A CENSUS OF SEMISYMMETRIC CUBIC GRAPHS ON UP TO 768 VERTICES

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Abstract

A list is given of all semisymmetric (edge- but not vertex-transitive) connected finite cubic graphs of order up to 768. This list was determined by the authors using Goldschmidt's classification of finite primitive amalgams of index $(3, 3)$, and a computer algorithm for finding all normal subgroups of up to a given index in a finitely-presented group. The list includes several previously undiscovered graphs. For each graph in the list, a significant amount of information is provided, including its girth and diameter, the order of its automorphism group, the order and structure of a minimal edge-transitive group of automorphisms, its Goldschmidt type, stabiliser partitions, and other details about its quotients and covers. A summary of all known infinite families of semisymmetric cubic graphs is also given, together with explicit rules for their construction, and members of the list are identified with these. The special case of those graphs having $K_{1,3}$ as a normal quotient is investigated in detail.

1   Introduction

A bipartite regular graph is called *semisymmetric* if its automorphism group has a single orbit on edges but two orbits on vertices (the two parts of the bipartition).

More generally, given a graph X, let $V(X)$, $E(X)$, $A(X)$ and Aut X be the vertex set, the edge set, the arc set and the automorphism group of X, respectively. If a subgroup G of Aut X acts transitively on $V(X)$, $E(X)$ and $A(X)$, then X is said to be is *G-vertex-transitive*, *G-edge-transitive* and *G-arc-transitive* (or *G-symmetric*), respectively. It is easily seen that a graph X which is G-edge- but not G-vertex-transitive is necessarily bipartite, with the two parts of the bipartition coinciding with

the orbits of G. In particular, if X is a regular graph, then these two parts have equal cardinalities, and such a graph is then referred to as being G-*semisymmetric*. In the case where $G = \operatorname{Aut} X$ the symbol G may be omitted from the definitions above, so that X is called symmetric if it is regular and $(\operatorname{Aut} X)$-arc-transitive, and semisymmetric if it is regular and $(\operatorname{Aut} X)$-edge-transitive but not $(\operatorname{Aut} X)$-vertex-transitive.

The study of semisymmetric graphs was initiated by Folkman [15], who gave a construction of several infinite families of such graphs and posed a number of open problems which spurred the interest in this topic. Several papers followed; see [2, 3, 13, 14, 24, 25, 26, 27, 33, 35].

This article is devoted to the case of cubic (trivalent) semisymmetric graphs. Motivated by the Foster census of arc-transitive cubic graphs on up to 512 vertices [4], the complete determination of all such graphs on up to 768 vertices [7], and the extended Foster census of these up to 998 vertices [6] (incomplete in the range 770-998), we give a complete list of all connected finite semisymmetric cubic graphs of order up to 768. This list contains several previously undiscovered graphs; the third smallest graph on the list, on 112 vertices, is the subject of a separate paper [8].

The construction of this list of semisymmetric cubic graphs was achieved using Goldschmidt's classification [20] of finite primitive amalgams of index $(3,3)$. Roughly speaking, an amalgam of index $(3,3)$ is a pair of groups P_1 and P_2 having a common subgroup of index 3 in both, such as the stabilisers in $\operatorname{Aut} X$ of the vertices u and v of some edge $\{u,v\}$ in a cubic graph X. Goldschmidt's classification implies that the automorphism group of every edge-transitive cubic graph is obtainable as a homomorphic image of the universal completion of one of 15 finite amalgams, denoted by G_1, G_1^1, G_1^2, G_1^3, G_2, G_2^1, G_2^2, G_2^3, G_2^4, G_3, G_3^1, G_4, G_4^1, G_5 and G_5^1. Each universal completion is essentially an amalgamated free product of the two subgroups P_1 and P_2 (with their common subgroup of index 3 amalgamated), and hence has a finite presentation in terms of generators and defining relations. By abuse of notation we use the same symbols (G_1, G_1^1, and so on) to denote these 15 finitely-presented groups, and call each such symbol the *Goldschmidt type* of the corresponding amalgam.

Note that in all of these 15 finite primitive amalgams of index $(3,3)$, the orders of the two subgroups P_1 and P_2 are at most $384 = 3 \cdot 2^7$, and it follows that in the automorphism group of any finite connected edge-transitive cubic graph X, the stabiliser of any vertex has order $3 \cdot 2^{s-1}$ for some $s \le 8$. A now classical theorem of Tutte [37] states that $s \le 5$ in the special case where X is symmetric.

Adopting the same approach as taken in [7] to determine all connected finite symmetric cubic graphs, we used a computerised algorithm for finding all normal subgroups of up to a certain index in each of the 15 finitely-presented groups. In fact again we used the "mixed" version of this algorithm (as explained in [7]), involving coset enumeration over the subgroup P_1 in each case, so that the upper bound on the index was taken as $768/2 = 384$ in each case, in order to obtain all finite homomorphic images representable as an edge-transitive group of automorphisms of some connected finite cubic graph. A more detailed description of this algorithm and unix implementations of it are available from Peter Dobcsányi [12]. The MAGMA system was used to construct and analyse the resulting graphs, and to determine vertex-transitivity (and hence distinguish between symmetric and semisymmetric cases).

This article is organised as follows. Background material on amalgams (including a table showing the 15 types of primitive amalgam of index $(3,3)$) is given in Section 2, followed by some basic theory of quotients and covering projections in Section 3. A

brief summary of all known semisymmetric cubic graphs (together with a number of new constructions of such graphs) is given in Section 4. In Section 5, a more detailed analysis of one of these families is undertaken, namely of those semisymmetric cubic graphs having $K_{1,3}$ as a normal quotient. The list itself is presented in Section 7, including detailed information on several graph- and group-theoretic parameters for each of the graphs, and is preceded by some explanation in Section 6. Finally in Section 8 we show how some of the graphs in the list can be obtained as covers of smaller symmetric cubic graphs using voltage constructions.

For general background on graphs and groups, as well as for the concepts and notation not defined here, we refer the reader to [10, 19, 21, 38].

2   Amalgams of index (3,3)

Following [20], an *amalgam* is a pair (ϕ_1, ϕ_2) of group monomorphisms $\phi_1 : B \to P_1$ and $\phi_2 : B \to P_2$ with the same domain B. The amalgam is said to be *finite* if the codomains P_1 and P_2 are finite. The pair $(|P_1 : \mathrm{im}\phi_1|, |P_2 : \mathrm{im}\phi_2|)$ of indices is called the *index* of the amalgam. The amalgam is said to be *primitive* if for each $i \in \{1, 2\}$ and for each non-trivial subgroup K of B such that $\phi_{3-i}(K)$ is normal in P_{3-i}, the normaliser of $\phi_i(K)$ in P_i is $\phi_i(B)$. Two amalgams $(\phi_1 : B \to P_1, \phi_2 : B \to P_2)$ and $(\phi_1' : B' \to P_1', \phi_2' : B' \to P_2')$ are isomorphic if there exist group isomorphisms $\alpha : B \to B'$, $\alpha_1 : P_1 \to P_1'$ and $\alpha_2 : P_2 \to P_2'$, making the obvious diagrams commutative.

A *completion* of the amalgam (ϕ_1, ϕ_2) is a pair of group homomorphisms $(\psi_1 : P_1 \to G, \psi_2 : P_2 \to G)$ into the same group G, such that $\psi_1 \circ \phi_1 = \psi_2 \circ \phi_2$, and such that the images of ψ_1 and ψ_2 generate G. If ψ_1 and ψ_2 are monomorphisms, then the completion is said to be *faithful*. The completion is called *finite* if the group G is finite. For every amalgam $(\phi_1 : B \to P_1, \phi_2 : B \to P_2)$, there is a so-called *universal completion* $(\psi_1^\circ : P_1 \to G^\circ, \psi_2^\circ : P_2 \to G^\circ)$ with the following defining property: for every completion $(\psi_1 : P_1 \to G, \psi_2 : P_2 \to G)$ of (ϕ_1, ϕ_2) there exists a group homomorphism $\alpha : G^\circ \to G$, such that $\psi_i = \alpha \circ \psi_i^\circ$ for $i = 1, 2$. The group G° is usually called the *amalgamated product* of P_1 and P_2 (with amalgamated subgroup B), and denoted by $P_1 *_B P_2$. In particular, every faithful finite completion can be obtained from the universal completion by taking as the kernel of α some normal subgroup of finite index in G° that intersects $\psi_i^\circ(P_i)$ trivially for $i = 1, 2$.

Now let $(\psi_1 : P_1 \to G, \psi_2 : P_2 \to G)$ be a faithful, finite completion of a finite amalgam $(\phi_1 : B \to P_1, \phi_2 : B \to P_2)$. By abuse of notation, we may identify the groups B, P_1 and P_2 with their images in G. We then define the *graph of the completion* as the bipartite graph $X = X(G, P_1, P_2)$ with vertex set V the union of the two (right) coset spaces $[G : P_1]$ and $[G : P_2]$, and with adjacency given by $P_1 g \sim P_2 h$ if and only if $P_1 g \cap P_2 h$ is non-empty. It is easy to see that the group G acts (via right multiplication) as a group of automorphisms of X, and that this action is transitive on edges but not on vertices; moreover, this action is faithful if and only if $\mathrm{core}_G(B) = \underset{g \in G}{\cap} g^{-1} B g = 1$, which is equivalent to the primitivity of the amalgam (ϕ_1, ϕ_2).

Note that the valence of any vertex $P_i g$ is $|P_i : B|$, for $i = 1, 2$. This implies that the graph of a faithful finite completion of a finite amalgam of index $(3, 3)$ is a cubic G-semisymmetric graph, with stabilisers of any two adjacent vertices isomorphic to P_1 and P_2 in some order. The converse of this fact together with the classification of all

finite primitive amalgams of index $(3,3)$ achieved in [20, Theorem A] is crucial for our analysis of semisymmetric cubic graphs:

Proposition 2.1 *Let X be a finite cubic G-semisymmetric graph other than $K_{3,3}$. Let $\{u, v\}$ be any edge of X, and let G_u and G_v be the stabilisers in G of u and v respectively. Then the pair $(\psi_1 \colon G_u \to G, \psi_2 \colon G_v \to G)$ of inclusion maps is a faithful finite completion of the finite primitive amalgam (ϕ_1, ϕ_2), of index $(3,3)$, given by the inclusion maps of the intersection $G_u \cap G_v$ into G_u and G_v respectively, and X is isomorphic to the graph of this completion.*

Theorem 2.2 [20, Theorem A] *There are precisely fifteen isomorphism classes of finite primitive amalgams of index $(3,3)$. The corresponding fifteen universal completions are given in Table 1.*

Table 1: Universal completions of finite primitive amalgams of index $(3,3)$

Type	Universal completion
G_1	$\langle\, x, y \mid x^3, y^3 \,\rangle$
	$P_1 = \langle x \rangle \cong \mathbb{Z}_3, \quad P_2 = \langle y \rangle \cong \mathbb{Z}_3, \quad B = \{1\}$
G_1^1	$\langle\, c, x, y \mid c^2, x^3, y^3, (cx)^2, (cy)^2 \,\rangle$
	$P_1 = \langle c, x \rangle \cong S_3, \quad P_2 = \langle c, y \rangle \cong S_3, \quad B = \langle c \rangle \cong \mathbb{Z}_2$
G_1^2	$\langle\, c, x, y \mid c^2, x^3, y^3, (cx)^2, [c, y] \,\rangle$
	$P_1 = \langle c, x \rangle \cong S_3, \quad P_2 = \langle c, y \rangle \cong \mathbb{Z}_6, \quad B = \langle c \rangle \cong \mathbb{Z}_2$
G_1^3	$\langle\, c, d, x, y \mid c^2, d^2, [c, d], x^3, y^3, (cx)^2, [d, x], [c, y], (dy)^2 \,\rangle$
	$P_1 = \langle c, d, x \rangle \cong D_{12}, \quad P_2 = \langle c, d, y \rangle \cong D_{12}, \quad B = \langle c, d \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
G_2	$\langle\, c, d, x, y \mid c^2, d^2, [c, d], x^3, y^3, (cx)^2, [d, x], (cy)^3, dy^{-1}cy \,\rangle$
	$P_1 = \langle c, d, x \rangle \cong D_{12}, \quad P_2 = \langle c, d, y \rangle \cong A_4, \quad B = \langle c, d \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
G_2^1	$\langle\, c, d, x, y \mid c^2, d^4, (cd)^2, x^3, (cx)^2, [d, x], y^3, (dy^{-1})^2, cydy \,\rangle$
	$P_1 = \langle c, d, x \rangle \cong D_{24}, \quad P_2 = \langle c, d, y \rangle \cong S_4, \quad B = \langle c, d \rangle \cong D_8$
G_2^2	$\langle\, c, d, x, y \mid c^2, d^4, (cd)^2, x^3, [c, x], [d^2, x], d^{-1}xdx, y^3, (dy^{-1})^2, cydy \,\rangle$
	$P_1 = \langle c, d, x \rangle \cong \mathbb{Z}_3 \rtimes D_8, \quad P_2 = \langle c, d, y \rangle \cong S_4, \quad B = \langle c, d \rangle \cong D_8$
G_2^3	$\langle\, c, d, e, x, y \mid c^2, d^2, e^2, [c, d], [c, e], [d, e], x^3, (cx)^2, [d, x], [e, x], y^3, [c, y], y^{-1}cdye, (ye)^3 \,\rangle$
	$P_1 = \langle c, d, e, x \rangle \cong D_{12} \times \mathbb{Z}_2, \ P_2 = \langle c, d, e, y \rangle \cong A_4 \times \mathbb{Z}_2, \ B = \langle c, d, e \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
G_2^4	$\langle\, c, d, e, x, y \mid c^2, d^4, e^2, (cd)^2, [c, e], [d, e], x^3, [ce, x], [d, x], (ex)^2, y^3, (dy^{-1})^2, cydy, [y, e] \,\rangle$
	$P_1 = \langle c, d, e, x \rangle \cong D_8 \times S_3, \quad P_2 = \langle c, d, e, y \rangle \cong S_4 \times \mathbb{Z}_2, \quad B = \langle c, d, e \rangle \cong D_8 \times \mathbb{Z}_2$
G_3	$\langle\, c, d, x, y \mid c^2, d^4, (cd)^2, x^3, (dx^{-1})^2, cxdx, y^3, (dy^{-1})^2, cdyd \,\rangle$
	$P_1 = \langle c, d, x \rangle \cong S_4, \quad P_2 = \langle c, d, y \rangle \cong S_4, \quad B = \langle c, d \rangle \cong D_8$
G_3^1	$\langle\, c, d, e, x, y \mid c^2, d^4, e^2, (cd)^2, [c, e], [d, e], x^3, (dx^{-1})^2, cxdx, [x, e], y^3,$ $(dy^{-1})^2, cd^{-1}ydy, [y, ed^2] \,\rangle$
	$P_1 = \langle c, d, e, x \rangle \cong S_4 \times \mathbb{Z}_2, \quad P_2 = \langle c, d, e, y \rangle \cong S_4 \times \mathbb{Z}_2, \quad B = \langle c, d, e \rangle \cong D_8 \times \mathbb{Z}_2$
G_4	$\langle\, a, b, s, x, y \mid a^4, b^4, s^2, [a, b], sasb^{-1}, sbsa^{-1}, x^3, x^{-1}axb^{-1}, x^{-1}bxba,$ $(xs)^2, y^3, y^{-1}absysa^2, y^{-1}sa^2yba^{-1}, a^{-1}sysay \,\rangle$
	$P_1 = \langle a, b, s, x \rangle$ of order 96, $\quad P_2 = \langle a, b, s, y \rangle$ of order 96, $\quad B = \langle a, b, s \rangle$ of order 32
G_4^1	$\langle\, a, b, s, t, x, y \mid a^4, b^4, s^2, t^2, [a, b], sasb^{-1}, sbsa^{-1}, (ta)^2, (tb)^2, [s, t], x^3, x^{-1}axb^{-1},$ $x^{-1}bxba, (xs)^2, [x, t], y^3, y^{-1}absysa^2, y^{-1}sa^2yba^{-1}, a^{-1}sysay, tytsa^2y^{-1}a^2s \,\rangle$
	$P_1 = \langle a, b, s, t, x \rangle$ of order 192, $\ P_2 = \langle a, b, s, t, y \rangle$ of order 192, $\ B = \langle a, b, s, t \rangle$ of order 64
G_5	$\langle\, a, b, s, t, x, y \mid a^4, b^4, s^2, t^2, [a, b], sasb^{-1}, (ta)^2, (tb)^2, [s, t], x^3, x^{-1}axb^{-1}, x^{-1}bxba, (xs)^2,$ $[x, t], y^3, y^{-1}absyba^{-1}, y^{-1}sa^2ysb^{-1}a^{-1}, y^{-1}tsa^2yb^{-1}a^{-1}, y^{-1}abyb^{-1}ast, tb^{-1}ybty \,\rangle$
	$P_1 = \langle a, b, s, t, x \rangle$ of order 192, $\ P_2 = \langle a, b, s, t, y \rangle$ of order 192, $\ B = \langle a, b, s, t \rangle$ of order 64
G_5^1	$\langle\, a, b, s, t, v, x, y \mid a^4, b^4, s^2, t^2, v^2, [a, b], sasb^{-1}, (ta)^2, (tb)^2, [s, t], vavb^{-2}a^{-1}, vbva^{-2}b,$ $(vs)^2t, [v, t], x^3, x^{-1}axb^{-1}, x^{-1}bxba, (xs)^2, [x, t], [x, v], y^3, y^{-1}absyba^{-1}, y^{-1}sa^2ysb^{-1}a^{-1},$ $y^{-1}tsa^2yb^{-1}a^{-1}, y^{-1}abyb^{-1}ast, tb^{-1}ybty, b^{-1}tvbtvtb^3a^2tb^{-1}, [vtb, y] \,\rangle$
	$P_1 = \langle a, b, s, t, v, x \rangle$ of order 384, $\quad P_2 = \langle a, b, s, t, v, y \rangle$ of order 384, $\ B = \langle a, b, s, t, v \rangle$ of order 128

