CHAPTER 1

1. (a) $x \,\nu\, A$ because if $r > 0$ then $(r/2,0)$ is an element of A within r of x.

   (b) $x \not\nu A$ because A contains only points with integer coordinates, which are at least one unit apart, and $x \notin A$.

   (c) $x \,\nu\, A$ because if $r > 0$ then we can find a rational number q with $0 < q < r$. Then $(q,0) \in A$ and $(q,0)$ is within r of x.

   (d) $x \,\nu\, A$ because if $r > 0$ then we can find an irrational number s with $0 < s < r$. Then $(s,0) \in A$ and $(s,0)$ is within r of x.

   (e) $x \,\nu\, A$ because if $r > 0$ then $(1 - s,0)$ is an element of A within r of x, where $s = r/2$ if $r \le 1$ and $s = 1$ if $r > 1$.

   (f) $x \,\nu\, A$ because if $r > 0$ then $(s,-\sqrt{1 - s^2})$ is an element of A within r of x, where $1 - r/2 < s < 1$.

   (g) $x \,\nu\, A$ because if $r > 0$ then $(1/n\pi,0)$ is an element of A within r of x, where n is an integer, with $n > 1/\pi r$.

2. <u>Usual nearness relation on $\mathbb{R}^n$:</u>

Near 1: If $x \,\nu\, A$ then $\exists\, a \in A$ with $|x - a| < 1$ (take $r = 1$ in the definition). Since $a \in A$, we have $A \neq \phi$.

Near 2: If $x \in A$, then $\forall\, r > 0$, letting $a = x$, we have $a \in A$ and $|x - a| (= 0) < r$, so $x \,\nu\, A$.

Near 3: We verify the equivalent condition: $x \,\nu\, (A \cup B)$ and $x \not\nu A \Rightarrow x \,\nu\, B$. Suppose $x \,\nu\, (A \cup B)$ and $x \not\nu A$. To show $x \,\nu\, B$, let $r > 0$: we must find $b \in B$ with $|x - b| < r$. Since $x \not\nu A$, $\exists\, s > 0$ so that whenever $a \in A$, $|x - a| \ge s$. Since $x \,\nu\, (A \cup B)$, $\exists\, c \in A \cup B$ with $|x - c| < r$. If $r \le s$, then $c \notin A$, so $c \in B$ and we could take $b = c$. If $r > s$, then it might happen that $c \in A$ so we need to try something different. Note that since $s > 0$, $\exists\, d \in A \cup B$ with $|x - d| < s$. Again $d \notin A$ so $d \in B$; moreover $|x - d| < r$, so we may take $b = d$. Thus in either case $\exists\, b \in B$ with $|x - b| < r$.

Near 4: Suppose $x \, \nu \, A$ and $\forall \, a \in A$, $a \, \nu \, B$. Let $r > 0$. Then $r/2 > 0$ so $\exists \, a \in A$ with $|x - a| < r/2$. Also $\exists \, b \in B$ with $|a - b| < r/2$. Since $|x - b| \leq |x - a| + |a - b|$, by the triangle inequality, we deduce that $|x - b| < r$.

Discrete nearness relation:

Near 1: If $x \, \nu_d \, A$ then $x \in A$, so $A \neq \phi$.

Near 2: If $x \in A$ then $x \, \nu_d \, A$ by definition.

Near 3: If $x \, \nu_d \, (A \cup B)$ then $x \in A \cup B$ so either $x \in A$ (in which case $x \, \nu_d \, A$) or $x \in B$ (in which case $x \, \nu_d \, B$).

Near 4: If $x \, \nu_d \, A$ then $x \in A$ from which the condition $\forall \, a \in A$, $a \, \nu_d \, B$ immediately implies $x \, \nu_d \, B$.

Concrete nearness relation:

Near 1: If $x \, \nu_c \, A$ then $A \neq \phi$ by definition.

Near 2: If $x \in A$ then $A \neq \phi$, so $x \, \nu_c \, A$.

