

1 BASIC NOTIONS

Definition 1.1 A topological space is a pair (X, \mathcal{T}) (usually abbreviated to X) where X is a set and \mathcal{T} a family of subsets of X such that:

1. $\emptyset \in \mathcal{T}$;
2. $X \in \mathcal{T}$;
3. for each $U, V \in \mathcal{T}$, $U \cap V \in \mathcal{T}$;
4. for each $\mathcal{F} \subset \mathcal{T}$, $\cup \mathcal{F} \in \mathcal{T}$, where $\cup \mathcal{F} = \{x \in X \mid \text{there is } U \in \mathcal{F} \text{ such that } x \in U\}$.

\mathcal{T} is called a topology for X and the elements of \mathcal{T} are called open sets.

Definition 1.2 Let (X, \mathcal{T}) and (Y, \mathcal{U}) be topological spaces and $f : X \rightarrow Y$ a function. Then f is continuous iff for each $U \in \mathcal{U}$, we have $f^{-1}(U) \in \mathcal{T}$. A continuous function is also called a map. A continuous bijection whose inverse is also continuous is called a homeomorphism. Spaces X and Y are homeomorphic iff there is a homeomorphism from X to Y (equivalently, from Y to X).

Definition 1.3 A property of topological spaces is called a topological property iff whenever it is possessed by a given space it is also possessed by all homeomorphic spaces.

The fundamental objects of study in topology are the topological spaces and maps: they form a category. The primary goal of topology is to classify topological spaces up to homeomorphism and the principal tool is the topological property. The general task is hopeless, but there has been success in special cases: moreover, the attempt has led to many interesting and valuable results and applications. For example, the study of fixed points is useful in the theory of differential equations; the study of singularities of differentiable maps has resulted in a classification of these in low dimensions, the finiteness of this classification leading to the practicality of catastrophe theory with its applications in diverse areas such as Physics, Biology, Economics, Linguistics; the study of the placement of sets, especially of curves, in 3-dimensional space has led to knot theory with applications in Biology and Physics.

Any attempt to classify in some suitable way the finite topological spaces is likely to be unsuccessful, and as the following table shows, even the number of topologies on a finite set follows a strange pattern; indeed there seems to be no simple formula for this number. In this table, the number of distinct topologies on a set of n elements is denoted by $t(n)$ and the number of homeomorphism classes of such topologies is denoted by $h(n)$.

n	1	2	3	4	5	6	7	8	9	10
$t(n)$	1	4	29	355	6942	209527	9535241	642779354	63260289423	8977053873043
$h(n)$	1	3	9	33	139	718	4535	35979	363083	4717687

(Source, in part, <http://www.research.att.com/njas/sequences/>)

Example 1.4 Examples of topological spaces.

1. Declare any subset of \mathbb{R} to be open iff it is a union of open intervals. Then the set of open subsets of \mathbb{R} forms a topology on \mathbb{R} , called the *usual topology*. More generally, declare any subset of \mathbb{R}^n to be open iff it is a union of open “cubes”, i.e. sets of the form $\{(x_1, x_2, \dots, x_n) \mid a_i < x_i < b_i\}$, for some a_i and b_i . Again the family of open sets forms a topology on \mathbb{R}^n called the *usual topology*.

- Let (X, d) be any metric space. Declare a subset $U \subset X$ open iff for each $x \in U$ there is $r > 0$ such that $B_d(x; r) \subset U$, where $B_d(x; r)$ consists of all $y \in X$ for which $d(x, y) < r$. This is the *topology induced by the metric d* . It is the same as the usual topology on \mathbb{R} and \mathbb{R}^n if we take as d the Pythagorean metric.

If (X, \mathcal{T}) is a topological space such that there is a metric d on X for which \mathcal{T} is induced by d then we say that (X, \mathcal{T}) is *metrisable*. A big sub-problem of topology is to decide when a given space is metrisable.

- Let X be any set and topologise X by declaring only \emptyset and X to be open. This, the smallest possible topology on X , is called the *indiscrete topology*.
- Topologise any set X by declaring all subsets to be open. This, the largest topology on X , is called the *discrete topology*. Unlike the indiscrete topology, it is a special case of a metric topology, where the metric is the discrete metric in which the distance between two different points is always 1.
- Declare any subset of \mathbb{R} to be open iff it is a union of intervals of the form $[a, b)$. This, the *right half-open interval topology*, is not metrisable, but obviously it differs from the indiscrete topology.

Example 1.5 *Examples of continuous functions.*

- Any function whose range is indiscrete is continuous.
- Any function whose domain is discrete is continuous.
- Any function between metric spaces is continuous in the metric sense iff it is continuous in the sense of Definition 1.2 when the spaces are given the topologies induced by the metrics.