3 Quotients and coverings

In order to facilitate easier reading, we give a short account on quotient projections, and regular covering projections in particular. (For a more extensive treatment of coverings, see [23, 31, 32] for example.)

Quotients and coverings We say that two graph morphisms $f\colon X \to \bar{X}$ and $f'\colon X' \to \bar{X}'$ are *isomorphic*, written $f \cong f'$, if there exist graph isomorphisms $\alpha\colon X \to X'$ and $\bar{\alpha}\colon \bar{X} \to \bar{X}'$ such that $\bar{\alpha} \circ f = f' \circ \alpha$. (We tacitly assume that graph morphisms are composed on the <u>left</u>, with composition written as $\circ$. However, graph automorphisms are occasionally composed on the <u>right</u>; this is indicated by juxtaposition, that is, by using no special sign for the composition.) If $X = X'$ and $\alpha = \mathrm{id}$ we call $f \cong f'$ an *equivalence below X* (or just *equivalence from below*). If $\bar{X} = \bar{X}'$ and $\bar{\alpha} = \mathrm{id}$ we call $f \cong f'$ an *equivalence above $\bar{X}$* (or just *equivalence from above*).

Let X be a graph and $N \leq \mathrm{Aut}\, X$. The *quotient graph X_N* has the vertex orbits $\{[v] : v \in V(X)\}$ of N as the vertex set, the arc orbits $\{[a] : a \in A(X)\}$ as the arc set (and hence the edge orbits $\{[e] : e \in E(X)\}$ as the edge set), and the adjacencies in X_N are defined in a natural way so that the mappings $v \mapsto [v]$ and $a \mapsto [a]$ on vertices and arcs (and hence $e \mapsto [e]$ on edges) define a graph epimorphism $q_N\colon X \to X_N$, the *quotient projection by the action of N*. (Note that X_N may have multiple edges, loops or even semiedges — a semiedge is created whenever two inverse arcs of X are in the same orbit of N.) By abuse of terminology, any graph epimorphism $q\colon X \to \bar{X}$ which is equivalent below X to a quotient projection $q_N\colon X \to X_N$ is also called a quotient projection and often identified with q_N – although a quotient projection might arise from several distinct groups; this fact is a source of many anomalies to be discussed further below.

The *kernel* of a quotient projection $q_N\colon X \to X_N$ is the largest subgroup $\mathrm{Ker}\, q_N \leq \mathrm{Aut}\, X$ having the same orbits as the group N. This group is uniquely defined and $\mathrm{Ker}\, q_N = \{\alpha \in \mathrm{Aut}\, X \mid q_N \circ \alpha = q_N\}$. Thus, $q_N = q_{N'}$ if and only if $\mathrm{Ker}\, q_N = \mathrm{Ker}\, q_{N'}$ (for instance, we do have $q_N = q_{N'}$ whenever $N \leq N' \leq \mathrm{Ker}q_N$, but the converse might fail to be true). Also, two quotient projections q_N and $q_{N'}$ defined on a graph X are isomorphic if and only if $\mathrm{Ker}\, q_N$ and $\mathrm{Ker}\, q_{N'}$ are conjugate subgroups of $\mathrm{Aut}\, X$; in particular q_N and $q_{N'}$ are equivalent below X if and only if $q_N = q_{N'}$. Finally, q_N is an isomorphism if and only if N is the trivial group (because $\mathrm{Aut}\, X$ acts faithfully on the set of arcs).

Let X be a graph. We say that a group $N \leq \mathrm{Aut}\, X$ acts *semiregularly on X* if it acts semiregularly on the vertex set $V(X)$ and on the arc set $A(X)$. The respective quotient projection $X \to X_N$ is called a *regular N-covering projection* and is usually denoted by $\wp_N$. If N belongs to some specific class of groups such as cyclic, elementary abelian etc., then the respective regular covering covering projection is also called *cyclic, elementary abelian* etc. By abuse of terminology, any graph epimorphism $\wp\colon X \to \bar{X}$ which is equivalent below X to a regular N-covering projection $\wp_N\colon X \to X_N$ is itself called a regular covering projection (and often identified with $\wp_N$). Note that the group $\mathrm{CT}(\wp) = N = \mathrm{Ker}\,\wp$, also known as the *group of covering transformations*, is now uniquely determined by $\wp$. This is one of the many nice properties that regular covering projections share and which makes them rather distinguished among quotient projections in general. Another such (characteristic) property is the following: for each arc starting at $[v] \in V(X_N)$ and each vertex $u \in [v]$, there exists a unique arc in

X starting at u which projects to the selected arc in X_N — this property extends to walks and it is known as the *unique walk lifting property*; in other words: a quotient projection is a regular covering projection if and only if it is valency preserving.

A regular covering projection $\wp\colon X \to \bar{X}$ can be reconstructed from $\bar{X}$, up to equivalence above $\bar{X}$, as follows. Consider $\mathrm{CT}(\wp) = N$ as an abstract group, now called the *voltage group*. In each *vertex fibre* $\wp^{-1}(\bar{v})$, $\bar{v} \in V(\bar{X})$, choose a vertex and label it by $1 \in N$. Then transfer the labelling by the action of N (considered as a group of automorphisms of X acting on the left) to all the vertices of X. Let $\bar{a} \in A(\bar{X})$. Then there exists an element $\zeta(\bar{a}) \in N$, called the *voltage of $\bar{a}$*, such that all arcs in the *arc fibre* $\wp^{-1}(\bar{a})$ over $\bar{a} \in A(\bar{X})$ have origins and termini labelled by g and $g\,\zeta(\bar{a})$, where $g \in N$. Obviously, inverse arcs receive inverse voltages. The graph $\mathrm{Cov}(X;\zeta)$ with vertex set $V(\bar{X}) \times N$ and edge set $E(\bar{X}) \times N = \{\{(\bar{u},g),(\bar{v},g\,\zeta(\bar{a})\} \mid \bar{a} \in A(\bar{X})$ with origin $\bar{u}$ and terminus $\bar{v}$, $g \in N\}$ is isomorphic to X. Obviously, N acts by left multiplication on the second coordinate as a group of automorphisms of $\mathrm{Cov}(\bar{X};\zeta)$, and gives rise to a regular covering projection $\wp_\zeta = \wp_N\colon \mathrm{Cov}(\bar{X};\zeta) \to \bar{X}$ which is equivalent above $\bar{X}$ to $\wp\colon X \to \bar{X}$. As an abstract voltage group, N acts by right multiplication on itself as the set of labels. It is well known that the voltages can be assigned in such a way that the arcs of an arbitrary chosen spanning tree of $\bar{X}$ receive the trivial voltage.

Conversely, given a graph $\bar{X}$, an abstract group N, and a voltage assignment $\zeta\colon A(\bar{X}) \to N$ such that the inverse arcs receive inverse voltages, the construction of $\mathrm{Cov}(\bar{X};\zeta)$ as above gives a regular covering projection $\wp_\zeta = \wp_N\colon \mathrm{Cov}(\bar{X};\zeta) \to \bar{X}$. The only inconvenience with ζ being arbitrary is that the resulting *derived* graph might be disconnected. The additional "if and only if" condition which should be imposed on ζ such that $\mathrm{Cov}(\bar{X};\zeta)$ is connected, is well known: each element of N should appear as the voltage of some closed walk. If we assume that the arcs of an arbitrarily chosen spanning tree carry the trivial voltage then the derived covering graph is connected if and only if the voltages of the "off-tree arcs" generate the group N.

Projecting and lifting automorphisms Let $H \le \mathrm{Aut}\, X$. We say that H *projects along* $q_N\colon X \to X_N$ if, for each $h \in H$, there exists an automorphism $\bar{h} \in \mathrm{Aut}\, X_N$ such that $\bar{h} \circ q_N = q_N \circ h$, called the *projection* of h *along* q_N. Note that if $\bar{h}$ exists then it is unique. So if H projects then we have a naturally defined group $\bar{H} \le \mathrm{Aut}\, X_N$, the *projection* of H *along* q_N, and we have an *induced group epimorphism* $q_N^*\colon H \to \bar{H}$, taking $h \mapsto \bar{h}$. We say that $G \in \mathrm{Aut}\, X_N$ *lifts along* q_N if there exists a group $H \le \mathrm{Aut}\, X$ which projects to G, that is, $\bar{H} = G$. The largest group that projects onto G is called the *lift* of G *along* q_N and is denoted by $\tilde{G}$ (such a group is unique). In particular, the kernel $\mathrm{Ker}\, q_N$ is the lift of the trivial group.

If a group $H \le \mathrm{Aut}\, X$ normalizes N, then H projects along q_N, and the groups $H/(H \cap N)$ and $\bar{H}$ have the same permutation representation on X_N; but the action of $H/(H \cap N)$ might not be faithful. On the other hand, H may project even if H does not normalize N. In particular, N might not be a normal subgroup of $\mathrm{Ker}\, q_N$. These anomalies are not present with regular covering projections, since $\mathrm{Ker}\, \wp_N = N$ in their case.

Proposition 3.1 *Let X be a graph and $N \le \mathrm{Aut}\, X$. Then a group $H \le \mathrm{Aut}\, X$ projects along $q_N\colon X \to X_N$ if and only if H normalizes $\mathrm{Ker}\, q_N$. In particular, the largest group that projects along q_N is the normalizer of $\mathrm{Ker}\, q_N$ within $\mathrm{Aut}\, X$.*

Moreover, if a group $H \leq \operatorname{Aut} X$ projects, then the kernel of the induced group epimorphism $q_N^* \colon H \to \bar{H}$ is $H \cap \operatorname{Ker} q_N$. In particular, if $G \leq \operatorname{Aut} X_N$ lifts, then the kernel of the group epimorphism $\tilde{G} \to G$ is $\operatorname{Ker} q_N$. ∎

A quotient projection $q_N \colon X \to X_N$ is called *normal* if $\operatorname{Aut} X$ projects along q_N, that is, if $\operatorname{Ker} q_N \triangleleft \operatorname{Aut} X$. In general, if H projects along q_N then a conjugate subgroup $\alpha \circ H \circ \alpha^{-1}$ projects along $q_{(\alpha \circ N \circ \alpha^{-1})}$ (which might not be equal to q_N). From Proposition 3.1 it follows that, given a graph X and $H \leq \operatorname{Aut} X$, one can find all quotient projections $q_K \colon X \to X_K$ along which H projects — by letting K run through all normal subgroups of all subgroups $H' \leq \operatorname{Aut} X$ containing H. (The resulting projections are not necessarily pairwise distinct, nonequivalent or nonisomorphic; we do find all morphisms, but not necessarily all groups realising these morphisms.) Among these, all regular covering projections are obtained by selecting those subgroups K which act semiregularly on X.

If an automorphism lifts along a quotient projection then it lifts along any quotient projection equivalent to it from above. Consequently, if a regular covering projection $\wp_N \colon X \to \bar{X}$ is reconstructed as $\wp_\zeta \colon \operatorname{Cov}(\bar{X}; \zeta) \to \bar{X}$, then a necessary and sufficient condition for $G \leq \operatorname{Aut} \bar{X}$ to lift along $\wp_N$ can be expressed combinatorially in terms of voltages (in order for G to lift along $\wp_\zeta$). Also, it is enough to require that each automorphism in some generating set of G lifts; see Proposition 3.2 below. Equivalence and isomorphism of regular covering projections can also be studied along these lines; see Proposition 3.3 below.

Proposition 3.2 *An automorphism $\alpha \in \operatorname{Aut} \bar{X}$ lifts along a regular covering projection $\wp_\zeta \colon \operatorname{Cov}(\bar{X}; \zeta) \to \bar{X}$ if and only if each closed walk with trivial voltage is mapped to a walk with trivial voltage.* ∎

Proposition 3.3 *Regular covering projections $\wp_{N_i} \colon \operatorname{Cov}(\bar{X}; \zeta_i) \to \bar{X}$, for $i = 1, 2$, are isomorphic if and only if there exists a group isomorphism $\tau \colon N_1 \to N_2$ and a graph automorphism $\alpha \in \operatorname{Aut} \bar{X}$ such that for each closed walk W in $\bar{X}$ we have $\tau(\zeta_1(W)) = \zeta_2(\alpha(W))$. In particular, $\wp_{N_1}$ and $\wp_{N_2}$ are equivalent from above if and only if there exists an isomorphism $\tau \colon N_1 \to N_2$ such that $\tau \circ \zeta_1 = \zeta_2$.* ∎

Decomposition of quotient projections We say that a quotient projection $q_N \colon X \to X_N$ has a *decomposition* if it can be written as $q_N = q_M \circ q_K$ where $N, K \leq \operatorname{Aut} X$ and $M \leq \operatorname{Aut} X_K$. Such a decomposition is said to be *nontrivial* if neither of the projections q_K or q_M is an isomorphism.

Observe that the K-orbits are contained in the orbits of N if and only if $\operatorname{Ker} q_K \leq \operatorname{Ker} q_N$. Consequently, a necessary condition for a nontrivial decomposition of q_N to exist is that there be a subgroup $1 \neq K \leq \operatorname{Ker} q_N$ such that $\operatorname{Ker} q_K < \operatorname{Ker} q_N$. But the condition is not sufficient. Specifically, if the K-orbits are contained in the orbits of N, then one can define a graph morphism $f \colon X_K \to X_N$ such that $q_N = f \circ q_K$, however, this morphism does not necessarily arise from taking the quotient by the action of some group of automorphisms of X_K. Now suppose that $\operatorname{Ker} q_K < \operatorname{Ker} q_N$ is normalized by N, that is, let N project along q_K. Then the projection $\bar{N} \cong N/(N \cap \operatorname{Ker} q_K)$ of N along q_K projects along f; in fact, the graph epimorphism f is isomorphic to the projection $q_{\bar{N}}$. Such a decomposition is called *strong*. In particular, if $\operatorname{Ker} q_K$ is a nontrivial proper normal subgroup of $\operatorname{Ker} q_N$, then the decomposition is nontrivial and

strong. But if a nontrivial decomposition is strong, then $\operatorname{Ker} q_K$ is not necessarily a normal subgroup of $\operatorname{Ker} q_N$, that is, $\operatorname{Ker} q_N$ might not project along q_K.