Near 3: If $x \, \nu_c \, (A \cup B)$ then $A \cup B \neq \phi$, so either $A \neq \phi$ (in which case $x \, \nu_c \, A$) or $B \neq \phi$ (in which case $x \, \nu_c \, B$).

Near 4: If $x \, \nu_c \, A$ then $A \neq \phi$, say $a' \in A$. The condition $\forall \, a \in A$, $a \, \nu \, B$ implies that $a' \, \nu \, B$, from which $B \neq \phi$ and hence $x \, \nu_c \, B$.

Cofinite nearness relation:

Near 1: If $x \, \nu \, A$ then either A is infinite or $x \in A$, either of which implies that $A \neq \phi$.

Near 2: If $x \in A$ then $x \, \nu \, A$ by definition.

Near 3: If $x \, \nu \, (A \cup B)$ then either $A \cup B$ is infinite or $x \in A \cup B$. If $A \cup B$ is infinite then at least one of A and B is infinite, say A, so $x \, \nu \, A$. If $x \in A \cup B$ then either $x \in A$ (so $x \, \nu \, A$) or $x \in B$ (so $x \, \nu \, B$).

Near 4: Suppose $x \nu A$ and $\forall a \in A$, $a \nu B$. Either A is infinite or $x \in A$. In the latter case $x \nu B$. If A is infinite then so is B for either $A \subset B$ or $\exists a \in A - B$ from which $a \nu B$ implies that B is infinite.

3. By Near 1, $0 \not\nu \phi$, and by Near 2, $0 \nu \{0\}$ and $0 \nu \{0,1\}$, for any nearness relation ν on $\{0,1\}$. Thus the only subset A of $\{0,1\}$ for which it is uncertain whether $0 \nu A$ is the set $A = \{1\}$. Similarly it is uncertain whether $1 \nu \{0\}$. Either $0 \nu \{1\}$ or $0 \not\nu \{1\}$, and either $1 \nu \{0\}$ or $1 \not\nu \{0\}$, giving rise to four distinct possibilities, although we must check whether they all give rise to nearness spaces.

(i) $0 \nu \{1\}$ and $1 \nu \{0\}$. This gives rise to the discrete nearness relation.

(ii) $0 \not\nu \{1\}$ and $1 \not\nu \{0\}$. This gives rise to the concrete nearness relation.

(iii) $0 \nu \{1\}$ and $1 \not\nu \{0\}$. This gives rise to a nearness relation. It is routine to verify Near 3 and Near 4 as in exercise 2.

(iv) $0 \not\nu \{1\}$ and $1 \nu \{0\}$. This also gives rise to a nearness relation.

The only distinct pair of the four relations above giving rise to homeomorphic nearness spaces is that defined by (iii) and (iv), the function $f : \{0,1\} \to \{0,1\}$ defined by $f(0) = 1$ and $f(1) = 0$ being a homeomorphism between these two nearness spaces.

4. Let $A = \{(x_1,x_2) \in S^1 \,/\, x_2 < 0\}$. The solution to exercise 1(f) tells us that $(1,0) \nu A$. Now $f(0) = (1,0)$, so $g(1,0) = 0$, and $f(\pi,2\pi) = A$, so $g(A) = (\pi,2\pi)$. Thus $g(1,0) \not\nu g(A)$ and so g is not continuous at $(1,0)$.

5. Suppose $x \in X$ and $A \subset X$ with $x \nu A$. Since $A = (A \cap X_1) \cup (A \cap X_2)$, by Near 3 $x \nu (A \cap X_1)$ or $x \nu (A \cap X_2)$: suppose the former. Since $A \cap X_1 \subset X_1$, by Near 4 $x \nu X_1$ and hence $x \in X_1$. Thus $x \in X_1$ and $A \cap X_1 \subset X_1$ satisfy $x \nu (A \cap X_1)$ and so by continuity of $f|X_1$ we have $f(x) \nu f(A \cap X_1)$ and hence $f(x) \nu f(A)$.

