x - y|)^2 = b^2n^2|x - y|^2 ,$$

so $$|f(x) - f(y)| \leq bn|x - y| .$$

2. It is sufficient to consider sets in $\mathbb{R}^q$ for some q. Let (S_i) be a sequence of sets $\ni \forall\ i\ \ \theta_n(S_i) = 0$, and let $S = \cup_{i=1}^{\infty} S_i$. Given $\varepsilon > 0$, $\forall\ i\ \exists$ open balls $\{B(x_{ij};r_{ij}) \ /\ j = 1,2,\dots\}$ covering $S_i \ni \Sigma_{j=1}^{\infty} r_{ij}^n < \varepsilon / 2^i$. Then the balls $\{B(x_{ij};r_{ij}) \ /\ i, j = 1,2,\dots\}$ cover S and $\Sigma_{i,j=1}^{\infty} r_{ij}^n < \varepsilon$, so $\theta_n(S) = 0$.

3. As a closed ball in $\mathbb{R}^n$, each member of A is compact and convex. Clearly A is countable since $\mathbb{Q}$ is, and the union of the members of A lies in U. On the other hand, if $x \in U$, then $\exists\ r > 0 \ni B(x;r) \subset U$. Let $q \in \mathbb{Q} \cap (0,r/2)$. Then $B(x;q) \cap A \neq \phi$, say $y \in B(x;q) \cap A$. We have $x \in C\ell\ B(y;q) \subset B(x;2q) \subset B(x;r) \subset U$. Thus $C\ell\ B(y;q)$ is a member of A containing x, so U is the union of the members of A.

4. If $C\ell\ (M - S) \neq M$, say $x \in M - C\ell\ (M - S)$, then $\exists$ chart (U,φ) about $x \ni U \subset S$. Thus $\theta_m(U) = 0$, which means that $\theta_m(\varphi(U)) = 0$. But $\varphi(U)$ is a non-empty open subset of $\mathbb{R}^m$ and no such set has m-dimensional Hausdorff measure 0. Thus $C\ell\ (M - S) = M$.