Regular covering projections are much better behaved with respect to the questions addressed above. We point out, firstly, that any decomposition $\wp_N = \wp_M \circ \wp_K$ (where all projections are required to be regular covering projections) is necessarily strong. Secondly, since $N = \operatorname{Ker} \wp_N$ and $K = \operatorname{Ker} q_K$, we see that K must be normal in N, and consequently, such a decomposition takes the form $\wp_N = \wp_{N/K} \circ \wp_K$. Also, a decomposition is trivial if and only if $K = 1$ or $K = N$.

We close this section by introducing some additional terminology for later reference. If $\wp_N \colon X \to X_N$ is a regular covering projection and a group $G \leq X_N$ lifts along $\wp_N$ we say that $\wp_N$ is G-admissible. If $\wp_N$ is G-admissible but there exists no G-admissible regular covering projection $\wp_{N/K}$ such that $\wp_N = \wp_{N/K} \circ \wp_K$, then we say that $\wp_N$ is minimally G-admissible. Loosely speaking, $\wp_N$ is minimally G-admissible whenever G lifts along $\wp_N$ but does not lift along any "smaller" regular covering projection which decomposes $\wp_N$. We also remark that if ζ denotes the voltage assignment realising $\wp_N$, then the voltage assignment which realises, up to equivalence, the covering projection $\wp_{N/K}$ (such that $\wp_N = \wp_{N/K} \circ \wp_K$), is given by $q \circ \zeta$, where $q \colon N \to N/K$ is the quotient projection. Further details on decomposing quotient and regular covering projections will be given in Section 6.

4 Families of cubic semisymmetric graphs

In this section we give an overview of known families of cubic semisymmetric graphs. With the exception of families **I**, **II** and **VII**, they all arise as regular covers of certain cubic arc-transitive graphs of small orders.

Let n be an integer such that there exists a unique cubic semisymmetric graph on n vertices. We use the symbol **Sn** to denote this graph. As a natural extension of this notation we use the symbols **Sna**, **Snb**, ... in the case when nonisomorphic cubic semisymmetric graphs on n vertices exist. For example, the smallest cubic semisymmetric graph, the so called Gray graph on 54 vertices, is denoted with symbol **S54**, while the three pairwise non-isomorphic cubic semisymmetric graphs on 336 vertices are denoted with symbols **S336a**, **S336b** and **S336c**. The proposed notation is analogous to that used in the Foster census [4] for cubic arc-transitive graphs on n vertices, namely **Fn** (or **Fna**, **Fnb**, ...).

I. Let $n \geq 9$ be an integer divisible by 3, let A_n denote the alternating group acting on the set $\{1, 2, \ldots, n\}$, and let $\alpha, \beta \in A_n$ be the following permutations:

$$\alpha = (1, 2, 3)(4, 5, 6)(7, 8, 9) \cdots (n-2, n-1, n),$$

$$\beta = (1, 2, 6)(4, 5, 9)(7, 8, 12) \cdots (n-5, n-4, n)(3)(n-2)(n-1).$$

Then the Cayley graph of A_n relative to the generating set $\{\alpha, \beta\}$ is the line graph of a cubic semisymmetric graph of order $n!/3$ (see [33]). The vertex stabilisers are all isomorphic to S_3 and so the Goldschmidt type is G_1^1. The smallest graph in the family has 60480 vertices.

II. Two families of cubic semisymmetric graphs given in [28] can be defined using the amalgam completion construction described in Section 2. Namely, let $G =$

$PGL(2,p)$ for p a prime congruent to 11 or 13 modulo 24 and let $G = PSL(2,p)$ for a prime p congruent to ± 1 modulo 24. Then G contains a subgroup P_1 isomorphic to D_{24}, and a maximal subgroup P_2 isomorphic to S_4, which generate G and intersect in a subgroup B isomorphic to D_8. The pair of inclusion maps from B into P_1 and P_2 is a finite primitive amalgam of index $(3,3)$ and type G_2^1 (see 2.2). Moreover, the pair of inclusion maps from P_i, for $i = 1, 2$, into G is its faithful finite completion. As observed by Goldschmidt [20], the graph of this completion is a cubic G-semisymmetric graph. The semisymmetry of these graphs was proved in [28]. The smallest members, arising for primes $p = 11, 13, 23$, are the three biprimitive graphs in our list: **S110**, **S182** and **S506**. Note that other graphs in these two families are not biprimitive (since the group $P_1 \cong D_{24}$ is not maximal in G). But as $P_2 \cong S_4$ is maximal in G, the action of G on one part of bipartition is primitive in all cases.

III. The next two families arise as covers of $K_{3,3}$ with a cyclic or elementary abelian voltage group N. In order to describe them via voltage assignments we choose the spanning tree carrying trivial voltages as shown in Figure 1. Furthermore, let $a, b, c, d \in N$ be the voltages of the remaining cotree arcs $(3,2)$, $(3,4)$, $(2,5)$ and $(4,5)$, respectively.

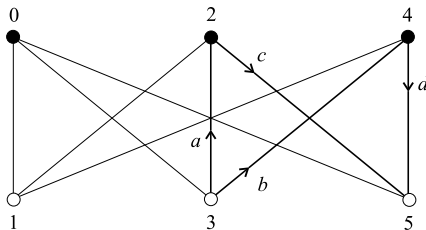Figure 1.

Let $n = 3^\varepsilon p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$, $k \geq 2$, where $\varepsilon \in \{0, 1\}$ and p_i, $i = 1, \ldots, k$, are distinct primes congruent to 1 modulo 3, and let $N = \mathbb{Z}_n$. In this case there are two solutions $r, s \in \mathbb{Z}_n^*$ of the equation $x^2 + x + 1 = 0$ such that $r \neq s, s^{-1}$. Then, letting $a = 1$, $b = -r$, $c = s$, and $d = -rs$, the corresponding regular N-cover of $K_{3,3}$ is semisymmetric with vertex stabilisers isomorphic to $\mathbb{Z}_3$ (see [30]). Hence the Goldschmidt type is G_1. The smallest graph in this family has 546 vertices, appearing in the list as **S546a**.

Along similar lines, let $p \equiv 1 \pmod{3}$ be a prime, let $N = \mathbb{Z}_p^2$, and let $r \in \mathbb{Z}_p$ be a primitive third root of unity. Then, by letting $a = (1,0)$, $b = (0,1)$, $c = (r,0)$ and $d = (0,r)$, the corresponding regular N-cover of $K_{3,3}$ is semisymmetric (see [29]). We note that vertex stabilisers are isomorphic to S_3 for $p > 7$, and so the Goldschmidt type is G_1^1 in such cases. On the other hand, $p = 7$ gives rise to the smallest graph in the family on 294 vertices, appearing in the list as **S294**. Its Goldschmidt type is G_1^2.

IV. There are four infinite families of pairwise nonisomorphic regular elementary abelian covers of the Heawood graph (**F14**), where the maximal lifted subgroup acts semisymmetrically on the covering graph. The first two families arise, respectively, as $\mathbb{Z}_p^3$-covers and $\mathbb{Z}_p^5$-covers for $p \equiv 1, 2$ or $4 \pmod{7}$, whereas the remaining two arise as $\mathbb{Z}_p^4$-covers when p satisfies the additional condition $p \equiv 1 \pmod{3}$. Furthermore, there are three sporadic examples of regular $\mathbb{Z}_7^6$-covers of **F14** satisying the above-mentioned property. The minimal members of the first two families are the graphs **S112** and **S448**

in the list. Their Goldschmidt type is G_1. We remark that for the remaining graphs the semisymmetry of the full automorphism group has not yet been verified.

In order to describe these constructions, we let the arcs of the Hamilton path $5, 0', 6, 1', \ldots, 3, 5', 4, 6'$ of **F14** carry the trivial voltage, and we let $a_7, a_0, a_1, \ldots, a_6$ denote the voltages of the arcs $(5, 6')$, $(0, 4')$, $(1, 5')$, $(2, 6')$, $(3, 0')$, $(4, 1')$, $(5, 2')$, $(6, 3')$, respectively (see Figure 2). To obtain the three $\mathbb{Z}_7^6$-covers, let $a_7 = (1, 0, 0, 0, 0, 0), a_0 = (0, 1, 0, 0, 0, 0), \ldots, a_4 = (0, 0, 0, 0, 0, 1)$ be the six standard basis vectors of $\mathbb{Z}_7^6$, and let $a_5 = (x, 1, 2, 3, 4, 5)$ and $a_6 = (y, 5, 4, 3, 2, 1)$, where $(x, y) = (6, 6)$, $(3, 2)$ or $(0, 5)$, respectively. The corresponding three covering graphs on $14 \cdot 7^6 = 1647086$ vertices arise from pairwise non-isomorphic covering projections.

For the remaining infinite families, the prime p satisfies the condition $p \equiv 1, 2$ or $4 \pmod{7}$. In this case there exists an element $s \in \mathbb{Z}_p$ satisfying $s^2 + s + 2 = 0$. The infinite family of $\mathbb{Z}_p^3$-covers is obtained by letting $a_7 = (0, 0, 0)$, $a_0 = (1, 0, 0)$, $a_1 = (0, 1, 0)$, $a_2 = (0, 0, 1)$, $a_3 = (1, s + 1, s)$, $a_4 = (s, -1, -1)$, $a_5 = (-1, -1, -s - 1)$, and $a_6 = (-s - 1, -s, 1)$. Similarly, the infinite family of $\mathbb{Z}_p^5$-covers is obtained by letting $a_7 = (1, 0, 0, 0, 0), a_0 = (0, 1, 0, 0, 0)$, $a_1 = (0, 0, 1, 0, 0)$, $a_2 = (0, 0, 0, 1, 0)$, $a_3 = (0, 0, 0, 0, 1)$ be the standard basis vectors of $\mathbb{Z}_p^5$, and by setting $a_4 = (s, -1, -s, 1, s+1)$, $a_5 = (s, -s - 1, 1, 1, s)$, and $a_6 = (-1, -s, 1, s + 1, -1)$. Finally, let p satisfy the additional condition $p \equiv 1 \pmod{3}$ and let r_1, r_2 be the two primitive cube roots of unity in $\mathbb{Z}_p$. Then the two infinite families of $\mathbb{Z}_p^4$-covers are obtained for $i = 1, 2$ by letting $a_7 = (r_i + 3, 0, 0, 0)$, $a_0 = (1, 1, 0, 0)$, $a_1 = (1, 0, 1, 0)$, $a_2 = (1, 0, 0, 1)$, $a_3 = (-r_i - 2, 1, s + 1, s)$, $a_4 = (-r_i - 2, s, -1, -1)$, $a_5 = (1, -1, -1, -s - 1)$ and $a_6 = (1, -s - 1, -s, -1)$.
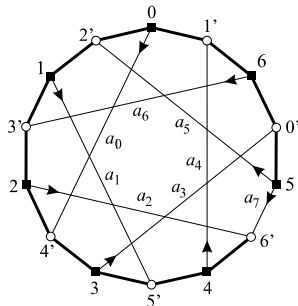
Figure 2.

V. Apart from the graph **S144**, with Goldschmidt type G_1^1, which is a regular $\mathbb{Z}_3^2$-cover of the Moebius-Kantor graph (**F16**), there is an infinite family of regular elementary abelian covers of **F16** where the maximal lifted subgroup acts semisymmetrically on the covering graphs. These graphs arise as $\mathbb{Z}_p^2$-covers, where $p \equiv 1 \pmod{4}$ is a prime. The smallest graph in this family is **S400**. Its Goldschmidt type is G_1. As in **IV** above, the semisymmetry of the full automorphism group for the remaining graphs in the family has not been verified.

Figure 3.

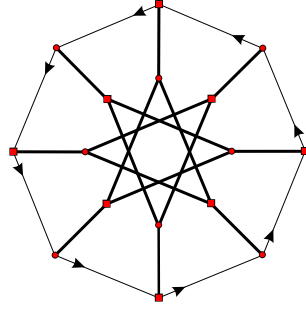In order to describe these constructions we observe that **F16** is isomorphic to the generalised Petersen graph $GP(8,3)$, thus allowing us to identify its vertex set with $\mathbb{Z}_8 \times \mathbb{Z}_2$ and its edge set with

$$E = \{\{(i,0),(i+1,0\} \mid i \in \mathbb{Z}_8\} \cup \{\{(i,1),(i+3,1)\} \mid i \in \mathbb{Z}_8\} \cup \{\{(i,0),(i,1)\} \mid i \in \mathbb{Z}_8\}.$$

Define the voltages on the arcs of the inner cycle and on the spokes to be trivial. As for the arcs of the oriented outer cycle, we define these to be, respectively, $(1,0)$, $(0,1)$, $(0,1)$, $(-1,0)$, $(-1,0)$, $(0,-1)$, $(0,-1)$, $(1,0)$, in the case $p=3$. Similarly, for the infinite family of regular $\mathbb{Z}_p^2$-covers, $p \equiv 1 \,(\mathrm{mod}\,4)$, we let the above voltages be, respectively, $(1,0)$, $(0,1)$, $(r,r-1)$, $(0,-r)$, $(-1,0)$, $(0,-1)$, $(-r,-r-1)$, $(0,r)$, where $r^2 = -1$. (see Figure 3.)

VI. There exists an infinite family of regular elementary abelian covers of the Pappus graph (**F18**) where the maximal lifted subgroup acts semisymmetrically on the covering graph. The graphs in this family arise as $\mathbb{Z}_p^2$-covers, where $p \equiv 1 \,(\mathrm{mod}\,3)$ is a prime. With the exception of the smallest graph ($p=7$), which has 882 vertices and is indeed semisymmetric, the semisymmetry of the full automorphism group for the remaining graphs in the family has not been verified.
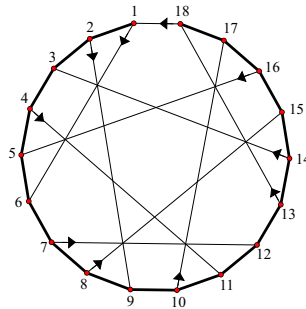
Figure 4.

To describe this construction we identify the vertex set of **F18** with $\{1, 2, \ldots, 18\}$ and its edge set with the union of the sets $\{\{i, i+1\} \mid i \in \{1, 2, \ldots, 18\}\}$, $\{\{i, i+5\} \mid i \in \{1, 7, 13\}\}$, $\{\{i, i+7\} \mid i \in \{2, 8, 14\}\}$, and $\{\{i, i+7\} \mid i \in \{4, 10, 16\}\}$, where the arithmetic is performed modulo 18. We let the arcs of the Hamilton path $(1, 2, \ldots, 18)$ carry trivial voltages (see Figure 4), and we let the arcs $\{(i, i+5) \mid i \in \{1, 7, 13\}\}$, $\{(i, i+7) \mid i \in \{2, 8, 14\}\}$, $\{(i, i+7) \mid i \in \{4, 10, 16\}\}$, and $(18, 1)$, respectively, have

the following voltages: $(1,0)$, $(1,0)$, $(1,0)$, $(0,1)$, $(-r+1,r)$, $(-1,-r-1)$, $(-r,r+1)$, $(2,-1)$, $(-2,-r)$, and $(-r-3,0)$, where r is an element of order 3 in $\mathbb{Z}_p^*$.

VII. There are eight graphs in the list with the tripod $K_{1,3}$ as a normal quotient: **S120**, **S330**, **S336b**, **S336c**, **S504**, **S546b**, **S660** and **S720b**. These graphs may be obtained using a general approach described in the next section.