6. Let $r > 0$, and set $B = \{a \in A \,/\, |f(x) - f(a)| < r\}$. If $x \not\nu B$ then by Near 3, $x \nu (A - B)$ so that $f(x) \nu f(A - B)$, which contradicts the definition of B: thus $x \nu B$. Continuity of g implies that $g(x) \nu g(B)$, so $\exists\, a \in B \ni |g(x) - g(a)| < r$. This $a \in A$ and also satisfies $|f(x) - f(a)| < r$.

7. (a) Suppose $(x,y) \in \mathbb{R}^2$ and $A \subset \mathbb{R}^2$ are such that $(x,y) \nu A$. To verify that $s(x,y) \nu s(A)$ and $p(x,y) \nu p(A)$, let $r > 0$.

Firstly, $\exists\, (a,b) \in A$ with $|(x,y) - (a,b)| < r/2$, so $|x - a| < r/2$ and $|y - b| < r/2$ and hence $|s(x,y) - s(a,b)| \le |x - a| + |y - b| < r$.

Secondly, if $x \neq 0$ then $\min\{r/2|y|, r/3|x|, |x|/2\}$ is positive so $\exists\, (a,b) \in A$ with $|(x,y) - (a,b)|$ less than this positive number. In particular, $|x - a| < r/2|y|$, $|x - a| < |x|/2$ and $|y - b| < r/3|x|$, from which $|a| < 3|x|/2$ and $|y - b| < r/2|a|$, and hence

$$\begin{aligned}|p(x,y) - p(a,b)| &= |xy - ay + ay - ab| \\ &\le |x - a| \,.\, |y| + |y - b| \,.\, |a| \\ &< r.\end{aligned}$$

This reasoning requires a slight modification if $y = 0$. Interchanging the roles of x and y allows us to assert that $\exists\, (a,b) \in A \ni |p(x,y) - p(a,b)| < r$ when either $x \neq 0$ or $y \neq 0$. If $(x,y) = 0$, then choosing $(a,b) \in A$ with $|(x,y) - (a,b)| < \min\{1,r\}$, we have $|a| = |x - a| < r$ and $|b| = |y - b| < 1$, so again $|p(x,y) - p(a,b)| = |a| \,.\, |b| < r$

(b) <u>Method I</u>. We could generalise the solution to (a) to cover this situation. Suppose $x \in X$ and $A \subset X$ satisfy $x \nu A$, and that $r > 0$.

Firstly, by exercise 6, $\exists\, a \in A \ni |f(x) - f(a)| < r/2$ $|g(x) - g(a)| < r/2$. Then $|(f + g)(x) - (f + g)(a)| \le |f(x) - f(a)| + |g(x) - g(a)| < r$.

Secondly, if $f(x) \neq 0$, then exercise 6 assures us that $\exists\, a \in A \ni |f(x) - f(a)| < \min\{r/2|g(x)|, |f(x)|/2\}$ and $|g(x) - g(a)| < r/3|f(x)|$, from which $|(f \,.\, g)(x) - (f \,.\, g)(a)| \le |f(x) - f(a)| \,.\, |g(x)| + |g(x) - g(a)| \,.\, |f(a)| < r$, with the appropriate modification if $g(x) = 0$. As in (a), we can interchange the roles of f and g to take care of the case $g(x) \neq 0$. The case $f(x) = g(x) = 0$ is similar to the corresponding case in (a).

Method II. We can express $f + g$ and $f \cdot g$ as compositions

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} ,$$

where the last map is s for $f + g$ and p for $f \cdot g$. The map $\Delta : X \to X \times X$ is defined by $\Delta(x) = (x,x)$ and $f \times g : X \times X \to \mathbb{R} \times \mathbb{R}$ is defined by $(f \times g)(x,y) = \big(f(x), g(y)\big)$. One can impose natural nearness relations on $X \times X$ and $\mathbb{R} \times \mathbb{R}$ so that the composant functions are continuous from which the result follows.