Definition 1.6 *Let (X, \mathcal{T}) be a topological space. A subfamily \mathcal{B} of \mathcal{T} is a basis for \mathcal{T} provided every member of \mathcal{T} is a union of members of \mathcal{B} . [Thus $\{[a, b) / a, b \in \mathbb{R}\}$ is a basis for the right half-open interval topology of 5 of Example 1.4.] A subfamily \mathcal{S} of \mathcal{T} is a sub-basis provided the family of all finite intersections of members of \mathcal{S} is a basis for \mathcal{T} . Any family \mathcal{F} of subsets of X is a sub-basis for a unique topology on X , called the topology generated by \mathcal{F} .*

Proposition 1.7 *A family \mathcal{B} of subsets of a set X is a basis for a topology on X if and only if:*

- For each $x \in X$ there is $B \in \mathcal{B}$ such that $x \in B$; and
- For each $B_1, B_2 \in \mathcal{B}$ and each $x \in B_1 \cap B_2$ there is $B_3 \in \mathcal{B}$ so that $x \in B_3 \subset B_1 \cap B_2$.

Proof. \Rightarrow : Suppose \mathcal{B} is a basis for a topology \mathcal{T} . By definition, X is open so is a union of members of \mathcal{B} ; this is equivalent to 1. Given $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, it follows that $B_1, B_2 \in \mathcal{T}$, so that $B_1 \cap B_2 \in \mathcal{T}$. Thus $B_1 \cap B_2$ is a union of members of \mathcal{B} , one of which must contain x and lie in $B_1 \cap B_2$.

\Leftarrow : Given a collection \mathcal{B} satisfying 1 and 2, let \mathcal{T} denote the family of all subsets of X which are a union of some members of \mathcal{B} .

- $\emptyset \in \mathcal{T}$ since \emptyset is the union of the subcollection \emptyset of \mathcal{B} .
- $X \in \mathcal{T}$ by condition 1 on \mathcal{B} .

3. If $U, V \in \mathcal{T}$ and $x \in U \cap V$ then there are $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subset U$ and $x \in B_2 \subset V$. By condition 2 on \mathcal{B} there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$; thus $x \in B_3 \subset U \cap V$. This shows that for each $x \in U \cap V$ there is $B_x \in \mathcal{B}$ such that $x \in B_x \subset U \cap V$, hence that $U \cap V$ is a union of members of \mathcal{B} and so $U \cap V \in \mathcal{T}$.
4. For each $\mathcal{F} \subset \mathcal{T}$, $\cup \mathcal{F} \in \mathcal{T}$ is obvious. ■

Often it is more convenient to specify a topology by describing a basis for the topology. The criterion of Proposition 1.7 is useful in this regard.

Definition 1.8 Let (X, \mathcal{T}) be a topological space. A subset N of X is a neighbourhood of $x \in X$ iff there is $U \in \mathcal{T}$ such that $x \in U \subset N$. Let $\mathcal{N}(x)$ denote the collection of all neighbourhoods of x . A subfamily \mathcal{M} of $\mathcal{N}(x)$ is a basis of neighbourhoods at x provided for each $N \in \mathcal{N}(x)$ there is $M \in \mathcal{M}$ such that $M \subset N$.

Note that there is no requirement that neighbourhoods be open. Occasionally authors do insist that their neighbourhoods are open.

Proposition 1.9 Let (X, \mathcal{T}) and $\mathcal{N}(x)$ be as above. Then

1. $N \in \mathcal{N}(x) \Rightarrow x \in N$;
2. $N_1 \in \mathcal{N}(x)$ and $N_1 \subset N_2 \subset X \Rightarrow N_2 \in \mathcal{N}(x)$;
3. $N_1, N_2 \in \mathcal{N}(x) \Rightarrow N_1 \cap N_2 \in \mathcal{N}(x)$;
4. For each $N \in \mathcal{N}(x)$ there is $M \in \mathcal{N}(x)$ such that $y \in M \Rightarrow N \in \mathcal{N}(y)$.

Proof. Easy. ■

Lemma 1.10 Let (X, \mathcal{T}) and $\mathcal{N}(x)$ be as above and let $U \subset X$. Then $U \in \mathcal{T}$ iff whenever $x \in U$, it follows that $U \in \mathcal{N}(x)$.

Proof. \Rightarrow : obvious.

\Leftarrow : For each $x \in U$ there is $U_x \in \mathcal{T}$ such that $x \in U_x \subset U$. By Definition 1.1, $\cup_{x \in U} U_x \in \mathcal{T}$. But $U = \cup_{x \in U} U_x$, so $U \in \mathcal{T}$. ■

Proposition 1.11 Let \mathcal{T}_1 and \mathcal{T}_2 be two topologies on a set X with $\mathcal{N}_i(x)$ the corresponding families of neighbourhoods. Then $\mathcal{T}_1 = \mathcal{T}_2$ iff for each $x \in X$, $\mathcal{N}_1(x) = \mathcal{N}_2(x)$.

Proof. \Rightarrow : trivial.

\Leftarrow : Let $U \subset X$. By Lemma 1.10 and hypothesis, $U \in \mathcal{T}_1$ iff for each $x \in U$, $U \in \mathcal{N}_1(x)$ iff for each $x \in U$, $U \in \mathcal{N}_2(x)$ iff $U \in \mathcal{T}_2$, so $\mathcal{T}_1 = \mathcal{T}_2$ as required. ■

Theorem 1.12 Let X be a set and suppose that to each $x \in X$, there is assigned a non-empty family $\mathcal{N}(x)$ of subsets of X satisfying 1 to 4 of Proposition 1.9. Then there is a unique topology on X having $\mathcal{N}(x)$ as its family of neighbourhoods for each $x \in X$.