5   Semisymmetric actions arising from $K_{1,3}$

In this section we prove a theorem which enables us to construct several infinite families of semisymmetric cubic graphs. It deals with the situation where a semisymmetric group of automorphisms contains a normal simple subgroup acting transitively on one part of the bipartition and intransitively on the other. It came to our notice while preparing the final version of this paper that a similar problem was considered by Giudici, Li and Praeger [17].

First we prove the following, which relies heavily on the analysis and classification of finite arc-transitive cubic graphs by Tutte[37] and Djoković and Miller [11], and on an in-depth analysis of their automorphism groups in terms of generators and relations by Conder and Lorimer [5]. In particular, it is known that an arc-transitive group of automorphisms of a finite cubic graph X acts regularly on the set of k-arcs of X for some $k \in \{1, 2, 3, 4, 5\}$, and its action falls into one of seven possible classes, denoted in [5] by G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2 and G_5. We shall call these classes *Djoković-Miller classes*, to distinguish them from the Goldschmidt types of semisymmetric group actions.

Proposition 5.1 *Let X be a connected cubic bipartite graph admitting an arc-transitive action of a group $G \le \operatorname{Aut} X$, and a semisymmetric action of a subgroup $H \le G$. Also let T be the maximal subgroup of G preserving both parts of the bipartition of X. If H is self-normalising in G, that is, if $N_G(H) = H$, then H is of Goldschmidt type G_1, and G belongs to the Djoković-Miller class G_4^2, and the permutation representation induced by T (by right multiplication) on right cosets of H is isomorphic to the natural action of $PSL(2,7)$ on the projective line $PG(1,7)$.*

PROOF. The proof is divided into several cases, dealing in turn with the seven possible Djoković-Miller classes to which G might belong. It follows from [5] that in each of these cases, the group G is generated by an element h (of order 3) fixing a vertex v, an element a (of order 2 or 4) reversing an arc (v, w) incident with v, and up to four involutions (denoted by p, q, r and s) which generate the stabiliser of the arc (v, w). Also the vertex-stabilisers G_v and G_w are generated by the stabiliser of the arc (v, w) and elements h and $h^a = a^{-1}ha$, respectively. Moreover, since the order of G_v divides $3 \cdot 2^4$ (by Tutte's theorem), all cyclic subgroups of order 3 fixing the vertex v are conjugate in G_v.

We consider possibilities for H in terms of two elements x and y, each of order 3 in H, and stabilizing v and w respectively. Without loss of generality we may suppose that x is the element h used as a generator for G. Furthermore, since $v^a = w$ we know that aya^{-1} fixes v, and therefore also aya^{-1} is an element of order 3 in G_v.

Now if G belongs to one of the Djoković-Miller classes G_1, G_2^1 or G_2^2, then $|T_v| = |G_v| = 3$ or 6, so the index $|T : H| = |T_v : H_v|$ is either 1 or 2, and hence H is normal in T, contradicting the assumption $N_G(H) = H$.

If G belongs to the Djoković-Miller class G_3, then $G_v \cong S_3 \times C_2$ is generated by h and two involutions p and q such that $pq = qp$, $php = h$, $qhq = h^{-1}$ and $ap = qa$, and a itself is an involution. Clearly, G_v has only two elements of order 3, so $aya^{-1} = h$ or h^{-1}. In both cases, the group $\langle h, h^a \rangle = \langle x, y \rangle \le H$ has index dividing 4 in T with transversal a subset of $\{1, p, q, pq\}$. Since $N_G(H) = H$, the index of H in T is at least 4. It follows that $H = \langle h, h^a \rangle$, but this subgroup is normalised by a, contradiction.

If G belongs to the Djoković-Miller class G_4^1 or to the class G_4^2, then $G_v \cong S_4$ is generated by h and three involutions p, q and r such that $pq = qp$, $pr = rp$, $(qr)^2 = p$, $h^{-1}ph = q$, $h^{-1}qh = pq$, $rhr = h^{-1}$, $ap = pa$ and $aq = ra$, with $a^2 = 1$ in the case of Djoković-Miller class G_4^1, while $a^2 = p$ in the case of Djoković-Miller class G_4^2. This time we find that G_v has eight elements of order 3, and so $aya^{-1} = h^{\pm 1}$ or $ph^{\pm 1}p$ or $qh^{\pm 1}q$ or $qph^{\pm 1}pq$. Thus H contains one of the four subgroups $L_1 = \langle h, a^{-1}ha \rangle$, $L_2 = \langle h, a^{-1}phpa \rangle$, $L_3 = \langle h, a^{-1}qhqa \rangle$ or $L_4 = \langle h, a^{-1}qphpqa \rangle$. These subgroups all have index dividing 16 in G, and index dividing 8 in $T = \langle h, a^{-1}ha, p, q, r \rangle$, with transversal in T being a subset of the dihedral subgroup $A = \langle p, q, r \rangle$ of order 8. Since $h^{-1}ph = q$, $h^{-1}qh = pq$, $h^{-1}pqh = p$, it follows that if H contains one of the elements p, q and pq, then it contains all of them, and has therefore index at most 2 in T, a contradiction. Observe also that $r^{-1}L_1r = L_3$ while $r^{-1}L_2r = L_4$.

In the case of G_4^1, the subgroup L_1 is normalised by a, while L_2, L_3 and L_4 are normalised by ap, aqr and arq respectively. Hence the index of H in T has to be 4, and H must be of the form $\langle L_i, u \rangle$ where u is an element of $A \setminus \langle p, q \rangle = \{r, pr, qr, pqr\}$ normalizing L_i. It is easy to check, however, that this is impossible for all four subgroups L_i; for example, $prhrp = ph^{-1}p = pqh^{-1}$ but $pq \notin H$, and although $rhr = h^{-1}$ we have also $r(a^{-1}ha)r = ra^{-1}hqa = ra^{-1}pha = rpa^{-1}ha = p(a^{-1}ha)$ but $p \notin H$.

In the case of G_4^2, observe that $a^{-1}pL_1pa = \langle a^{-1}phpa, a^{-1}pahapa \rangle = \langle a^{-1}phpa, h \rangle = L_2$ since $apa = a^4 = p^2 = 1$, and hence that all four subgroups L_1, L_2, L_3 and L_4 are conjugate to $L = L_1 = \langle h, a^{-1}ha \rangle$ in G. We can therefore assume without loss of generality that $L \le H$. Now consider the action of the group T on the right cosets of its subgroup L. The permutations induced by its generators h, $a^{-1}ha$, p, q and r are as follows:

- h induces the permutation $(L)(Lp, Lq, Lpq)(Lpr, Lpqr, Lqr)(Lr)$;
- $a^{-1}ha$ induces the permutation $(L)(Lp, Lr, Lpr)(Lq)(Lpq, Lqr, Lpqr)$;
- p induces the permutation $(L, Lp)(Lq, Lpq)(Lr, Lpr)(Lqr, Lpqr)$;
- q induces the permutation $(L, Lq)(Lp, Lpq)(Lr, Lpqr)(Lpr, Lqr)$; and
- r induces the permutation $(L, Lr)(Lp, Lpr)(Lq, Lqr)(Lpq, Lpqr)$.

These permutations generate a group of order 168 isomorphic to PSL(2,7) in its natural representation on the projective line over GF(7), as may be seen from the correspondence which takes $(L, Lp, Lqr, Lpq, Lq, Lr, Lpr, Lpqr)$ to $(\infty, 0, 1, 2, 3, 4, 5, 6)$, and the permutations induced by the generators h, $a^{-1}ha$, p, q and r of T to the transformations $z \mapsto 2z + 1$, $z \mapsto 2z - 1$, $z \mapsto 3/z$, $z \mapsto (z - 3)/(z - 1)$ and $z \mapsto (-z - 3)/(z + 1)$ respectively. In particular, as this permutation representation is primitive, we find that $L \le H$ is a maximal subgroup of T, implying $L = H$, as required.

14

Finally, if G belongs to Djoković-Miller class G_5, then $G_v \cong S_4 \times C_2$ is generated by h and four involutions p, q, r and s such that $pq = qp$, $pr = rp$, $ps = sp$, $qr = rq$, $qs = sq$, $(rs)^2 = pq$, $h^{-1}ph = p$, $h^{-1}qh = r$, $h^{-1}rh = pqr$, $shs = h^{-1}$, $ap = qa$ and $ar = sa$, and a itself is an involution. Again we find that G_v has eight elements of order 3, so $aya^{-1} = h^{\pm 1}$ or $qh^{\pm 1}q$ or $rh^{\pm 1}r$ or $rqh^{\pm 1}qr$, and hence H contains at least one of $M_1 = \langle h, a^{-1}ha \rangle$, $M_2 = \langle h, a^{-1}qhqa \rangle$, $M_3 = \langle h, a^{-1}rhra \rangle$ and $M_4 = \langle h, a^{-1}rqhqra \rangle$. These subgroups all have index dividing 32 in G, and index dividing 16 in T, with transversal in T being a subset of the subgroup $A = \langle p, q, r, s \rangle \cong D_4 \times C_2$. Also $p^{-1}M_1 p = M_2$ and $s^{-1}M_1 s = M_3$ and $(ps)^{-1}M_1 ps = M_4$, so that all four subgroups are conjugate in T, and hence we may suppose that H contains $M = M_1 = \langle h, a^{-1}ha \rangle$. Moreover, as M is normalised by a, we know $H \neq M$ and therefore H must contain at least one element of $A = \langle p, q, r, s \rangle$. But if H contains p, then H contains $(a^{-1}ha)^{-1}p(a^{-1}ha) = s$ and $(a^{-1}ha)p(a^{-1}ha)^{-1} = pqs$, and therefore contains also $p(pqs)s = q$ and $h^{-1}qh = r$, and so contains all of T, contradiction. Hence $p \notin H$. A similar argument shows that H contains none of $\{p, q, r, s, pqr, pqs\}$. Furthermore, this shows $prs \notin H$, for otherwise H would contain $hprsh = phrh^{-1}s = pqs$, and similarly $qrs \notin H$. On the other hand, if H contains pq, then H contains also $h^{-1}pqh = pr$ and $(a^{-1}ha)pq(a^{-1}ha)^{-1} = ps$, and therefore contains the subgroup $\langle h, a^{-1}ha, pq, pr, ps \rangle$, which has index 2 in T, contradiction. Hence $pq \notin H$, and the same sort of argument shows that H contains none of $\{pq, pr, ps, qr, qs\}$. In particular, $rs \notin H$ (for otherwise $pq = (rs)^2 \in H$), and similarly $pqrs = sr \notin H$.

This completes the proof. ∎

Theorem 5.2 *Let X be a connected cubic bipartite edge-transitive graph with automorphism group G, and let S be a simple group other than $\mathbb{Z}_3$ such that S is normal in some semisymmetric group $H \leq G$, and such that the quotient X_S is isomorphic to $K_{1,3}$. Then the following hold:*

(i) $N_G(H) \leq N_G(S)$;

(ii) $N_G(S)$ *acts semisymmetrically on* X;

(iii) $N_G(S)$ *is a self-normalising subgroup of* G, *that is,* $N_G(N_G(S)) = N_G(S)$;

(iv) X *is semisymmetric.*

PROOF. Observe first that H acts transitively on both parts of the bipartition of X, and that S acts transitively on one part and intransitively (with three orbits of equal length) on the other part. This implies that

$$|H : S| = |H_u : S_u| = 3|H_v : S_v| \qquad (*)$$

where u is a vertex of the part on which S acts transitively and v is a vertex of the other part. In particular, by Goldshmidt's theorem [20], the index $|H : S|$ divides $3 \cdot 2^{s-1}$ for some $s \leq 9$. (Moreover, if X were symmetric, then we would have $s \leq 5$ by the earlier theorem of Tutte [37]). Note also that since S is transitive on one part of the bipartition, and the cardinality of both parts is divisible by 3, the cardinality of S is divisible by 3, and $|S| > 3$.

Let us now prove part (i). Assume the contrary. Then there exists an element $g \in G$ which normalises H but does not normalise S. It follows that the conjugate S^g

is a normal subgroup of $H^g = H$, so also $S \cap S^g$ is a normal subgroup of S, and by simplicity of S, we find $S \cap S^g = 1$. This implies that the normal subgroup SS^g of H is isomorphic to the direct product $S \times S$, of order $|S|^2$. In particular, the index $|H : S|$ is divisible by $|S|$, and so by (∗), $|S|$ divides $3 \cdot 2^{s-1}$, and hence $S \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$, contradicting the fact that $|S| > 3$. Thus $N_G(H) \le N_G(S)$.

To prove parts (ii) and (iii), observe that S cannot be normal in any vertex-transitive subgroup of G, since its actions on the two parts of the bipartition are not equivalent. In particular, $N_G(S)$ is edge- but not vertex-transitive. But now applying part (i) to $N_G(S)$ in place of H, we find $N_G(N_G(S)) \le N_G(S)$, and thus $N_G(S)$ is a self-normalising semisymmetric subgroup of G.

Finally, let us prove that the graph X is not vertex-transitive. Assume the contrary. As a consequence of part (ii) we can assume without loss of generality that $N_G(H) = H$. Hence by Proposition 5.1, the group H has a homomorphic image isomorphic to a point-stabiliser in the natural action of $PSL(2, 7)$ on the projective line $PG(1, 7)$. This is a group of order 21, and so H must have both the cyclic groups C_3 and C_7 among its composition factors. On the other hand, we know that H contains a non-abelian simple normal subgroup S of index dividing $3 \cdot 2^{s-1}$ for some $1 \le s \le 5$, and so C_7 cannot be a composition factor of H. This contradiction completes the proof. ∎

In what follows we construct some infinite families of connected cubic graphs, the semisymmetry of which follows from Theorem 5.2.

Construction 5.3 Let G be the direct product $C \times S$, where C is a cyclic group generated by an element c of order 3, and S is a non-abelian simple group that can be generated by two elements a and b of order 3. Define $P_1 = \langle ca \rangle$ and $P_2 = \langle b \rangle$, and observe that the pair of inclusions $(P_1 \to G, P_2 \to G)$ is a faithful finite completion of the amalgam $(1 \to P_1, 1 \to P_2)$. Now take X to be the graph of this completion, with vertex-set the union $G/P_1 \cup G/P_2$ of right cosets of P_1 and P_2 in G, and with two cosets $P_1 x$ and $P_2 y$ adjacent in X if and only if $P_1 x \cap P_2 y \ne \emptyset$. Observe also that G acts on X by right multiplication of cosets, and that this action is semisymmetric (of Goldschmidt type G_1). Moreover, the subgroup S acts transitively on the coset space $(G : P_1)$ but has three orbits on $(G : P_2)$. The corresponding quotient graph X_S is therefore isomorphic to $K_{1,3}$, and so by Theorem 5.2, the graph X is semisymmetric.

Note that graphs **S120**, **S336b**, **S336c** and **S720b** in our list can be obtained in this way. To illustrate its generality, we remark that many simple groups can be generated by two elements of order 3, and can thus be used in the above construction. Among these are all alternating groups A_n for $n \ge 3$, all $PSL_2(p)$ and $PSL_3(p)$ for prime p, all $PSL_n(q)$ and $PSU_n(q)$ for large n and all prime powers q, many other families of simple groups of Lie type, and most of the sporadic simple groups, including the Monster. (Note that most of these groups can be generated by two elements x and y of orders 2 and 3, and then simplicity implies that these groups can also be generated by y and y^x.)

The next construction is a generalisation of 5.3.