A way of defining the natural nearness relation on $X_1 \times X_2$, where (X_1,ν) and (X_2,ν) are two nearness spaces is:

$$(x_1,x_2)\ \nu\ A \subset X_1 \times X_2 \quad \text{iff} \quad \forall\, N_i \subset X_i \ni x_i \not\nu\ (X_i - N_i), \quad i = 1, 2,$$

$$\text{we have} \quad (N_1 \times N_2) \cap A \neq \phi .$$

However, it is much easier to define products in the context of topological spaces: see exercise 2-6.

8. Discrete. Suppose that (X,ν) is a discrete nearness space and (Y,μ) a nearness space homeomorphic to (X,ν), say $h : X \to Y$ is a homeomorphism. We must show that (Y,μ) is discrete, i.e. $\forall\, y \in Y$, $\forall\, B \subset Y$, we have $y\ \mu\ B$ iff $y \in B$. Let $y \in Y$ and $B \subset Y$. If $y\ \mu\ B$ then by continuity of $h^{-1} : Y \to X$, we have $h^{-1}(y)\ \nu\ h^{-1}(B)$ and hence $h^{-1}(y) \in h^{-1}(B)$ since (X,ν) is discrete. This implies that $y \in B$. Conversely if $y \in B$ then Near 2 implies that $y\ \nu\ B$.

Concrete. Suppose (X,ν), (Y,μ) and $h : X \to Y$ are as above except that now (X,ν) is concrete. We show that for $y \in Y$ and $B \subset Y$, $y\ \mu\ B$ iff $B \neq \phi$. Let $y \in Y$ and $B \subset Y$. If $y\ \mu\ B$ then by Near 1, $B \neq \phi$. Conversely if $B \neq \phi$ then $h^{-1}(B) \neq \phi$ and so, since (X,ν) is concrete, $h^{-1}(y)\ \nu\ h^{-1}(B)$. Continuity of h now implies that $hh^{-1}(y)\ \mu\ hh^{-1}(B)$, i.e. $y\ \mu\ B$.

9. We must show that h is a bijection and that both $h : (-1,1) \longrightarrow \mathbb{R}$ and $h^{-1} : \mathbb{R} \longrightarrow (-1,1)$ are continuous. It is readily checked that if $g : \mathbb{R} \longrightarrow (-1,1)$ is defined by $g(y) = y / (1 + |y|)$ then g and h are mutual inverses. Thus h is a bijection with $h^{-1} = g$.

It is clear that if $x \in \mathbb{R}$ and $A \subset \mathbb{R}$ satisfy $x \nu A$ then $(1 \pm |x|) \nu (1 \pm |A|)$, where $1 \pm |A| = \{1 \pm |t| / t \in A\}$. Thus the functions $x \mapsto 1 - |x|$ and $x \mapsto 1 + |x|$ are continuous, and hence the functions h and h^{-1} are quotients of continuous functions in which the denominator is never 0. It is a standard result from elementary calculus that quotients of continuous functions are continuous. This may be proved as in exercise 7.

10. (a) ϕ and $\mathbb{R}$ are connected: they are intervals. The sets $\{0\} \cup \{1/k / k = 1,2,\ldots\}$ and $\{x \in \mathbb{Q} / 0 \le x \le 1\}$ are not connected because they are not intervals.

(b) S^1 is connected by theorem 4, since it is the union of the two connected sets $\{(x_1,x_2) \in S^1 / x_2 \ge 0\}$, $\{(x_1,x_2) \in S^1 / x_2 \le 0\}$ [each being homeomorphic to the closed interval $[-1,1] \subset \mathbb{R}$] having the point $(1,0)$ in common.

The set $\{(x_1,x_2) \in \mathbb{R}^2 / x_1x_2 > 0\}$ is not connected, the function δ defined by $\delta(x_1,x_2) = 0$ if $x_1 > 0$ and $x_2 > 0$ and $\delta(x_1,x_2) = 1$ if $x_1 < 0$ and $x_2 < 0$, being a disconnection.

$\{(x_1,x_2) \in \mathbb{R}^2 / x_1x_2 \ge 0\}$ is connected by theorem 4. Indeed, it is the union of all lines through the origin having positive slope together with the x_2-axis. As lines, these sets are connected, and they have $(0,0)$ in common.