Proof. Let $\mathcal{T} = \{U \subset X / x \in U \Rightarrow U \in \mathcal{N}(x)\}$. Then \mathcal{T} is clearly a topology for X . For each $x \in X$, let $\mathcal{M}(x)$ denote the family of neighbourhoods of x in (X, \mathcal{T}) . We show that $\mathcal{M}(x) = \mathcal{N}(x)$.

Suppose $M \in \mathcal{M}(x)$. Then there is $U \in \mathcal{T}$ such that $x \in U \subset M$. By definition of \mathcal{T} , $x \in U \Rightarrow U \in \mathcal{N}(x)$, so by 1 of Proposition 1.9, $M \in \mathcal{N}(x)$. Thus $\mathcal{M}(x) \subset \mathcal{N}(x)$.

Suppose $N \in \mathcal{N}(x)$. Let $U = \{y \in X / N \in \mathcal{N}(y)\}$. If $y \in U$, then by 4 of Proposition 1.9 there is $M_y \in \mathcal{N}(y)$ such that $z \in M_y \Rightarrow N \in \mathcal{N}(z)$. By definition of U , we have $M_y \subset U$, so, since $M_y \in \mathcal{N}(y)$, we also have $U \in \mathcal{N}(y)$, so by definition of \mathcal{T} , $U \in \mathcal{T}$. Now by Lemma 1.10, $U \in \mathcal{M}(x)$, so, since $U \subset N$, $N \in \mathcal{M}(x)$ by 2 of Proposition 1.9. Hence $\mathcal{N}(x) \subset \mathcal{M}(x)$, and so $\mathcal{M}(x) = \mathcal{N}(x)$ as required. ■

The significance of Theorem 1.12 is that we could have defined a topological space by axiomatising its neighbourhood systems using conditions 1 to 4 of Proposition 1.9. Often this approach is more convenient.

Definition 1.13 *A subset A of a topological space (X, \mathcal{T}) is closed iff $(X - A) \in \mathcal{T}$.*

Proposition 1.14 *Closed sets satisfy the following:*

1. X is closed;
2. \emptyset is closed;
3. the union of any two closed sets is closed;
4. the intersection of an arbitrary family of closed sets is closed.

Proof. Easy. ■

Compare conditions 1 to 4 above with those of Definition 1.1 for open sets.

Definition 1.15 *Let $A \subset X$. By 4 of Proposition 1.14, the intersection of the (non-empty!) family $\{C \subset X / A \subset C \text{ and } C \text{ is closed}\}$ is a closed set containing A : in fact the smallest closed set containing A . This set is called the closure of A , denoted \bar{A} or clA . Dually, using 4 of Definition 1.1, there is a largest open set contained in A , called the interior of A , denoted A or $intA$.*

Of course, we cannot expect the intersection of all open sets containing A to be open nor the union of all closed sets contained in A to be closed. For example, let $A = [0, 1)$, a subset of \mathbb{R} with the usual topology. Then A is neither open nor closed although it is both the intersection of all open sets containing A and the union of all closed sets contained in A .

Definition 1.16 *Let $A \subset X$. A point x of X is an accumulation point of A iff $x \in \overline{A - \{x\}}$. The derived set, A' , of A is the set of all accumulation points. The frontier, frA , of A is the set $\bar{A} \cap \overline{X - A}$.*

Proposition 1.17 *Let $A \subset X$. Then*

$$\bar{A} = \{x \in X / \text{for each } N \in \mathcal{N}(x), A \cap N \neq \emptyset\} = A \cup A' = A \cup frA.$$

Proof. If $x \notin \bar{A}$, then by Lemma 1.10, $X - \bar{A} \in \mathcal{N}(x)$. Since $A \cap (X - \bar{A}) \subset \bar{A} \cap (X - \bar{A}) = \emptyset$, we deduce that $\{x \in X / \text{for each } N \in \mathcal{N}(x), A \cap N \neq \emptyset\} \subset \bar{A}$.

Suppose $x \in X$ and $N \in \mathcal{N}(x)$ are such that $A \cap N = \emptyset$. Then there is open U such that $x \in U \subset N \subset X - A$. Thus A is contained in the closed set $X - U$, so $\bar{A} \subset X - U$ and hence $x \notin \bar{A}$. Thus $\bar{A} \subset \{x \in X / \text{for each } N \in \mathcal{N}(x), A \cap N \neq \emptyset\}$.

If $x \in \bar{A} - A$, then $\overline{A - \{x\}} = \bar{A}$, so $x \in \overline{A - \{x\}}$, i.e. $x \in A'$. Thus $\bar{A} \subset A \cup A'$. Clearly $A \subset \bar{A}$ and if $x \in A'$, then $x \in \overline{A - \{x\}} \subset \bar{A}$, so $A' \subset \bar{A}$.

By definition, $A \cup frA \subset \bar{A}$. If $x \in \bar{A} - A$, then $x \in X - A$, so $x \in \overline{X - A}$ and hence $x \in frA$: thus $\bar{A} \subset A \cup frA$. ■

Definition 1.18 Let $f : X \rightarrow Y$ be a function between topological spaces. Then f is continuous at $x \in X$ iff for each neighbourhood N of $f(x)$ in Y , $f^{-1}(N)$ is a neighbourhood of x in X .