Construction 5.4 Let $(\phi_1 \colon B \to P_1, \phi_2 \colon B \to P_2)$ be a finite primitive amalgam of index $(3, 3)$, such that for $i = 1$ or 2 (say $i = 2$), the image $\mathrm{Im}\,\phi_i$ is normal in P_i. By Theorem 2.2 there are exactly four isomorphism classes of such amalgams, namely G_1,

G_1^2, G_2 and G_2^3. Further, let $(\psi_1 \colon P_1 \to S, \psi_2 \colon P_2 \to S)$ be a faithful finite completion of the above amalgam, where S is a simple group. Without loss of generality we may assume that homomorphisms ϕ_i and ψ_i are inclusions (for $i = 1$ and 2), so that P_1 and P_2 are subgroups generating S, and $B = P_1 \cap P_2$. Note that by Proposition 2.1 this is equivalent to the following situation: S is a simple group acting semisymmetrically on a connected bipartite cubic graph, such that the intersection B of the stabilisers P_1 and P_2 of two adjacent vertices is normal in P_2.

Now let C be a cyclic group of order 3 generated by an element z, and let G be the direct product $C \times S$. Since B is a normal 2-subgroup of index 3 in P_2, the group P_2 is isomorphic to the semidirect product $B \rtimes \langle y \rangle$, where $y^3 = 1$, and there exists a natural epimorphism $\pi \colon P_2 \to \langle y \rangle$ with kernel B. We can now define group homomorphisms

$$f_1 \colon P_1 \to C \times S \;\; \text{by} \;\; f_1(g) = (1, g), \quad \text{and} \;\; f_2 \colon P_2 \to C \times S \;\; \text{by} \;\; f_2(g) = (\pi(g), g).$$

These homomorphisms are clearly injective, and the simplicity of S implies that their images generate $C \times S$. Indeed, let H be the group generated by $\operatorname{Im} f_1$ and $\operatorname{Im} f_2$. Since P_1 and P_2 generate S, the projection of H onto the second factor of $C \times S$ is S itself, and so H has index 1 or 3 in $C \times S$. In the latter case, H would be isomorphic to S, and therefore simple. Also $S \cap H$ would be a normal subgroup of H of index 1 or 3, and since H is simple and $|H| = |S| > 3$, the index cannot be 3, so $S \cap H = H$. This is impossible, however, because $f_2(x) = (z, x) \notin S$. Thus $H = G = C \times S$.

The above implies that the pair of homomorphisms (f_1, f_2) gives a finite faithful completion of the amalgam (ϕ_1, ϕ_2). Let X be the graph of this completion, with vertex-set $(G : \operatorname{Im} f_1) \cup (G : \operatorname{Im} f_2)$. By Proposition 2.1, the group $G = C \times S$ acts semisymmetrically on X. Observe also that its subgroup S acts transitively on $(G : \operatorname{Im} f_2)$, but has three orbits on $(G : \operatorname{Im} f_1)$. The quotient graph X_S is therefore isomorphic to $K_{1,3}$, and by Theorem 5.2, X is semisymmetric.

This construction may be used to obtain the graphs **S330**, **S546b** and **S660** in our list. (Also Construction 5.3 can be obtained as a special case of this one, by choosing the amalgam (ϕ_1, ϕ_2) to be of type G_1.)

The following proposition introduces a third construction, for producing a connected bipartite cubic graph that admits a semisymmetric action of a semidirect product of a simple group by a cyclic group of order 3.

Proposition 5.5 *Let S be a simple group which has an automorphism ω of order 3, and let B and P be proper subgroups of S such that:*

(i) *B is a subgroup of P of index 3, and B is either trivial or not normal in P;*

(ii) *$B^\omega = B$;*

(iii) *The subgroup generated by P, P^ω and P^{ω^2} is S.*

Further, let G be the semi-direct product $S \rtimes \langle \omega \rangle$, and define $P_1 = P$ (viewed as a subgroup of G) and $P_2 = \langle B, \omega \rangle$, the subgroup of G generated by B and ω (isomorphic to the semi-direct product $B \rtimes \langle \omega \rangle$). Then:

(a) *$B = P_1 \cap P_2$, and the pair of inclusions $(\phi_1 \colon B \to P_1, \phi_2 \colon B \to P_2)$ is a primitive amalgam of index $(3, 3)$;*

(b) *the pair of inclusions $(\psi_1 \colon P_1 \to G, \psi_2 \colon P_2 \to G)$ is a faithful finite completion of the amalgam (ϕ_1, ϕ_2);*

(c) *the graph X of this completion admits a semisymmetric action of G, such that the quotient graph X_S is isomorphic to $K_{1,3}$;*

(d) *X is a connected semisymmetric cubic graph.*

PROOF. Everything in part (a) is obvious except for the fact that the amalgam (ϕ_1, ϕ_2) is primitive. So let us assume that (ϕ_1, ϕ_2) is not primitive. Then there exists a non-trivial subgroup K of B such that either:

- K is normal in P_1 and the normaliser $N_{P_2}(K)$ properly contains B, or

- K is normal in P_2 and the normaliser $N_{P_1}(K)$ properly contains B.

In the former case $N_{P_2}(K) = P_2$, since B is maximal in P_2, and so $\omega \in P_2$ normalises K and therefore K is normal in $\langle P_1, P_1^\omega, P_1^{\omega^2} \rangle = S$, but that is impossible since S is simple. In the latter case $N_{P_1}(K) = P_1$ since B is maximal in P_1, while ω normalises K because $\omega \in P_2$, and again K is normal in $\langle P_1, P_1^\omega, P_1^{\omega^2} \rangle = S$, which is impossible because S is simple.

To prove part (b) we need only show that P_1 and P_2 generate G. This follows directly from the fact that $\langle P, P^\omega, P^{\omega^2} \rangle = S$ implies $G = \langle S, \omega \rangle = \langle P, \omega \rangle = \langle P_1, P_2 \rangle$.

For part (c), recall that the two parts of the bipartition of X consist of right cosets of P_1 and P_2 in G, respectively, and that G acts as a group of automorphisms of X by right multiplication. Clearly $S \leq G$ acts transitively on the cosets of $P_2 = \langle \omega \rangle B$ in $G = \langle \omega \rangle S$, in essentially the same way as S acts on the cosets of B. On the other hand, S acts intransitively on the cosets of $P_1 = P$ in G, with three orbits corresponding to the cosets S, ωS and $\omega^2 S$. The quotient graph X_S is thus indeed isomorphic to $K_{1,3}$.

Finally, part (d) follows immediately from Theorem 5.2. ∎

Example 5.6 Let $F = GF(2^3)$ be the finite field with 8 elements, let λ be the generator of the multiplicative group F^* satisfying $\lambda^3 + \lambda + 1 = 0$, and let

$$b = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix} \lambda & \lambda^{-1} \\ \lambda^{-1} & \lambda^3 \end{bmatrix}$$

be two elements of $S = SL(2, F) \cong PSL(2, 8)$. Also let $B = \langle b \rangle$ and $P = \langle B, a \rangle$. Note that P is a subgroup of S isomorphic to S_3. Next let ω be the automorphism of F mapping λ to λ^2, which has order 3 since $\lambda^{\omega^3} = \lambda^8 = \lambda$. It is easy to check that $\langle a, a^\omega, a^{\omega^2} \rangle = S$, so all conditions of Proposition 5.5 are satisfied. The resulting graph X is a connected semisymmetric cubic graph on 504 vertices, isomorphic to the graph **S504** in our list.

Example 5.7 Let S, a and ω be as in the previous example, but let B be the trivial subgroup of S, and $P = \langle a \rangle$. Then the conditions of Proposition 5.5 are again satisfied, and the resulting graph is a connected semisymmetric cubic graph on 1008 vertices.

6 The diagrams — explaining the list

In this section we explain some of the structure and number of aspects of the list of semisymmetric graphs provided in Section 7. To do this, we need to discuss relationships between members of a class of graphs, as follows.

Irreducible morphisms and direct objects Let $\mathcal{C}$ be a class of graphs and graph morphisms (not necessarily a subcategory of graphs). A graph in $\mathcal{C}$ is called *basic* if it has no morphism in $\mathcal{C}$ to another graph in $\mathcal{C}$. A morphism $X \to Y$ in $\mathcal{C}$ is *irreducible* if it cannot be written as a composition of two morphisms (neither being an isomorphism) in $\mathcal{C}$. A graph X is a *direct object* over a graph Y if the set $\mathrm{Hom}_{\mathcal{C}}(X, Y) \neq \emptyset$ consists of irreducible morphisms only (which are then called *direct morphims*). For example, in the context of all graphs and regular covering projections, $\wp\colon X \to Y$ is irreducible if and only if $\mathrm{CT}(\wp)$ is a simple group; in addition, if $\mathrm{CT}(\wp)$ is abelian then X is a direct object over Y, but might fail to be direct if $\mathrm{CT}(\wp)$ is nonabelian. (We remark that in this setting the concept of a direct object corresponds to that of a *direct cover* as defined in the Foster census [4].)

When restricting to a specific subclass of graphs (and/or specific regular covering projections), the characterisation of irreducible morphisms, let alone direct ones, largely depends on the chosen subclass — as taking a quotient by the action of a semiregular group may result in a graph (and/or a morphism) outside that subclass; on one hand, some of the irreducible morphisms might not be included in the class at all, and on the other hand, a previously reducible projection might become irreducible. For instance, within the class of cubic arc-transitive graphs and regular covering projections there exist irreducible morphisms, even direct ones such as **F8** $\to$ **F2**, with a nonsimple group as the group of covering transformations.

When speaking of cubic semisymmetric graphs it is necessary to consider all cubic edge-transitive graphs as objects and a certain subclass of regular covering projections as morphisms. This needs further explanation. First, note that there are semisymmetric cubic graphs which have a regular covering projection onto a smaller cubic arc-transitive graph (for example, the smallest one, the Gray graph **S54**, has a projection onto the Pappus graph **F18**). Second, as for the choice of morphisms we have restricted ourselves to consider only edge-transitive regular covering projections. By definition, a regular covering projection $\wp_N\colon X \to X_N$ is *edge-transitive* if at least one (minimal) edge-transitive subgroup of $\mathrm{Aut}\, X$ projects along $\wp_N$ (implying that both graphs in question must be edge-transitive). By Proposition 3.1, $\wp_N$ is edge transitive if and only if N is normalized by some (minimal) edge-transitive subgroup of $\mathrm{Aut}\, X$; this is equivalent to requiring that N is normal in some edge-transitive subgroup of $\mathrm{Aut}\, X$ or, that the normalizer of N within $\mathrm{Aut}\, X$ is edge-transitive. In view of the above morphism restrictions it follows that an edge-transitive graph is *basic* whenever it has no edge-transitive covering projection onto a smaller edge-transitive graph, that is, if and only if no edge-transitive subgroup of $\mathrm{Aut}\, X$ has a semiregular normal subgroup.

Note that each edge-transitive graph is a quotient graph arising from an edge-transitive covering projection (for instance, by taking an appropriate voltage assignment with values in the first homology group of the graph, the respective covering projection is edge-transitive). Therefore by restricting ourselves to edge-transitive covering projections, no basic graph (in the wider context of all regular covering projections) is

lost; the set of basic graphs could just be larger. (No example is known to the authors, however, of a cubic edge-transitive graph covering another cubic edge-transitive graph with no covering projection between them being edge-transitive.)

Finally, we remark that in a later subsection we shall extend our class of morphisms to include all edge-transitive quotient projections.

Excluding strongly reducible morphisms Clearly we would like to omit redundant morphisms (ideally, we would keep just the direct ones, up to isomorphism of projections) and even the unnecessary arc-transitive graphs, and still be able to generate all semisymmetric graphs starting from the set of basic graphs (by reconstructing each of the remaining covering projections in terms of voltages).

Recall that an edge-transitive regular covering projection $\wp_N \colon X \to X_N$ is reducible if there exists a decompostion $\wp_N = \wp_{N/K} \circ \wp_K$ where $\wp_K$ and $\wp_{N/K}$ are edge-transitive. In particular, $\wp_N$ is *strongly reducible* whenever there exists $K \triangleleft N$ such that some edge-transitive subgroup projects simultaneously along $\wp_N$ and $\wp_K$. Unfortunately, reducible projections which are not strongly reducible may exist.

Lemma 6.1 *An edge-transitive regular covering projection $\wp_N \colon X \to X_N$ is strongly reducible if and only if there exists an edge-transitive group $H \leq \operatorname{Aut} X$ such that N is normal but not a minimal normal subgroup of H.* ∎

By Lemma 6.1, it suffices to consider all edge-transitive subgroups $H \leq \operatorname{Aut} X$ and then take quotients by the action of all subgroups in H which are minimal with respect to being normal and semiregular on X. Since each subgroup of a semiregular group is semiregular, these minimal subgroups are minimal normal subgroups of H which are, in addition, semiregular on X. Moreover, we can consider edge-transitive subgroups and their minimal normal subgroups only up to conjugation in $\operatorname{Aut} X$. Namely, normality and semiregularity are preserved under conjugation, and the projection relative to conjugate subgroups results in isomorphic covering projections. Now, if a selected subgroup N is normal and not minimal normal in some other edge-transitive subgroup (where "some other" should be understood as relative to all edge-transitive subgroups, not just subgroups up to conjugation), then such a projection is discarded. Repeating the procedure recursively on all graphs thus obtained, no "smaller object below X" is lost, and all morphisms are either irreducible, or at least not strongly reducible.

Strong reducibility can also be studied in terms of lifting edge-transitive subgroups along $\wp \colon Y \to X$ (usually given by means of a voltage assignment on X). Observe that if $\wp \colon Y \to X$ is a regular covering projection and if some (minimal) edge-transitive subgroup of $G \leq \operatorname{Aut} X$ lifts, then the lift $\tilde{G} \leq \operatorname{Aut} Y$ is edge-transitive and hence $\wp$ is edge-transitive. The next lemma says that $\wp$ is strongly reducible if and only if there exists a decomposition $\wp = \wp_N = \wp_{N/K} \circ \wp_K$ and a (minimal) edge-transitive subgroup of $\operatorname{Aut} X$ which lifts both along $\wp_N$ and $\wp_{N/K}$.

Lemma 6.2 *An edge-transitive regular covering projection $\wp \colon Y \to X$ is strongly reducible if and only if there exists a minimal edge-transitive subgroup $G \leq \operatorname{Aut} X$ such that $\wp$ is G-admissible but not minimally G-admissible.* ∎

In view of Lemma 6.2, it suffices to consider all minimal edge-transitive subgroups $G \leq \operatorname{Aut} X$, and then find all minimally G-admissible covers of X (up to isomorphism of covering projections). Moreover, we can consider minimal edge-transitive subgroups

only up to conjugation in $\operatorname{Aut} X$. Namely, if $G \leq \operatorname{Aut} X$ lifts along $\wp_\zeta \colon \operatorname{Cov}(X; \zeta) \to X$ then a conjugate subgroup G^α lifts along an isomorphic covering projection ζ^α, defined by $\zeta^\alpha(a) = \zeta(\alpha^{-1}(a))$ for each arc $a \in A(X)$. (Note that G^α might not lift along $\wp_\zeta$.) Repeating the procedure recursively on all larger graphs thus obtained, no "object above X" is lost. (If there is an edge-transitive graph which covers X but does not cover X edge-transitively, then such a graph is basic and needs to be found by other means.) All morphisms constructed this way are either irreducible, or at least not strongly reducible.

Characterising irreducible morphisms   Let $\wp_N \colon X \to X_N$ be a reducible but not strongly reducible edge-transitive regular covering projection, and let $\wp_N = \wp_{N/K} \circ \wp_K$ be an edge-transitive depomposition. In such a case, edge-transitive subgroups of $\operatorname{Aut} X_K$ projecting along $\wp_{N/K}$ are not related to edge-transitive-subgroups of $\operatorname{Aut} X$ (via their projections along $\wp_K$). Similarly, no lift along $\wp_{N/K}$ of any edge-transitive group lifts along $\wp_K$. This makes the characterisation of irreducible morphisms rather difficult if not impossible.