$\{(x_1,x_2) \in \mathbb{R}^2 / x_1 = 0$ or $x_2 = 0$ or $x_1x_2 = 1\}$ is not connected, the function δ defined by $\delta(x_1,x_2) = 0$ if $x_1x_2 = 0$ and $\delta(x_1,x_2) = 1$ if $x_1x_2 \ne 0$ being a disconnection.

$\{(x_1,x_2) \in \mathbb{R}^2 / x_1 = 0$ and $-1 \le x_2 \le 1\} \cup \{(x_1,x_2) \in \mathbb{R}^2 / x_1 > 0$ and $x_2 = \sin(1/x_1)\}$ is connected. Indeed, the two pieces are homeomorphic to $[-1,1]$ and $(0,\infty)$ respectively and hence, connected, by theorem 5. Let δ be a continuous function from that set to 2. We may assume that $\delta(\{(x_1,x_2) \in \mathbb{R}^2 / x_1 > 0$ and $x_2 = \sin(1/x_1)\}) = \{0\}$. Now $(0,0) \nu \{(x_1,x_2) \in \mathbb{R}^2 / x_2 > 0$ and $x_2 = \sin(1/x_1)\}$, by exercise 1(g). Thus $\delta(0,0) \nu \{0\}$ and hence

$\delta(0,0) = 0$. Connectedness of $\{(x_1,x_2) \in \mathbb{R}^2 / x_1 = 0$ and $-1 \le x_2 \le 1\}$ now implies that δ is constant.

(c) Both S^{n-1} (assuming $n > 1$) and the x -axis in $\mathbb{R}^n$ are connected. S^{n-1} is connected because it is the union of the two connected sets $\{(x_1,\dots,x_n) \in S^{n-1} / x_n \ge 0\}$ and $\{(x_1,\dots x_n) \in S^{n-1} / x_n \ge 0\}$ and $\{(x\ ,\dots,x_n) \in S^{n-1} / x_n \le 0\}$ having the point $(1,0,\dots,0)$ in common: these sets are connected because they are homeomorphic to B^{n-1} which is the union of all line segments (connected!) of unit length emanating from $0 \in \mathbb{R}^{n-1}$. The x_1-axis is homeomorphic to $\mathbb{R}$ and hence is connected.

11. (i) Inspection of figure 3 suggests that the only connected subsets of the discrete space are the empty set and the one-point subsets, and this is the case. If A is any subset of a discrete space, with A containing at least two points, then $\delta : A \to 2$ defined by $\delta(x) = 0$ and $\delta(A - \{x\}) = \{1\}$, where x is some element of A, is a disconnection.

(ii) Inspection of figure 4 suggests that there is no way of disconnecting any subset of a concrete space, i.e. all subsets of a concrete space are connected. Indeed, since every subspace of a concrete space is itself concrete, it is enough to show that concrete spaces are connected. Let (X,ν) be a concrete space and let $f : X \to 2$ be continuous. Let $x \in X$. If $y \in X$, then $y \ \nu \ \{x\}$, so $f(y) \ \nu \ \{f(x)\}$, i.e. $f(y) = f(x)$. Thus f is constant, hence cannot be a disconnection.

CHAPTER 2

1. A is both open and closed; $\text{Int } A = C\ell\ A = \text{Fr } A = \phi$.
B is both open and closed (in $\mathbb{R}$); $\text{Int } B = C\ell\ B = \mathbb{R}$, $\text{Fr } B = \phi$.
C is closed but not open and D is open but not closed;
$\text{Int } C = \text{Int } D = (0,1)$, $C\ell\ C = C\ell\ D = [0,1]$, $\text{Fr } C = \text{Fr } D = \{0,1\}$.
E is closed but not open and F is neither open nor closed;
$\text{Int } E = \text{Int } F = (-184,405)$, $C\ell\ E = C\ell\ F = [-184,405] \cup \{1000\}$,
$\text{Fr } E = \text{Fr } F = \{-184,405,1000\}$.