Theorem 1.19 Let $f : X \rightarrow Y$ be a function between topological spaces. Then the following conditions are equivalent:

- (i) f is continuous;
- (ii) for each $x \in X$, f is continuous at x ;
- (iii) if \mathcal{B} is a basis for Y then for each $U \in \mathcal{B}$, $f^{-1}(U)$ is open in X ;
- (iv) if \mathcal{S} is a sub-basis for Y then for each $U \in \mathcal{S}$, $f^{-1}(U)$ is open in X ;
- (v) for each closed subset C of Y , $f^{-1}(C)$ is closed in X ;
- (vi) for each $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$;
- (vii) for each $B \subset Y$, $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$;
- (viii) for each $B \subset Y$, $f^{-1}(\overset{\circ}{B}) \subset \text{int}(f^{-1}(B))$.

Proof. (i) \Rightarrow (ii): let $x \in X$ and $N \in \mathcal{N}(f(x))$. Then by definition there is U , open in Y , such that $f(x) \in U \subset N$. By (i), $f^{-1}(U)$ is open in X and contains x , so $f^{-1}(U) \in \mathcal{N}(x)$. Since $f^{-1}(U) \subset f^{-1}(N)$, 2 of Proposition 1.9 tells us that $f^{-1}(N) \in \mathcal{N}(x)$.

(ii) \Rightarrow (iii): let \mathcal{B} be a basis for the topology of Y and let $U \in \mathcal{B}$. By Lemma 1.10, it suffices to show that $f^{-1}(U)$ is a neighbourhood of each of its points. But if $x \in f^{-1}(U)$ then $f(x) \in U$, so $U \in \mathcal{N}(f(x))$. Hence by (ii), $f^{-1}(U) \in \mathcal{N}(x)$ as required.

(iii) \Rightarrow (iv): trivial.

(iv) \Rightarrow (v): suppose C is closed in Y and let $x \in X - f^{-1}(C)$. Then $f(x) \in Y - C$ which is open, so there are $U_1, \dots, U_n \in \mathcal{S}$ such that $f(x) \in U_1 \cap \dots \cap U_n \subset Y - C$. By (iv), $f^{-1}(U_i)$ is open, so $f^{-1}(U_1) \cap \dots \cap f^{-1}(U_n)$ is open in X . But $f^{-1}(U_1) \cap \dots \cap f^{-1}(U_n) = f^{-1}(U_1 \cap \dots \cap U_n)$, and $x \in f^{-1}(U_1 \cap \dots \cap U_n) \subset X - f^{-1}(C)$, so $X - f^{-1}(C) \in \mathcal{N}(x)$. Hence by Lemma 1.10, $X - f^{-1}(C)$ is open and hence $f^{-1}(C)$ is closed in X .

(v) \Rightarrow (vi): given $A \subset X$, $\overline{f(A)}$ is closed in Y so by (v), $f^{-1}(\overline{f(A)})$ is closed in X . But $A \subset f^{-1}(f(A)) \subset f^{-1}(\overline{f(A)})$, so $\bar{A} \subset f^{-1}(\overline{f(A)})$ and hence $f(\bar{A}) \subset \overline{f(A)}$.

(vi) \Rightarrow (vii): given $B \subset Y$, letting $A = f^{-1}(B)$ in (vi), we have $f(\overline{f^{-1}(B)}) \subset \overline{f(f^{-1}(B))} \subset \bar{B}$, so $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$.

(vii) \Rightarrow (viii): given $B \subset Y$, apply (vii) to $Y - B$ to obtain $f^{-1}(\overset{\circ}{B}) = f^{-1}(Y - \overline{Y - B}) = X - f^{-1}(\overline{Y - B}) \subset X - \overline{f^{-1}(Y - B)} = X - \overline{X - f^{-1}(B)} = \text{int}(f^{-1}(B))$.

(viii) \Rightarrow (i): let U be open in Y : thus $\overset{\circ}{U} = U$, so by (viii), $f^{-1}(U) \subset \text{int}(f^{-1}(U))$. Clearly $\text{int}(f^{-1}(U)) \subset f^{-1}(U)$, so $f^{-1}(U) = \text{int}(f^{-1}(U))$ and hence is open. ■

Proposition 1.20 Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous. Then $gf : X \rightarrow Z$ is continuous.

Proof. Obvious from the definition. ■

Let A be a subset of the space (X, \mathcal{T}) . By declaring open all subsets of A of the form $U \cap A$ for $U \in \mathcal{T}$ we obtain a topology for A with which A is called a *subspace* of X . Let $i : A \rightarrow X$ denote the inclusion function.

Proposition 1.21 $i : A \rightarrow X$ is continuous. Hence if $f : X \rightarrow Y$ is continuous then so is $f|_A : A \rightarrow Y$.

Proof. If U is open in X then $i^{-1}(U) = U \cap A$ is open in A , so i is continuous. The second part follows from Proposition 1.20 since $f|_A = fi$. ■

Theorem 1.22 *Suppose $\{X_\alpha / \alpha \in A\}$ is a collection of subspaces of a topological space X whose union is X and $f : X \rightarrow Y$ is a function. Let $f_\alpha = f|_{X_\alpha}$ and suppose each f_α is continuous. Then f is continuous provided either:*

- (i) each X_α is open; or
- (ii) each X_α is closed and A is finite.