We discuss the matter in some detail from the point of view of projecting groups first. Let $\mathcal{N}_{\min} = \cup \operatorname{Min}_\lhd H$ be the set of all minimal normal subgroups of all edge-transitive subgroups $H \leq \operatorname{Aut} X$, and let nInc denote the "normal subgroup" relation. Observe that the set of nInc-minimal elements

$$\mathcal{M} = \min_{\mathrm{nInc}} \bigcup_{H \text{ edge-transitive}} \operatorname{Min}_\lhd H$$

can also be described as the set of all nInc-minimal elements in the set of all subgroups of $\operatorname{Aut} X$ which are normalized by some minimal edge-transitive group, that is,

$$\mathcal{M} = \min_{\mathrm{nInc}}\{M \mid M \text{ is normalized by a minimal edge-transitive subgroup}\}.$$

After calculation of the nInc-minimal elements, the set $\mathcal{M}$ can be reduced up to conjugation — but not before. Now, if $N \in \mathcal{M}$ acts semiregularly on the graph, then $\wp_N$ is edge-transitive and irreducible. However, the set

$$\min_{\mathrm{nInc}} \{\mathcal{N}_{\min} \setminus \mathcal{M}\}$$

may contain groups, acting semiregularly on X, which do not give rise to a strongly reducible projection. To decide whether such a projection $\wp_N$ is reducible or not, one has to consider $K \lhd N$, $K \in \mathcal{M}$ and the quotient graph X_K. Therefore, by restricting just to groups in $\mathcal{M}$, the reduction has gone too far. This can happen even in such special cases where $\operatorname{Aut} X$ has a single conjugacy class of minimal edge-transitive subgroups.

Specialising further we now consider the case where $\operatorname{Aut} X$ has a unique minimal edge-transitive subgroup (which is necessarily normal in $\operatorname{Aut} X$). In such a case, an edge-transitive regular covering projection is either irreducible or strongly reducible, implying that irreducible projections can be characterised.

Lemma 6.3 *Let X be an edge-transitive graph such that* $\operatorname{Aut} X$ *has a unique minimal edge-transitive subgroup, H. Let $\mathcal{M}$ be the set of all* nInc-*minimal elements*

$$\mathcal{M} = \min_{\mathrm{nInc}}\{M \mid \text{M normalized by H}\} = \min_{\mathrm{nInc}} \bigcup \operatorname{Min}_\lhd H',$$

where the union is taken over all subgroups H' of $\operatorname{Aut} X$ containing H. Then the quotient projection by a group $N \leq \operatorname{Aut} X$ is an irreducible edge-transitive regular covering projection if and only if N is semiregular on X and $N \in \mathcal{M}$. ∎

Note that, in contrast with the general case, $\mathcal{M}$ now does contain all minimal normal subgroups of H (if H is not simple). The remaining groups in $\mathcal{M}$, if any, must intersect H trivially.

As from the point of view of lifting groups we consider only the most particular case when $\operatorname{Aut} X$ has a unique minimal edge-transitive subgroup, H. Like above, an edge-transitive regular covering projection $\wp \colon Y \to X$ is now either irreducible or strongly reducible. Hence $\wp$ is irreducible if and only if it is minimally H-admissible, by Lemma 6.2.

Our reduced list of direct covers Returning to our list, a further removal of many arc-transitive graphs can be accomplished: if a graph X is arc-transitive, we keep only those graphs below X obtained by direct morphisms as found in the Foster census [4] (such morphisms are indeed edge-transitive). Moreover, there should be a semisymmetric cubic graph Y on up to 768 vertices and an edge-transitive regular covering projection $Y \to X$.

As for the semisymmetric ones, recall that all such graphs in the list have been computed separately by other methods (so that we do have all the objects on up to 768 vertices). Checking their automorphism groups (using MAGMA [1]) reveals that, save for one single exception — the Gray graph — all have a unique minimal edge-transitive group which is, moreover, of index 1, 2, or 4 in the full automorphism group. As for the Gray graph, its minimal edge-transitive subgroups are precisely the Sylow 3-subgroups of its full automorphism group, and are thus pairwise conjugate.

The graphs **F2**, **F4**, **F10**, **F28**, **F56b**, **F110**, **F182d**, **S110**, **S126**, **S182**, **S210**, **S504**, **S506** and **S650** are basic, for they share the following property: no edge-transitive subgroup of the respective automorphism groups has a normal semiregular subgroup. This clearly holds for the arc-transitive ones, in view of the Foster census [4]. The minimal edge-transitive subgroup of **S504** is its full automorphism group itself, isomorphic to $\operatorname{PSL}(2,8) \rtimes \mathbb{Z}_3$, and $\operatorname{PSL}(2,8)$ is its only proper nontrivial normal subgroup which is not semiregular since the respective quotient graph is $K_{1,3}$. The unique minimal edge-transitive subgroup H for each of the remaining graphs X is simple and of index at most 2 in $\operatorname{Aut} X$. Therefore, a normal subgroup $N \triangleleft \operatorname{Aut} X$ either contains H or intersects H trivially. In the first case, N is edge-transitive and hence not semiregular on X. In the second case, N must be isomorphic to $\mathbb{Z}_2$. However, all these graphs are of the form twice an odd number, and so quotienting by $\mathbb{Z}_2$ cannot result in a cubic graph.

As for the irreducible morphisms, in the case of Gray graph we can take just the quotient by the minimal normal subgroup, isomorphic to $\mathbb{Z}_3$, of some Sylow 3-subgroup. Namely, since the Gray graph has order $54 = 2 \cdot 3^3$ and since cubic graphs have even order, the group by which we take the quotient must be a 3-group, and hence contained in a minimal edge-transitive subgroup. The respective covering projection is **S54** $\to$ **F18**. All other irreducible morphisms are found using Lemma 6.3.

There are a few cases where $N \in \mathcal{M}$ intersects H trivially. In this case N must be a minimal normal subgroup of $\operatorname{Aut} X$ and isomorphic to $\mathbb{Z}_2$. At most three such subgroups exist. This holds true because H is normal and of index 1, 2 or 4 in $\operatorname{Aut}X$.

The only remaining possiblity that $N \cong \mathbb{Z}_2^2$ cannot occur for the following reason. If $N \cong \mathbb{Z}_2^2$ then $|\operatorname{Aut} X : H| = 4$ and N is a minimal normal subgroup of $\operatorname{Aut} X$. But there is always a subgroup $H' \leq \operatorname{Aut} X$ of index 2 containing H, and $N \cap H'$ must be a proper and nontrivial subgroup of N. So $N \cap H' \triangleleft \operatorname{Aut} X$, a contradiction to the minimality of N. Thus, $N \cong \mathbb{Z}_2$ is the only possibility. Clearly, N is the only subgroup of order 2 complementary to H in the group $N \times H$ (a subgroup of index 2 in $\operatorname{Aut} X$). Hence N is also normal in $\operatorname{Aut} X$. As there are at most three subgroups of index 2 containing H, there are at most three minimal normal subgroups of $\operatorname{Aut} X$ isomorphic to $\mathbb{Z}_2$ and having trivial intersection with H. However, a close inspection of our list shows that we only have cases where H is of index 2 and H is a simple group. Hence N is always unique. These are the cases **S220** $\to$ **F110**, **S364** $\to$ **F182d** and **S420** $\to$ **S210**. In all other cases, the group $N \in M$ is actually a minimal normal subgroup of H. There are never more than two such subgroups, and with a few exceptions, they are of coprime orders. Apart from **S294** $\to$ **S42**, the minimal normal subgroups of H are always normal in $\operatorname{Aut} X$.

Having found all irreducible morphisms, all graphs in the list can be constructed by a sequence of irreducible covering projections starting from the basic ones. Each semisymmetric graph projects onto at most two distinct smaller graphs in the list, and there are no nonisomorphic projections between the same pair of graphs in the list. Note that any composition of two or more irreducible morphisms is either a strongly reducible edge-transitive regular covering projection — or a covering projection which is not of our concern: a regular but not edge-transitive, or a nonregular covering projection.

We now show that all covering projections constructed as described above are actually direct. By the opening remarks at the beginning of this subsection, all morphisms between arc-transitive graphs are direct. This trivially holds also in all cases where the group of covering transformations is of prime order. So what we need to check are the following projections — those with $|\operatorname{Aut} X : H| = 1$ (namely **S112** $\to$ **F14**, **S336a** $\to$ **F42**, **S400** $\to$ **F16**, **S448** $\to$ **F56a**, **S448** $\to$ **S112**, and **S702a** $\to$ **F26**), and those with $|\operatorname{Aut} X : H| = 2$ (namely **S144** $\to$ **F16**, **S216** $\to$ **S54**, **S432b** $\to$ **F48**, **S576** $\to$ **F64**, **S576** $\to$ **S144**, and **S768a** $\to$ **F192c**), and finally, those with $|\operatorname{Aut} X : H| = 4$ (namely **S384** $\to$ **F96b** and **S768b** $\to$ **F192a**). All these covering projections arise as factorisations by a minimal normal subgroup of H.

If one of the above irreducible morphisms $X \to X_N$ is not direct, then there exists a subgroup $N' \leq \operatorname{Aut} X$, acting semiregularly on X, such that N' (together with one of its normal subgroups) is normalized by H and $X_{N'} \cong X_N$. In particular, $|N'| = |N|$.

Now, if any two minimal normal subgroups of H have either equal or coprime cardinality, then $N' \cap H = 1$. So if $|\operatorname{Aut} X : H| \leq 2$, then N' is either trivial or isomorphic with $\mathbb{Z}_2$, a contradiction since $\wp_{N'}$ was assumed strongly reducible. This allows us to dispose of the cases with $|\operatorname{Aut} X : H| = 1$, except possibly for the projections **S448** $\to$ **F56a** and **S448** $\to$ **S112**, and of the cases with $|\operatorname{Aut} X : H| = 2$, except possibly for the projections **S432b** $\to$ **F48** and **S768a** $\to$ **F192c**. We now consider these four projections separately.

The minimal normal subgroups of $H = \operatorname{Aut}$ **S448** are $N_1 \cong \mathbb{Z}_2^3$ and $N_2 \cong \mathbb{Z}_2^2$, giving rise to the two irreducible projections in the list. Clearly, no projection $\wp_{N'}$: **S448** $\to$ **S112** is strongly reducible, for otherwise H would contain a minimal normal subgroup isomorphic to $\mathbb{Z}_2$. So $\wp_{N_2}$: **S448** $\to$ **S112** is direct.

Suppose that $\wp_{N'}$: **S448** $\to$ **F56a** is strongly reducible. As N' is of order 8 and is

not minimal normal in $H = \text{Aut}\,\mathbf{S448}$, it contains N_2. Thus, $\wp_{N'}$ would factorise via $\wp_{N_2}$. But this is not possible because $\text{Aut}\,\mathbf{S112}$ (the minimal edge-transitive group of $\mathbf{S112}$) does not have a normal subgroup isomorphic to $\mathbb{Z}_2$ (as in fact its minimal normal subgroup is $\mathbb{Z}_2^3$). Consequently, $\wp_{N_1} : \mathbf{S448} \to \mathbf{F56a}$ is direct.

Consider a strongly reducible projection $\wp_{N'} : \mathbf{S432b} \to \mathbf{F48}$. The group N' is of order 9, and since $|\text{Aut}\,X : H| = 2$, the group N' is contained in H and is not minimal normal in H. The minimal normal subgroups of H are $N_1 \cong \mathbb{Z}_3$ and $N_2 \cong \mathbb{Z}_3^2$. Thus N' contains $N_1 \cong \mathbb{Z}_3$, the group giving rise to the projection $\mathbf{S432b} \to \mathbf{S144}$. The quotient group $N'/N_1 \cong \mathbb{Z}_3$ must be normalized by the minimal edge-transitive subgroup of $\text{Aut}\,\mathbf{S144}$. But this minimal subgroup is also of index 2 in $\text{Aut}\,\mathbf{S144}$, and must therefore contain the projection N'/N_1. However, the minimal edge-transitive subgroup of $\text{Aut}\,\mathbf{S144}$ has only one minimal normal subgroup, and this is isomorphic to $\mathbb{Z}_3^2$. Therefore, $\wp_{N_2} : \mathbf{S432b} \to \mathbf{F48}$ is direct.

Consider a strongly reducible projection $\wp_{N'} : \mathbf{S768a} \to \mathbf{F192c}$. The minimal normal subgroups of H are $N_1 \cong \mathbb{Z}_2$ and $N_2 \cong \mathbb{Z}_2^2$. As $|N'| = 4$ and $|\text{Aut}\,X : H| = 2$, the group N' has a nontrivial intersection with H, implying that $\wp_{N'}$ factorises through $\wp_{N_1} : \mathbf{S768a} \to \mathbf{S384}$. Thus $\text{Aut}\,\mathbf{S384}$ has a group $N'/N_1 \cong \mathbb{Z}_2$ which is normalized by the minimal edge-transitive subgroup $H' \leq \text{Aut}\,\mathbf{S384}$. The group N'/N_1 is not contained in H' since the minimal normal subgroup of H' is isomorphic to $\mathbb{Z}_2^2$. But neither can N'/N_1 have trivial intersection with H' for the following reason. First note that H' has index 4 in $\text{Aut}\,\mathbf{S384}$, and that there would be a group $\mathbb{Z}_2 \times H'$ of index 2 in $\text{Aut}\,\mathbf{S384}$ with a nontrivial centre. However, any of the three index 2 subgroups of $\text{Aut}\,\mathbf{S384}$ are, according to MAGMA, centreless. This contradiction proves that $\wp_{N_2} : \mathbf{S768a} \to \mathbf{F192c}$ is direct.

To complete the proof we need to discuss the two cases where H is of index 4 in $\text{Aut}\,X$. First we consider a strongly reducible projection $\wp_{N'} : \mathbf{S384} \to \mathbf{F96b}$. Now, if $N' \cap H = 1$ then $\text{Aut}\,\mathbf{S384} = N' \times H$, and the centre of $\text{Aut}\,\mathbf{S384}$ would be nontrivial. That is a contradiction, by MAGMA. We cannot have $N' \cap H \cong \mathbb{Z}_2$ since the minimal normal subgroup of H is $N = \mathbb{Z}_2^2$, and $N' \leq H$ implies $N' = N$, a contradiction. Thus, $\wp_N : \mathbf{S384} \to \mathbf{F96b}$ is direct.

Finally, consider a strongly reducible projection $\wp_{N'} : \mathbf{S768b} \to \mathbf{F192a}$. If $N' \cap H = 1$, then $\text{Aut}\,\mathbf{S768b} = N' \times H$, a contradiction since the centre of $\text{Aut}\,\mathbf{S768b}$ coincides with that of H, by MAGMA. Now $N' \cap H \neq 1$ implies that $\wp_{N'}$ factorises through $\wp_{N_1} : \mathbf{S768b} \to \mathbf{S384}$. But this would mean that the minimal edge-transitive subgroup of $\text{Aut}\,\mathbf{S384}$ would normalize $N'/N_1 \cong \mathbb{Z}_2$, a contradiction since $\mathbf{S384}$ has a unique irreducible projection, the one by the action of $\mathbb{Z}_2^2$. So $\wp_{N_2} : \mathbf{S768b} \to \mathbf{F192a}$ is direct.

The above discussion is summarised in the following theorem.

Theorem 6.4 *Our list displays, up to isomorphism of covering projections, all possible constructions of all cubic semisymmetric graphs on up to to 768 vertices by taking successive direct regular covers of the following basic edge-transitive graphs:* **F2**, **F4**, **F10**, **F28**, **F56b**, **F110**, **F182d**, **S110**, **S126**, **S182**, **S210**, **S504**, **S506** *and* **S650**. ∎

As was shown in [29], every finite cubic graph admitting an edge-transitive solvable group may be obtained from either the 3-dipole Dip_3 (the graph **F2**), or the complete graph K_4 (the graph **F4**), by a sequence of elementary abelian covers. In other words,

F2 and **F4** are the only basic graphs needed to generate all finite edge-transitive cubic graphs having a solvable automorphism group. Thus each of the 21 cubic semisymmetric graphs in our list that has a solvable automorphism group can be obtained from one of these two graphs, as may be seen from our first diagram, see Figure 5.