G is open but not closed; $\text{Int } G = (-\infty,1000) \cup (1000,\infty)$, $C\ell\, G = \mathbb{R}$, $\text{Fr } G = \{1000\}$.
H is closed but not open; $\text{Int } H = \phi$, $C\ell\, H = S^{n-1}$, $\text{Fr } H = S^{n-1}$.
Assuming that i runs through the positive integers, I is open but not closed and J is closed but not open; $\text{Int } I = \text{Int } J = I$, $C\ell\, I = C\ell\, J = J$, $\text{Fr } I = \text{Fr } J = \{0\} \cup \bigcup_{i=1}^{\infty} \{x \in \mathbb{R}^n \;/\; |x| = 1/i\}$.

2. Only in cases (a) and (b), and case (c) when $n = 1$, are the given sets N neighbourhoods of 0. Using the criterion illustrated by figure 8, we show that in cases (c) (when $n > 1$), (d) and (e) the given sets N are not neighbourhoods of 0. Suppose $r > 0$. Then $(r/2,0,\ldots,0) \in B(0;r)$ but, when $n > 1$, $(r/2,0,\ldots,0)$ does not lie in the set N of (c). If $q \in \mathbb{Q}$ satisfies $0 < q < r$, then $(q,0,\ldots,0) \in B(0;r)$ but $(q,0,\ldots,0)$ does not lie in the set N of (d). If m is any positive integer with $1/m < r$, then $(1/m,0,\ldots,0) \in B(0;r)$ but $(1/m,0,\ldots,0)$ does not lie in the set N of (e).

3. Usual topology on $\mathbb{R}^n$:

Open 1: The condition $\forall\, x \in U,\ \exists\, r > 0 \ni B(x;r) \subset U$ is vacuously satisfied when $U = \phi$, so ϕ is open.

Open 2: Since $B(x;r) \subset \mathbb{R}^n$, the condition for $\mathbb{R}^n$ to be open is trivially satisfied.

Open 3: Suppose U and V are open in $\mathbb{R}^n$, and let $x \in U \cap V$. Since U is open and $x \in U$, $\exists\, s > 0 \ni B(x;s) \subset U$ and since V is open and $x \in V$, $\exists\, t > 0 \ni B(x;t) \subset V$. Let $r = \min\{s,t\}$. Then $r > 0$ and $B(x;r) \subset B(x;s) \subset U$ and $B(x;r) \subset B(x;t) \subset V$, so $B(x;r) \subset U \cap V$. Thus $U \cap V$ is open.

Open 4: Suppose $\{U_\alpha \;/\; \alpha \in A\}$ is a family of open subsets of $\mathbb{R}^n$ and let $x \in \bigcup_{\alpha \in A} U_\alpha$. For some $\beta \in A$, $x \in U_\beta$. Since U_β is open, $\exists\, r > 0 \ni B(x;r) \subset U_\beta$. Then $B(x;r) \subset \bigcup_{\alpha \in A} U_\alpha$, so the latter set is open.

Discrete topology: Let X be any set. Then ϕ and X are subsets of X, the intersection of any two subsets of X is again a subset of X and the union of any family of subsets of X is again a subset of X. Hence the collection of all subsets of X forms a topology on X.

Concrete topology: Let X be any set. The family $\{\phi, X\}$ clearly forms a topology on X.

4. Suppose that the cofinite and discrete topologies on X agree. If $X = \phi$ then X is finite. If $X \neq \phi$, pick $x \in X$. Then $\{x\}$, being open in the discrete topology is also open in the cofinite topology, i.e. has a finite complement. Thus X is finite, being the union of the two finite sets $\{x\}$ and $X - \{x\}$.

Conversely if X is finite then every subset of X has a finite complement hence is open in the cofinite topology: thus the two topologies are the same.