Proof. (i) Let U be open in Y . Then for each $\alpha \in A$, $f_\alpha^{-1}(U)$ is open in X_α and hence, since each X_α is open in X , $f_\alpha^{-1}(U)$ is open in X . But $f^{-1}(U) = \cup_{\alpha \in A} f_\alpha^{-1}(U)$, so $f^{-1}(U)$ is open and f is continuous.

(ii) Let C be closed in Y . Then for each $\alpha \in A$, $f_\alpha^{-1}(C)$ is closed in X_α and hence, since each X_α is closed in X , $f_\alpha^{-1}(C)$ is closed in X . But $f^{-1}(C) = \cup_{\alpha \in A} f_\alpha^{-1}(C)$, a finite union, so $f^{-1}(C)$ is closed and f is continuous. ■

Note that we cannot in general remove the finiteness condition from (ii). For example, let $X = \mathbb{R}$ and for each $\alpha \in \mathbb{R}$, let $X_\alpha = \{\alpha\}$. For any function $f : \mathbb{R} \rightarrow \mathbb{R}$, f_α is continuous, but f need not be continuous.

Definition 1.23 *Let $\{X_\alpha / \alpha \in A\}$ be a family of topological spaces and suppose that for each $\alpha, \beta \in A$ with $\alpha \neq \beta$ we have $X_\alpha \cap X_\beta = \emptyset$. By the topological sum, ΣX_α , of these spaces we mean the set $X = \cup_{\alpha \in A} X_\alpha$ topologised by declaring $U \subset X$ open iff for each $\alpha \in A$, $U \cap X_\alpha$ is open in X_α .*

Note that the inclusion functions $X_\alpha \rightarrow X$ are continuous: in fact X_α is a subspace of X .

Definition 1.24 *Let $\{X_\alpha / \alpha \in A\}$ be a family of topological spaces. By the topological product of this family we mean the set*

$$X = \prod X_\alpha = \{x : A \rightarrow \cup_{\alpha \in A} X_\alpha / \text{for each } \alpha \in A, x(\alpha) \in X_\alpha\},$$

furnished with the Tychonoff topology. The family

$$\{\prod U_\alpha / U_\alpha \text{ is open in } X_\alpha \text{ and } U_\alpha \neq X_\alpha \text{ for at most one index } \alpha \in A\}$$

forms a sub-basis for the Tychonoff topology, and is called the standard sub-basis.

For each $x \in X$ and $\alpha \in A$, denote $x(\alpha)$ by x_α . Define for each $\alpha \in A$, the α th projection $\pi_\alpha : X \rightarrow X_\alpha$ by $\pi_\alpha(x) = x_\alpha$.

$$\{\prod U_\alpha / U_\alpha \text{ is open in } X_\alpha \text{ and } U_\alpha \neq X_\alpha \text{ for at most finitely many indices } \alpha \in A\}$$

is called the standard basis for the Tychonoff topology.

If $A = \{1, \dots, n\}$ then $\prod X_\alpha$ is usually written $X_1 \times \dots \times X_n$.

Proposition 1.25 *The Tychonoff topology is the smallest topology on X for which the projections π_α are continuous.*

Proof. π_α is continuous when X is given the Tychonoff topology, for if U_α is open in X_α then $\pi_\alpha^{-1}(U_\alpha) = \prod_{\beta \in A} U_\beta$, where $U_\beta = X_\beta$ if $\beta \neq \alpha$. By definition, $\prod_{\beta \in A} U_\beta$ is open, so π_α is continuous.

Let \mathcal{T} be any topology on X with respect to which each projection is continuous. We must show that \mathcal{T} contains the Tychonoff topology. It suffices to show that \mathcal{T} contains the standard sub-basis. Given $\{U_\alpha\}$ with each U_α open in X_α and $U_\alpha = X_\alpha$ except possibly for the one index β . Then $\prod U_\alpha = \pi_\beta^{-1}(U_\beta)$. By continuity of π_β , the set $\pi_\beta^{-1}(U_\beta)$ is open in \mathcal{T} . Thus $\prod U_\alpha \in \mathcal{T}$. ■

Proposition 1.26 *A function $f : Y \rightarrow \prod X_\alpha$, where Y and each X_α are topological spaces, is continuous iff for each $\alpha \in A$, $\pi_\alpha f$ is continuous.*

Proof. We need only show that if $\pi_\alpha f$ is continuous for each α , then f is continuous.