Figure 5.

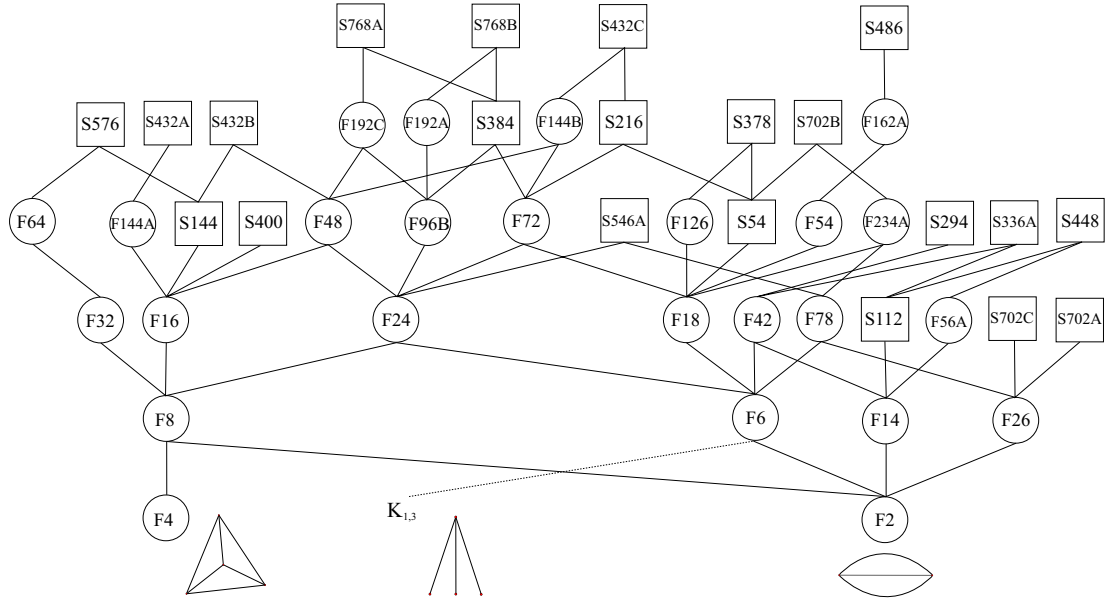The remaining 22 semisymmetric cubic graphs with a non-solvable automorphism group are displayed in Figures 6a and 6b. Note that every graph which reduces to **F6** (the complete bipartite graph $K_{3,3}$) reduces also to $K_{1,3}$, see Figure 5. In Figures 6a and 6b, the eight irreducible quotients of semisymmetric graphs onto $K_{1,3}$ are also displayed. These quotients are discussed below.
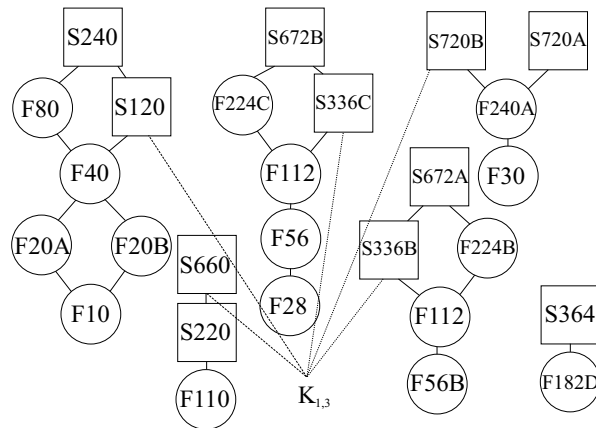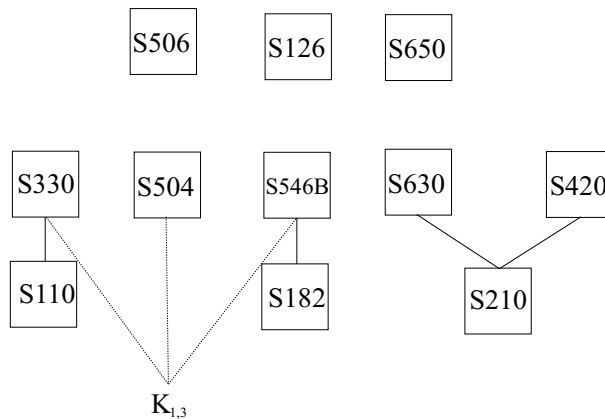

Figure 6a.

Figure 6b.

Quotients onto $K_{1,3}$ There are several examples of cubic semisymmetric graphs in our list having the tripod $K_{1,3}$ as a quotient, and as Section 5 demonstrates, they appear to be of particular interest. So let us enlarge our set of morphisms (and consequently, our set of objects). By definition, a quotient projection $q_N \colon X \to X_N$ is edge-transitive whenever at least one (minimal) edge-transitive subgroup of $\operatorname{Aut} X$ projects along q_N. By Proposition 3.1, q_N is edge-transitive if and only if $\operatorname{Ker} q_N$ is normalized by some (minimal) edge-transitive subgroup of $\operatorname{Aut} X$.

As for the objects of our study, we note the following. If N is edge-transitive, then the quotient graph is obviously isomorphic either to K_2 or to a one-vertex graph with a single loop or semiedge. Such quotients appear worthless and are exluded from further consideration. If N is not edge-transitive then an edge-transitive quotient projection is either a regular covering projection, or else X is bipartite and $X_N \cong K_{1,3}$. In view of these remarks, our extended class $\mathcal{C}$ of graphs and morphisms now includes: all edge-transitive cubic graphs as well as the graph $K_{1,3}$ as objects, and all edge-transitive regular covering projections together with all edge-transitive quotient projections onto $K_{1,3}$ as morphisms. The set of basic graphs consists of the graph $K_{1,3}$ and of all cubic edge-transitive graphs, which are basic relative to edge-transitive regular covering projections and do not have an edge-transitive quotient projection onto $K_{1,3}$. However, we still prefer to view all graphs that are basic in the old sense, as basic. (In the list on up to 768 vertices only the graph **S504** is, formally, not basic anymore.)

When discussing (ir)reducibility of morphisms, the situation with non-covering quotient projections is subtler and more difficult to analyse because of certain anomalies which never occur with regular covering projections. In particular, a reducible quotient projection $q_N \colon X \to X_N$ might not arise as a strong decomposition. This anomaly can occur even if q_N is strongly reducible (that is, one can have a decomposition $q_N = q_M \circ q_K$ where an edge transitive subgroup $H \leq \operatorname{Aut} X$ projects along q_N and along q_K — and hence $\bar{H} = q_K^*(H)$ projects along q_M — however N does not project along q_K). In our case the analysis is simplified to some extent — partly because noncovering quotients involve just a single graph, $K_{1,3}$, and partly because each semisymmetric graph X on up to 768 vertices has a single minimal edge-transitive group (save for the Gray graph). But even this does not seem to be enough to obtain a characterisation of irreducible morphisms comparable to that of Lemma 6.3 (and worse, not even a characterisation of strongly reducible morphisms as in Lemma 6.1). The crucial additional information used in the analysis below is the fact that the minimal

26

edge-transitive subgroup has a very small index in $\operatorname{Aut} X$. Thus, we do not attempt to give a general approach to this problem (not even in the case of cubic graphs), and the special case of the Gray graph is treated separately.

Recall that the Gray graph has four minimal edge-transitive groups, namely the four Sylow 3-subgroups of order 81 (thus acting regularly on the set of edges) having index 16 in the full automorphism group. Let N be a subgroup giving rise to an edge-transitive quotient onto $K_{1,3}$. Then, as N acts transitively on one bipartition set of 27 vertices, $|N|$ is divisible by 27. Now, a Sylow 3-subgroup of N is entirely contained in some Sylow 3-subgroup H of the full automorphism group, and must have order at least 27. So the Sylow 3-subgroup of N is precisely $H \cap N$. As $N \neq H$ we have that $|H \cap N| = 27$. Moreover, since $H \cap \operatorname{Ker} q_N$ cannot be H, we have $H \cap N = H \cap \operatorname{Ker} q_N$. Thus, $H \cap N$ is normal in H. The group $H \cap N$ cannot give rise to a covering projection for the respective quotient would be the dipole $\mathbf{F2}$ (which does not have a graph morphism onto $K_{1,3}$). Consequently, quotienting by $H \cap N$ gives us $K_{1,3}$, and $q_{H \cap N} = q_N$. Now H has a unique minimal normal subgroup $N_0 \cong \mathbb{Z}_3$ (contained in $H \cap N$) which gives rise to a regular covering projection onto $\mathbf{F18}$. So $q_N = q_{H \cap N / N_0} \circ q_{N_0}$ is an edge-transitive strong decomposition. Thus, no edge-transitive quotient of the Gray graph onto $K_{1,3}$ is irreducible.

Lemma 6.5 *Let X be an edge-transitive cubic graph having a unique minimal edge-transitive subgroup $H \leq \operatorname{Aut} X$ of index 1, 2 or 4 in $\operatorname{Aut} X$. If $q_N \colon X \to K_{1,3}$ is an irreducible edge-transitive quotient projection, then $q_N = q_{N_0}$, where N_0 is a minimal normal subgroup of H.*

PROOF. First observe that $N \cap H$ is nontrivial. For otherwise N is a group of order at most 4 and hence $|V(X)| \leq 2|N| \leq 8$ (the first inequality follows from the fact that N acts transitively on one part of the bipartition set of X). So X is either $\mathbf{F8}$ or $K_{3,3}$. Now, although $\mathbf{F8}$ and $K_{3,3}$ do have $K_{1,3}$ as an epimorphic image, neither can be realised as an edge-transitive quotient by a subgroup outside the minimal edge-transitive group. Thus, $N \cap H$ is indeed nontrivial. Consider $N' = \operatorname{Ker} q_N \cap H$. As H normalizes $\operatorname{Ker} q_N$ we have $N' \triangleleft H$. Also, N' is normal in $\operatorname{Ker} q_N$ (because H is normal in $\operatorname{Aut} X$) and hence N normalizes N'. Therefore, there exists a strong edge-transitive decomposition $q_N = q_{N/N \cap N'} \circ q_{N'}$. But q_N is assumed irreducible, implying that $q_{N'} = q_N$. Now let N_0 be a minimal normal subgroup of H contained in N'. Then $q_N = q_{N'} = q_{N'/N_0} \circ q_{N_0}$ is an edge-transitive strong decomposition. Since q_N is assumed irreducible, the decomposition is trivial. Therefore, $q_N = q_{N_0}$. ∎

There are eight semisymmetric graphs in our list having an edge-transitive quotient projection onto $K_{1,3}$ arising from a minimal normal subgroup N of H; namely, the following: **S120**, **S330**, **S336b**, **S336c**, **S504**, **S546b**, **S660** and **S720b**. Moreover, N is uniquely determined since in all cases $H = \mathbb{Z}_3 \times N$. Thus, N is always normal in $\operatorname{Aut} X$.

Note that if N is a minimal normal subgroup of H giving a quotient onto $K_{1,3}$, then q_N might still be reducible (but it is unclear whether such a case exists or not). This is in striking contrast with the situation encountered in Lemma 6.3. Fortunately, nothing like that happens in the case of our finite list.

Lemma 6.6 *Let X be an edge-transitive cubic graph having a unique minimal edge-transitive subgroup $H \leq \operatorname{Aut} X$ of index 1, 2 or 4 in $\operatorname{Aut} X$, and let N be a minimal*

normal subgroup of index 3 in H giving rise to an edge-transitive quotient projection $q_N \colon X \to K_{1,3}$. If q_N is reducible, then there exists a strong edge-transitive decomposition $q_N = q_{N/N \cap L} \circ q_L$, where $L \cong \mathbb{Z}_2$ intersects H trivially.

PROOF. Let $K = \operatorname{Ker} q_K$ be a nontrivial semiregular subgroup contained in $\operatorname{Ker} q_N$ giving rise to an edge-transitive decomposition of q_N. Consider $N' = \operatorname{Ker} q_N \cap H \lhd H$. Since $N \leq N' \leq \operatorname{Ker} q_N$ we have $q_{N'} = q_N$ (and N' is actually the largest subgroup of H with this property). Thus, $N' \neq H$. So $|H : N| = 3$ implies $N' = N$. It follows that $H \cap K \leq H \cap \operatorname{Ker} q_N = N$, and since K is normalized by H we have $H \cap K \lhd H$. But N is a minimal normal subgroup of H. As $H \cap K$ is also semiregular on X and hence not equal to N, we have $H \cap K = 1$. Consequently, K is a group of order 2 or 4. Now, if $H = \operatorname{Aut} X$ then $K = 1$, a contradiction. If $|\operatorname{Aut} X : H| = 2$, let $L = K \cong \mathbb{Z}_2$. If $|\operatorname{Aut} X : H| = 4$ then since H is normal in $\operatorname{Aut} X$, there is always a subgroup H' of index 2 in $\operatorname{Aut} X$ which contains H as an index 2 subgroup. In this case, let $L = K \cap H' \cong \mathbb{Z}_2$. In both cases the group L is normalized by H, and in particular, by N. So we do have an edge-transitive strong decomposition $q_N = q_{N/N \cap L} \circ q_L$. ∎

This lemma guarantees that, in our finite list, all quotients of semisymmetric graphs onto $K_{1,3}$ arising from minimal normal subgroups of H are indeed irreducible. Namely, the only cases where there exists an edge-transitive regular covering projection by the action of a group $\mathbb{Z}_2$ intersecting the minimal edge-transitive subgroup trivially, are the cases **S220 → F110**, **S364 → F182d** and **S420 → S210**. But in all these three cases the minimal edge-transitive subgroup is simple. Hence the respective graphs cannot have an edge-transitive quotient onto $K_{1,3}$.

There are other quotients onto $K_{1,3}$ in the list which are reducible. We now show that these cannot be defined on the semisymmetric graphs listed above, that is, all irreducible quotients in the list are indeed direct. So let $q_{N_0} \colon X \to K_{1,3}$ be an irreducible quotient projection, where N_0 is a minimal normal subgroup of index 3 in the minimal edge-transitive subgroup H of $\operatorname{Aut} X$. Further, let $q_N \colon X \to K_{1,3}$ be reducible. Consider $N' = \operatorname{Ker} q_N \cap H$ which must be normal in H. If $N_0 \leq N'$ then $q_N = q_{N_0}$, a contradiction since q_N is assumed reducible while q_{N_0} is not. So $N' \cap N = 1$, implying that $N' \cong \mathbb{Z}_3$. Hence X is isomorphic to $K_{3,3}$, a contradiction.

The preceeding discussion is summarised in the following theorem.

Theorem 6.7 *In our finite list, each semisymmetric graph having an irreducible edge-transitive quotient projection onto $K_{1,3}$ is a direct object over $K_{1,3}$.* ∎

7   The list

We are now ready to display the list of all connected cubic semisymmetric graphs on up to 768 vertices. For each graph in the list we have computed the order of the minimal edge-transitive group H. Recall that, except in the case of the Gray graph, this group is unique and characteristic in the full automorphism group. We have also computed (up to isomorphism of quotient/covering projections) all its direct quotient and covering projections. As explained in Section 6, these are found by taking the quotient relative to minimal normal subgroups of H (also in the case of the Gray graph), except for the three cases where the quotient is taken by an action of $\mathbb{Z}_2$ outside H. For each

projection we found the respective quotient graph, and the largest subgroup of the full automorphism group that projects. The use of MAGMA [1], and to some extent also the ATLAS [9], proved to be very helpful in our computations.