5. $\text{Int}\,(X - Y) \subset X - Y$, so $X - \text{Int}\,(X - Y) \supset X - (X - Y) = Y$. Thus $X - \text{Int}\,(X - Y)$, being closed, is one of the closed sets whose intersection forms $C\ell\, Y$. Hence $C\ell\, Y \subset X - \text{Int}\,(X - Y)$.

$C\ell\, Y \supset Y$ so $X - C\ell\, Y \subset X - Y$. Thus $X - C\ell\, Y$, being open, is one of the open sets whose union forms $\text{Int}\,(X - Y)$. Hence $X - C\ell\, Y \subset \text{Int}\,(X - Y)$ and hence $C\ell\, Y \supset X - \text{Int}\,(X - Y)$.

6. (a) We verify the criterion given in proposition 2. Let $\mathcal{F} = \{T \times U \subset X \times Y \,/\, T \in \mathcal{T} \text{ and } U \in \mathcal{U}\}$.
Since $X \in \mathcal{T}$ and $Y \in \mathcal{U}$, we have $X \times Y \in \mathcal{F}$, so $\bigcup \mathcal{F} = X \times Y$.
Suppose $T_1 \times U_1$, $T_2 \times U_2 \in \mathcal{F}$. Then $(T_1 \times U_1) \cap (T_2 \times U_2) = (T_1 \cap T_2) \times (U_1 \cap U_2) \in \mathcal{F}$, so the second part of the criterion is also satisfied.

(b) Let $\mathcal{T}$ denote the usual topology on $\mathbb{R}$, $\mathcal{U}$ the usual topology on $\mathbb{R}^2$ and $\mathcal{P}$ the product topology on $\mathbb{R}^2$. We must show that $\mathcal{P} = \mathcal{U}$.

To show that $\mathcal{P} \subset \mathcal{U}$ it is enough to show that the basis for $\mathcal{P}$ is contained in $\mathcal{U}$. Let $T, U \in \mathcal{T}$, so that $T \times U$ is a typical member of the basis for $\mathcal{P}$. Let $(x,y) \in T \times U$. Since T and U are open in the usual topology $\mathcal{T}$, $\exists\, r > 0 \ni (x - r, x + r) \subset T$

and $(y - r, y + r) \subset U$. Thus $B\big((x,y);r\big) \subset (x - r, x + r) \times (y - r, y + r) \subset T \times U$, so $T \times U \in \mathcal{U}$ and hence $\mathcal{P} \subset \mathcal{U}$.

On the other hand, if $V \in \mathcal{U}$ and $(x,y) \in V$ then $\exists\, r > 0 \ni B\big((x,y);r\big) \subset V$. Note that $(x - r/2, x + r/2) \times (y - r/2, y + r/2) \subset B\big((x,y);r\big)$, so that V is expressible as a union of members of the basis for $\mathcal{P}$ and hence $V \in \mathcal{P}$. Thus $\mathcal{U} \subset \mathcal{P}$.

(c) We use criterion (e) of theorem 4, taking as basis for the topology on $Y_1 \times Y_2$ the basis defined in part (a) of this exercise. If $V_1 \times V_2$ is a typical member of this basis then V_i is open in Y_i for $i = 1,2$. Further, $(f_1 \times f_2)^{-1}(V_1 \times V_2) = \big(f_1^{-1}(V_1)\big) \times \big(f_2^{-1}(V_2)\big)$, which is open in $X_1 \times X_2$ since continuity of f_i means that $f_i^{-1}(V_i)$ is open in X_i.

7. The sets divide into the following homeomorphism classes:

I. A, R;
II. B, 8;
III. C, J, L, M, N, S, U, V, W, 1, 2, 3, 5, 7, 2nd knot, 3rd knot;
IV. D, 0, 1st knot;
V. E, F, G, T, Y;
VI. H, I;
VII. K, X;
VIII. P, Q, 6, 9;
IX. 4 .