Let $\prod U_\beta$ be a member of the standard sub-basis of X , with $U_\beta = X_\beta$ unless $\beta = \alpha$. Now $\pi_\alpha(\prod U_\beta) = U_\alpha$ which is open in X_α so by continuity of $\pi_\alpha f$, $(\pi_\alpha f)^{-1}(U_\alpha)$ is open. Of course, $(\pi_\alpha f)^{-1}(U_\alpha) = f^{-1}(\prod U_\beta)$. ■

Corollary 1.27 *Let $\{f_\alpha : X_\alpha \rightarrow Y_\alpha / \alpha \in A\}$ be a family of continuous functions. Define the product of this family, $f : X \rightarrow Y$, by $f(x)(\alpha) = f_\alpha(x_\alpha)$. Then f is continuous.*

Proof. Let $\rho_\alpha : Y \rightarrow Y_\alpha$ be the α th projection. Then we have $\rho_\alpha f = f_\alpha \pi_\alpha$. Since $f_\alpha \pi_\alpha$ is continuous, so is f by Proposition 1.26. ■

Proposition 1.28 *For each $\alpha \in A$, $\pi_\alpha : X \rightarrow X_\alpha$ is an open map, i.e., for each open $U \subset X$, $\pi_\alpha(U)$ is open in X_α .*

Proof. Given $\alpha \in A$ and U open in X , suppose that $x_\alpha \in \pi_\alpha(U)$. Then there is $x \in U$ such that $\pi_\alpha(x) = x_\alpha$. Thus there are $\alpha_1, \dots, \alpha_n \in A$ and open subsets U_{α_i} of X_{α_i} such that $x \in \prod U_\beta \subset U$, where $U_\beta = X_\beta$ if $\beta \notin \{\alpha_1, \dots, \alpha_n\}$. Then $x_\alpha \in U_\alpha = \pi_\alpha(\prod U_\beta) \subset \pi_\alpha(U)$, so $\pi_\alpha(U)$ is open. ■

We cannot in general replace “open” by “closed” in Proposition 1.28, i.e. π_α is not in general a closed map. For example, \mathbb{R}^2 is (homeomorphic to) the product of \mathbb{R} with itself but projection onto either factor of the closed set $\{(x, y) \in \mathbb{R}^2 / xy = 1\}$ is the non-closed set $(-\infty, 0) \cup (0, \infty)$.

Definition 1.29 *Let X be any topological space and \sim an equivalence relation on X . Let X/\sim denote the set of equivalence classes and $\pi : X \rightarrow X/\sim$ the natural projection. By the quotient space of X by \sim is meant the set X/\sim topologised by declaring $U \subset X/\sim$ to be open iff $\pi^{-1}(U)$ is open.*

Proposition 1.30 *Let X and Y be spaces, \sim an equivalence relation on X and $f : X/\sim \rightarrow Y$ a function. Then f is continuous iff $f\pi$ is continuous.*

Proof. Easy. ■

Definition 1.31 *Let X be any space. By the cone on X is meant the space cX defined as follows: let v be any point not in $X \times [0, 1]$. Give $\{v\}$ the only possible topology and $X \times [0, 1]$ the Tychonoff topology. Define the equivalence relation \sim on the topological sum $\{v\} + (X \times [0, 1])$ to be that generated by $v \sim (x, 1)$, for each $x \in X$. Then cX is the quotient $[\{v\} + (X \times [0, 1])]/\sim$. The equivalence class of v is the vertex of the cone. We can think of X as a subspace of cX by identifying $x \in X$ with the equivalence class of $(x, 0)$.*

Definition 1.32 Let $f : X \rightarrow Y$ be a map. By the mapping cylinder of f is meant the space Z_f defined as follows: consider the topological sum $(X \times [0, 1]) + Y$ (if $(X \times [0, 1]) \cap Y \neq \emptyset$, replace Y by a homeomorph which is disjoint from $X \times [0, 1]$). Define the equivalence relation \sim on $(X \times [0, 1]) + Y$ to be that generated by $(x, 1) \sim f(x)$. Then Z_f is the quotient space $[(X \times [0, 1]) + Y] / \sim$. As in the case of cX , we can think of X as a subspace of Z_f . Y is also able to be treated as a subspace of Z_f .

Exercises

- Let X be a set and \mathcal{T} a family of subsets of X . Prove that \mathcal{T} is a topology for X iff
 - the intersection of any finite sub-family of \mathcal{T} is a member of \mathcal{T} and
 - the union of any sub-family of \mathcal{T} is a member of \mathcal{T} .
- Construct all possible topologies on a set of 3 elements and divide them into homeomorphism classes.
- Let $\mathcal{T} = \{U \subset \mathbb{Z} / \text{for each } x \in U \text{ there is } a \in \mathbb{Z} - \{0\} \text{ with } x + a\mathbb{Z} \subset U\}$. Prove that \mathcal{T} is a topology on \mathbb{Z} such that each arithmetic progression in \mathbb{Z} is both open and closed. Thus any finite union of arithmetic progressions is closed.
 - Let $A = \cup A_p$, where A_p consists of all multiples of p , and p runs through the set of all primes ≥ 2 . By identifying $\mathbb{Z} - A$, show that A is not closed. This proof that there are infinitely many primes is due to Furstenberg, *Amer. Math. Monthly*, 1955.
- Let X be a set. Let \mathcal{T} be the family of subsets of X consisting of \emptyset together with all subsets whose complement in X is finite. Verify that \mathcal{T} is a topology on X . \mathcal{T} is called the *cofinite* topology. We can replace “finite” by “countable” to get the *cocountable* topology.
- Construct a basis for the topology in Exercise 3 which differs from the topology itself.
- Show that the countable family $\mathcal{B} = \{(a, b) / a < b \text{ and } a \text{ and } b \text{ are rational}\}$ is a basis for the usual topology on \mathbb{R} .
- Show that the countable family $\mathcal{B} = \{[a, b) / a < b \text{ and } a \text{ and } b \text{ are rational}\}$ is a basis for a topology on \mathbb{R} . Is this the right half-open interval topology of Example 1.4?
- The subsets $A_{\pm}, \dots, I_{\pm}, X$ and Y of \mathbb{R}^3 are defined by