A number of topological invariants assist one in the division above. For example, A has a closed path and two 3-way junctions, both topological invariants. Thus A differs from H because H has no closed path and from P because P has only one 3-way junction. As an example, it is shown that L and 3 are homeomorphic. Firstly we identify these shapes algebraically as subsets of $\mathbb{R}^2$. Let

$$L = \{(x,y) \in \mathbb{R}^2 \ / \text{ either } x = 0 \text{ and } 0 \le y \le 1 \text{ or } y = 0 \text{ and } 0 \le x \le 1\}$$
$$3 = \{(x,y) \in \mathbb{R}^2 \ / \ x \ge 0 \text{ and either } x^2 + (y-1)^2 = 1 \text{ or } x + (y+1)^2 = 1\}.$$

Define $h : L \to 3$ by
$$h(x,y) = \begin{cases} (2\sqrt{y - y^2},\ 2y) & \text{if } x = 0 \\ (2\sqrt{x - x^2},\ -2x) & \text{if } y = 0 . \end{cases}$$

Note that h takes the vertical part of L and stretches it around the upper semi-circle of 3 and stretches the horizontal part of L around the lower semi-circle of 3. Using exercise 1-5 one can verify that h and h^{-1} are continuous.

8. We must show that (i) $\forall\ \nu \in N$, $\beta\,\alpha\,(\nu) = \nu$ and (ii) $\forall\ \mathcal{T} \in T$, $\alpha\,\beta\,(\mathcal{T}) = \mathcal{T}$.

(i) Suppose $\nu \in N$. By definition,

$x\ \beta\,\alpha\,(\nu)\ A$ iff $\forall\ U \subset X$ satisfying $x \in U$ and $\forall\ y \in U$, $y \not\!\nu\ (X-U)$, we have $U \cap A \neq \phi$.

If $x \in X$, A, $U \subset X$ satisfy $x \in U$ and $\forall\ y \in U$, $y \not\!\nu\ (X-U)$ but $U \cap A = \phi$, then $A \subset X - U$, so by Near 4 and Near 2, $\forall\ y \in U$, $y \not\!\nu\ A$; in particular $x \in U$, so $x \not\!\nu\ A$. Thus $x\ \nu\ A \Rightarrow x\ \beta\,\alpha\,(\nu)\ A$.

If $x\ \beta\,\alpha\,(\nu)\ A$ but $x \not\!\nu\ A$, then $U = \{y \in X \ /\ y \not\!\nu\ A\}$ contains x and satisfies $\forall\ y \in U$, $y \not\!\nu\ (X-U)$ [for $\forall\ z \in X - U$, $z\ \nu\ A$ so by Near 4, $y\ \nu\ (X-U) \Rightarrow y\ \nu\ A$]. Thus $U \cap A \neq \phi$ which contradicts Near 2. Thus $x\ \beta\,\alpha\,(\nu)\ A \Rightarrow x\ \nu\ A$.

(ii) Suppose $\mathcal{T} \in T$. By definition,

$$\alpha\,\beta\,(\mathcal{T}) = \{V \subset X \ /\ \forall\ x \in V,\ \exists\ U \in \mathcal{T} \ni x \in U \subset V\}.$$

By Open 4, $\alpha\,\beta\,(\mathcal{T}) \subset \mathcal{T}$ and clearly $\mathcal{T} \subset \alpha\,\beta\,(\mathcal{T})$. Thus $\alpha\,\beta\,(\mathcal{T}) = \mathcal{T}$.

CHAPTER 3

1. Let $x, y \in X$ be distinct points in an infinite space X having the cofinite topology and let U and V be open neighbourhoods of x and y respectively: thus $X - U$ and $X - V$ are finite. Hence $(X-U) \cup (X-V) = X - (U \cap V)$ is finite, so cannot be all of X and hence $U \cap V \neq \phi$.

2. Let X_1 denote X with the discrete topology and X_2 denote X with any Hausdorff topology. Define $h : X_1 \to X_2$ to be the identity function. Since it has a discrete domain, h is continuous. Clearly h is a bijection. Further, X_1 is compact since X is finite. Hence, since X_2 is Hausdorff, theorem 6 tells us that h is a homeomorphism, and hence by continuity of h^{-1}, every subset of X_2 is open, i.e. X_2 is discrete.