$$\begin{array}{ll}
 A_+ = [3, 5] \times [-1, 1] \times [-3, 7], & A_- = [-1, 1] \times [3, 5] \times [-7, 3], \\
 B_+ = [-5, -3] \times [-1, 1] \times [-3, 7], & B_- = [-1, 1] \times [-5, -3] \times [-7, 3], \\
 C_+ = [-5, 5] \times [-1, 1] \times [5, 7], & C_- = [-1, 1] \times [-5, 5] \times [-7, -5], \\
 D_+ = [-5, 5] \times [-1, 1] \times [-3, -1], & D_- = [-1, 1] \times [-5, 5] \times [1, 3], \\
 E_+ = [-1, 1] \times [-1, 1] \times [5, 11], & E_- = [-1, 1] \times [-1, 1] \times [-11, -5], \\
 F_+ = [-1, 9] \times [-1, 1] \times [9, 11], & F_- = [-1, 9] \times [-1, 1] \times [-11, -9], \\
 G_+ = [7, 9] \times [-1, 1] \times [-4, 11], & G_- = [7, 9] \times [-1, 1] \times [-11, 4], \\
 H_+ = A_+ \cup \dots \cup G_+, & H_- = A_- \cup \dots \cup G_-, \\
 I_+ = T(H_+), & I_- = T^{-1}(H_-), \\
 X = H_+ \cup H_-, & Y = I_+ \cup I_-.
 \end{array}$$

In the definition above, $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is defined by $T(x, y, z) = (x, y, z + 4)$.

Prove that

- X is homeomorphic to Y ;

- (b) there is a homeomorphism $h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $h(X) = Y$.
9. Let X be a topological space. Define two functions $c, cl : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, where $\mathcal{P}(X)$, the power set of X , consists of all subsets of X , by $c(A) = X - A$ and $cl(A) = \bar{A}$.
Prove that there are at most 14 distinct subsets of X obtained from a given subset A by successive applications of the functions c and cl . Give an example of a subset of \mathbb{R} for which there are 14 such distinct sets.
10. Is $\mathring{A} = A - \text{fr}A$?
11. Say that a point x is *near* a subset A of a topological space X and write $x\nu A$ iff $x \in \bar{A}$. Let $f : X \rightarrow Y$ be a function where X and Y are two topological spaces. Let $x \in X$. Prove that f is continuous at x iff for each $A \subset X$, $x\nu A \Rightarrow f(x)\nu f(A)$.
12. Suppose \mathcal{F} is a locally finite family of closed subspaces of a topological space X whose union is X and $f : X \rightarrow Y$ is a function such that $f|_F$ is continuous for each $F \in \mathcal{F}$. Prove that f is continuous. (The family \mathcal{F} is *locally finite* provided that each point of X has a neighbourhood meeting only finitely many members of \mathcal{F} .)
13. Let X and Y be two finite sets. It is desired to find all triples (f, g, h) of functions from X to Y such that whatever topologies are imposed on X and Y , if f and g are continuous then so is h . One way to find these is to use a computer to search for such triples. However, the number of topologies on a set of n elements grows rather quickly with n so one must look for ways of shortening the procedure. Prove that (f, g, h) is such a triple iff for each $V \subset Y$, $h^{-1}(V)$ is one of the following: $\emptyset, X, f^{-1}(V), g^{-1}(V), f^{-1}(V) \cap g^{-1}(V)$, and $f^{-1}(V) \cup g^{-1}(V)$.
14. Cantor's ternary set, C , is defined to be the intersection of the sequence C_0, C_1, \dots , where $C_0 = [0, 1]$ and for $i > 0$, C_i is the union of 2^i closed intervals obtained from the 2^{i-1} closed intervals of C_{i-1} by removing from each of these intervals the open middle third of the interval. Thus
- $$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right], C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right], \text{ etc.}$$
- Prove that C is homeomorphic to $\mathbf{2}^{\mathbb{N}}$ where $\mathbf{2}$ is the discrete space on two points, and $\mathbf{2}^{\mathbb{N}}$ denotes the product of countably many copies of the space $\mathbf{2}$ (i.e., each of the factor spaces is the space $\mathbf{2}$).
15. Let $\{X_\alpha / \alpha \in A\}$ be a family of topological spaces. Show that the family $\{\Pi U_\alpha / U_\alpha \text{ is open in } X_\alpha\}$ forms a basis for a topology on ΠX_α . This topology is called the *box topology* and seems to be a more natural generalisation of the topology on \mathbb{R}^2 having as basis the open rectangles. Show that for A finite, the Tychonoff topology and the box topology are the same, but that they can differ for A infinite. In the latter case, the box topology has too many open sets. Construct an example showing that Proposition 1.26 is false if we use the box topology in place of the Tychonoff topology.
16. Show that the cone cX on X is really just a particular example of a mapping cylinder.
17. Let $X = [-1, 1]$ and $Y = \{(x, y) \in \mathbb{R}^2 / 0 \leq y \leq 1 \text{ and } |x| \leq 1 - y\}$, with topologies inherited as subspaces of \mathbb{R} and \mathbb{R}^2 respectively. (Thus Y consists of the union of all line segments joining $(0, 1)$ to the closed line segment from -1 to 1 on the x -axis.) Prove that cX is homeomorphic to Y . What happens if we replace X by the open interval and let Y join $(0, 1)$ to the open line segment?

18. Let $X = \{(x, y, z) \in \mathbb{R}^3 / x^2 + y^2 = 1 \text{ and } 0 < z < 1\}$, $Y = [-1, 1] \times (0, 1)$ and $Z = (-1, 1) \times (0, 1) \cup \{2\} \times (0, 1)$. Give X the subspace topology inherited from the usual topology on \mathbb{R}^3 , and Y the subspace topology inherited from the usual topology on \mathbb{R}^2 .

Define \sim on Y by declaring $(x, y) \sim (x', y')$ if and only if either $(x, y) = (x', y')$ or $\{x, x'\} = \{-1, 1\}$ and $y = y'$.

For each $a, b, r \in (0, 1)$ with $a < b$, let $R_{a,b,r} = ((-1, -r) \cup (r, 1) \cup \{2\}) \times (a, b)$, and let

$$\mathcal{B} = \{U \cap (-1, 1) \times (0, 1) / U \text{ is open in the usual topology on } \mathbb{R}^2\} \\ \cup \{R_{a,b,r} / a, b, r \in (0, 1) \text{ and } a < b\}.$$

- (a) Prove that \sim is an equivalence relation on Y .
 (b) Construct a natural homeomorphism $X \rightarrow Y / \sim$.
 (c) Prove that \mathcal{B} is a basis for a topology on Z .
 (d) Construct a natural homeomorphism $Y / \sim \rightarrow Z$ when Z has the topology considered in (c).
19. Let $X = X_1 \cup \{0\} \times \{0\} \times \mathbb{R}$, where $X_1 = \{(x, y) \in \mathbb{R}^2 / x > 0\}$,
 $Y = \{(x, y) / 0 < x < 1 \text{ and } -x < y < x\} \cup \{0\} \times \{0\} \times (-1, 1)$ and
 $Z = [0, 1) \times (-1, 1)$.

Give Z the subspace topology inherited from the usual topology on \mathbb{R}^2 .

For each $a \in \mathbb{R}$ and $r > 0$, let

$$N_{a,r} = \{(x, y) \in \mathbb{R}^2 / 0 < x < r \text{ and } (a-r)x < y < (a+r)x\} \cup \{0\} \times \{0\} \times (a-r, a+r).$$

Let

$$\mathcal{B} = \{U \cap X_1 / U \text{ is open in the usual topology on } \mathbb{R}^2\} \cup \{N_{a,r} / a \in \mathbb{R} \text{ and } r > 0\}.$$

- (a) Draw pictures of $N_{a,r}$ for two different values of (a, r) .
 (b) Prove that \mathcal{B} is a basis for a topology on X .
 (c) Construct a natural homeomorphism $Y \rightarrow Z$ when Y has the subspace topology of the topology considered in (b).

In effect, X_1 is opened up at $(0, 0)$ to make room to insert a copy of \mathbb{R} there. Y is made up of an open triangle with $(0, 0)$ as one vertex. The triangle is opened up at this vertex to make room to insert the copy of $(-1, 1)$ giving the rectangle Z .

20. We can extend the ideas of Exercise 19 in two ways: replace X_1 by all of \mathbb{R}^2 except the y -axis; open up the now doubled X_1 at each point of the y -axis and insert a copy of \mathbb{R} at each of these points. The construction is very similar to that in Exercise 19. Let

$$N_{a,r} = \{(x, y) \in \mathbb{R}^2 / 0 < |x| < r \text{ and } (a-r)|x| < y < (a+r)|x|\} \cup \{0\} \times \{0\} \times (a-r, a+r).$$

With $X_1 = \{(x, y) \in \mathbb{R}^2 / y \neq 0\}$ and $X = X_1 \cup \{0\} \times \mathbb{R} \times \mathbb{R}$, let $T_\eta : X \rightarrow X$ be defined for each $\eta \in \mathbb{R}$ by $T_\eta(x, y) = (x, y + \eta)$ and $T_\eta(0, y, z) = (0, y + \eta, z)$. Let

$$\mathcal{B} = \{U \cap X_1 / U \text{ is open in the usual topology on } \mathbb{R}^2\} \cup \{T_\eta(N_{a,r}) / a, \eta \in \mathbb{R} \text{ and } r > 0\}.$$

- (a) Draw pictures of $T_\eta(N_{a,r})$ for two different values of (η, a, r) .
 (b) Prove that \mathcal{B} is a basis for a topology on X .
 (c) Prove that if X is topologised using the basis \mathcal{B} then each point of X has an open neighbourhood homeomorphic to \mathbb{R}^2 .