In summary the following information on these graphs is given in the list:

(i) the order;

(ii) the diameter;

(iii) the girth;

(iv) the order of the full automorphism group, as well as the order of the corresponding vertex stabilisers;

(v) the Goldschmidt type, together with additional relations which are satisfied by the generators of the automorphism group;

(vi) the two distance partitions with their refinements into orbits of the corresponding vertex stabilisers;

(vii) the order of a minimal edge-transitive subgroup (denoted by mET) with some additional information such as its solvability, simplicity, and isomorphism class;

(viii) all of its direct covers in the list, and the corresponding indices of the minimal edge-transitive subgroup in the maximal group that lifts;

(ix) all of its direct quotients (including the arc-transitive ones) and the maximal group that projects;

(x) possible membership of one of the families described in Section 7;

(xi) information on whether the graph is a basic semisymmetric graph, that is, a graph having no smaller regular graph as an edge-transitive quotient;

(xii) additional group relations which give, together with the Goldschmidt type, full information on the automorphism group of the graph and the stabilisers of two adjacent vertices.

The form of the information given in the list needs no explanation for items (i) to (v), (vii), (x) and (xi). The form in which the distance partitions (item (iv)) are given can be explained as follows. For a vertex u of a graph X, let $N_i(u)$ denote the set of vertices which are at distance i from u. Each such set $N_i(u)$ is clearly a union of orbits of the stabiliser $(\text{Aut } X)_u$, and sometimes just a single orbit. Let $\{n_{i,1}, n_{i,2}, \ldots, n_{i,t_i}\}$ be the set of lengths of the orbits of $N_i(u)$, and let $k_{i,j}$ denote the number of these orbits of length $n_{i,j}$, for $1 \leq j \leq t_i$. The distance partition of the graph X relative to the vertex u is then defined as the sequence

$$(n_{0,1}^{k_{0,1}}, \ n_{1,1}^{k_{1,1}} \ldots n_{1,t_i}^{k_{1,t_i}}, \ \ldots, \ n_{d,1}^{k_{d,1}} \ldots n_{d,t_i}^{k_{d,t_i}}),$$

where d is the diameter of the graph X.

Clearly, any two vertices in the same orbit of $\text{Aut } X$ have identical distance partitions, and since the automorphism group of a semisymmetric graph has two orbits on vertices, two sequences are given for every graph in our list. If there are no vertices at distance d from u (which can occur if the longest geodesics begin and end in the orbit not containing u), then we take the last symbol in the above sequence to be 0. For example, in the case of the graph **S112**, it can be read from the sequences $(1^1, 3^1, 3^2, 3^4, 3^8, 1^1 3^{11}, 3^8, 1^1 3^2, 1^1)$ and $(1^1, 3^1, 3^2, 3^4, 3^8, 1^1 3^{11}, 1^1 3^8, 1^1 3^2, 0)$

that the number of vertices at distance 6 from a given vertex u is either $8 \cdot 3 = 24$ or $8 \cdot 3 + 1 \cdot 1 = 25$. The vertex stabiliser $(\text{Aut } X)_u$ has 8 orbits of length 3 on $N_6(u)$ in the first case, and has 8 orbits of length 3 and one of length 1 in the second case.

Information regarding direct covers and direct quotients (items (viii) and (ix)) for each graph X in our list is given in a table with three sections: a top section, a middle section, and a bottom section, with the middle section containing only the name of the graph X.

The bottom section is divided by vertical lines into several subsections, one for each of the direct quotients of the graph X. (If there are no direct quotients, that is, if X is a basic graph, then the bottom section is omitted, and if there is exactly one direct quotient, there is no division into subsections.) Each subsection is again divided into an upper and lower part. The lower part gives the name of the quotient graph, which can be either a semisymmetric cubic graph (appearing elsewhere in the list), or a symmetric cubic graph (appearing in [7]), or $K_{1,3}$. In the upper part, three pieces of information are given. First, the symbol "Cov" is used to indicate that the quotient projection is a regular covering projection, and otherwise "Quo" is used to indicate that it is not. The second piece of information refers to the largest possible subgroup of $\text{Aut } X$ that projects along the quotient projection; "$\text{Aut } X$" is written if this is the full automorphism group $\text{Aut } X$ itself, and otherwise only the index of the minimal edge-transitive group of X in this maximal subgroup is given, and denoted by $[\_ : \text{mET}]$. The third piece of information is a description of the subgroup with respect to which the quotient projection is defined.

Similarly the top section is divided by vertical lines into several subsections, one for each graph in the list which is a direct cover of X. The upper part of each subsection gives the name of the covering graph, while the lower part gives two pieces of information: first, either "$\text{Aut } X$" if $\text{Aut } X$ lifts along the covering projection, or otherwise the index $|\_ : \text{mET}|$ of the minimal edge-transitive group in the largest subgroup of $\text{Aut } X$ that so lifts, and second, the voltage group (the group of covering transformations). If there are no such covering graphs in our list, then this is indicated by two sets of three dots.

S54 (The Gray graph), families **III** and **VII**
G-Type: G_2^4 additional relations: $xyx\bar{y}\bar{x}y\bar{x}\bar{y}$

Diam $X = 6$,   Girth $X = 8$
$|\mathrm{Aut}\, X| = 1296 = 2^4 \cdot 3^4$
$|(\mathrm{Aut}\, X)_u| = 48 = 2^4 \cdot 3$

Stab-Part:
$(1^1, 3^1, 6^1, 12^1, 4^1 12^1, 12^1, 4^1)$
$(1^1, 3^1, 6^1, 12^1, 12^1, 12^1, 8^1)$

$|\mathrm{mET}| = 81$, solvable
(The four Syl 3-subgroups)

S216	**S378**	**S702b**
$|\_ : \mathrm{mET}| = 2$ $\mathbb{Z}_2^2$	$|\_ : \mathrm{mET}| = 1$ $\mathbb{Z}_7$	$|\_ : \mathrm{mET}| = 1$ $\mathbb{Z}_{13}$
$X = $ **S54**		
Cov $|\_ : \mathrm{mET}| = 4$ $\mathbb{Z}_3$		
F18		

S110, family **II**, basic
G-Type: G_2^1 additional relations: $cd(\bar{x}\bar{y})^2 x\bar{y}(\bar{x}\bar{y})^2$

Diam $X = 7$,   Girth $X = 10$
$|\mathrm{Aut}\, X| = 1320 = 2^3 \cdot 3 \cdot 5 \cdot 11$, $PGL(2,11)$
$|(\mathrm{Aut}\, X)_u| = 2^3 \cdot 3$

StabPart:
$(1^1, 3^1, 6^1, 12^1, 12^2, 12^1 24^1, 12^2, 4^1)$
$(1^1, 3^1, 6^1, 12^1, 24^1, 12^1 24^1, 4^1 8^1 12^1, 4^1)$

$|\mathrm{mET}| = 660$, simple, PSL(2,11)

S330
Aut X $\mathbb{Z}_3$
$X = $ **S110**

S112, family **IV**
G-Type: G_1 additional relations: $xyx\bar{y}xy(\bar{x}\bar{y})^2$

Diam $X = 8$,   Girth $X = 10$
$|\mathrm{Aut}\, X| = 168 = 2^3 \cdot 3 \cdot 7$
$|(\mathrm{Aut}\, X)_u| = 3$

Stab-Part:
$(1^1, 3^1, 3^2, 3^4, 3^8, 1^1 3^{11}, 3^8, 1^1 3^2, 1^1)$
$(1^1, 3^1, 3^2, 3^4, 3^8, 1^1 3^{11}, 1^1 3^8, 1^1 3^2, 0)$

$|\mathrm{mET}| = 168$, solvable

S336a	**S448**
Aut X $\mathbb{Z}_3$	Aut X $\mathbb{Z}_2^2$
$X = $ **S112**	
Cov Aut X $\mathbb{Z}_2^3$	
F14	

S120, family **IIV**
G-Type: G_1^3 additional relations: $(x\bar{y}\bar{x}\bar{y})^2$

Diam $X = 8$,   Girth $X = 8$
$|\mathrm{Aut}\, X| = 720 = 2^4 \cdot 3^2 \cdot 5$
$|(\mathrm{Aut}\, X)_u| = 12 = 2^2 \cdot 3$

StabPart:
$(1^1, 3^1, 6^1, 12^1, 6^1 12^1, 6^2 12^1, 3^1 12^2, 3^1 6^1 12^1, 2^1 6^1)$
$(1^1, 3^1, 6^1, 12^1, 6^1 12^1, 6^2 12^1, 2^1 6^4, 3^1 6^1 12^1, 1^1 2^1 6^1)$

$|\mathrm{mET}| = 180$, nonsolvable, $\mathbb{Z}_3 \times A_5$

S240	
Aut X $\mathbb{Z}_2$	
$X = $ **S120**	
Cov Aut X $\mathbb{Z}_3$	Quo Aut X A_5
F40	$K_{1,3}$

S126 (Tutte's 12-cage), basic
G-Type: G_4^1 additional relations: $a^3b(xy)^3(\bar{x}\bar{y})^3$

  Diam $X = 6$,   Girth $X = 12$
  $|\text{Aut}\,X| = 12096 = 2^6 \cdot 3^3 \cdot 7,\ U_3(3)\dot{.}2$
  $|(\text{Aut}\,X)_u| = 192 = 2^6 \cdot 3$

  StabPart:
  $(1^1, 3^1, 6^1, 12^1, 24^1, 48^1, 32^1)$
  $(1^1, 3^1, 6^1, 12^1, 24^1, 48^1, 32^1)$

  $|\text{mET}| = 6048$, simple, $U_3(3)$

. . .
. . .
$X = \mathbf{S126}$

S144, family **V**
G-Type: G_1^1 additional relations: $(\bar{x}\bar{y})^4, xy\bar{x}yxy\bar{x}\bar{y}x\bar{y}\bar{x}\bar{y}$

  Diam $X = 8$,   Girth $X = 8$
  $|\text{Aut}\,X| = 432 = 2^4 \cdot 3^3$
  $|(\text{Aut}\,X)_u| = 6 = 2 \cdot 3$

  Stab-Part:
  $(1^1, 3^1, 6^1, 6^2, 3^16^3, 3^26^5, 3^16^6, 3^36^2, 1^23^1)$
  $(1^1, 3^1, 6^1, 6^2, 3^16^3, 3^26^5, 1^12^13^36^3, 3^36^2, 2^16^2)$

  $|\text{mET}| = 216$, solvable

S432b	**S576**
Aut X	Aut X
$\mathbb{Z}_3$	$\mathbb{Z}_2^2$
$X = \mathbf{S144}$	
Cov	
Aut X	
$\mathbb{Z}_3^2$	
F16	

S182, family **II**, basic
G-Type: G_2^1 additional relations: $(\bar{x}\bar{y})^6, cd\bar{x}yx\bar{y}x\bar{y}xy(\bar{x}\bar{y})^2$

  Diam $X = 8$,   Girth $X = 12$
  $|\text{Aut}\,X| = 2184 = 2^3 \cdot 3 \cdot 7 \cdot 13,\ PGL(2,13)$
  $|(\text{Aut}\,X)_u| = 24 = 2^3 \cdot 3$

  StabPart:
  $(1^1, 3^1, 6^1, 12^1, 12^2, 24^2, 12^224^1, 4^112^2, 12^1)$
  $(1^1, 3^1, 6^1, 12^1, 24^2, 24^2, 8^112^224^1, 4^112^2, 4^1)$

  $|\text{mET}| = 1092$, simple, $PSL(2,13)$

S546b
Aut X
$\mathbb{Z}_3$
$X = \mathbf{S182}$

S210, basic
G-Type: G_2^4 additional relations: $cd^2e\bar{x}\bar{y}(\bar{x}yx\bar{y})^2, (\bar{x}\bar{y})^7$

  Diam $X = 8$,   Girth $X = 10$
  $|\text{Aut}\,X| = 5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7,\ S_7$
  $|(\text{Aut}\,X)_u| = 48 = 2^4 \cdot 3$

  Stab-Part:
  $(1^1, 3^1, 6^1, 12^1, 12^2, 12^124^1, 12^224^1, 6^124^2, 2^124^1)$
  $(1^1, 3^1, 6^1, 12^1, 24^1, 12^124^1, 6^148^1, 6^124^2, 8^112^1)$

  $|\text{mET}| = 2520$, simple, A_7

S420	**S630**
Aut X	Aut X
$\mathbb{Z}_2$	$\mathbb{Z}_3$
$X = \mathbf{S210}$	

S216

G-Type: G_1^1 additional relations: $(xy)^3(\bar{x}\bar{y})^3, (x\bar{y}\bar{x}\bar{y})^3, x(y\bar{x})^4yx\bar{y}\bar{x}\bar{y}$

Diam $X = 8$,   Girth $X = 12$
$|\text{Aut } X| = 648 = 2^3 \cdot 3^4$
$|(\text{Aut } X)_u| = 6 = 2 \cdot 3$

StabPart:
$(1^1, 3^1, 6^1, 6^2, 6^4, 6^8, 2^1 3^2 6^{10}, 3^5 6^5, 1^2 2^2 3^1)$
$(1^1, 3^1, 6^1, 6^2, 6^4, 6^8, 2^1 3^2 6^{10}, 3^5 6^5, 3^3)$

$|\text{mET}| = 324$, solvable

S432c	
Aut X	
$\mathbb{Z}_2$	
$X = $ **S216**	
Cov	Cov
Aut X	Aut X
$\mathbb{Z}_3$	$\mathbb{Z}_2^2$
F72	**S54**

S220

G-Type: G_1^3 additional relations: $c\bar{x}\bar{y}(x\bar{y}\bar{x}\bar{y})^2, (\bar{x}\bar{y})^6$

Diam $X = 10$,   Girth $X = 10$
$|\text{Aut } X| = 1320 = 2^3 \cdot 3 \cdot 5 \cdot 11$
$|(\text{Aut } X)_u| = 12 = 2^2 \cdot 3$

StabPart:
$(1^1, 3^1, 6^1, 12^1, 12^2, 6^1 12^3, 6^3 12^3, 2^1 6^4 12^2, 6^2 12^1, 3^1, 1^1)$
$(1^1, 3^1, 6^1, 12^1, 12^2, 6^1 12^3, 6^5 12^2, 2^1 6^4 12^2, 6^4, 3^1, 1^1)$

$|\text{mET}| = 660$, simple, PSL$(2, 11)$

S660
Aut X
$\mathbb{Z}_3$
$X = $ **S220**
Cov
Aut X
$\mathbb{Z}_2$
F110

S240

G-Type: G_1^3 additional relations: $x\bar{y}(xy)^2\bar{x}y(\bar{x}\bar{y})^2, xy\bar{x}\bar{y}x\bar{y}xy\bar{x}\bar{y}$

Diam $X = 10$,   Girth $X = 12$
$|\text{Aut } X| = 1440 = 2^5 \cdot 3^2 \cdot 5$
$|(\text{Aut } X)_u| = 12 = 2^2 \cdot 3$

StabPart:
$(1^1, 3^1, 6^1, 12^1, 12^2, 12^4, 6^2 12^4, 3^1 6^2 12^3, 1^1 2^1 12^2, 6^1, 2^1)$
$(1^1, 3^1, 6^1, 12^1, 12^2, 12^4, 4^1 6^2 12^4, 3^1 6^2 12^3, 1^1 2^2 6^3, 6^1, 2^1)$

$|\text{mET}| = 360$, nonsolvable

$\cdots$	
$\cdots$	
$X = $ **S240**	
Cov	Cov
Aut X	Aut X
$\mathbb{Z}_2$	$\mathbb{Z}_3$
S120	**F80**

S294, family III

G-Type: G_1^2 additional relations: $(x\bar{y}\bar{x}\bar{y})^3, xy(\bar{x}yx\bar{y})^2\bar{x}\bar{y}, (\bar{x}y\bar{x}\bar{y})^3$

Diam $X = 9$,   Girth $X = 12$
$|\text{Aut } X| = 882 = 2 \cdot 3^2 \cdot 7^2$
$|(\text{Aut } X)_u| = 6 = 2 \cdot 3$

StabPart:
$(1^1, 3^1, 3^2, 6^2, 6^4, 6^8, 3^4 6^9, 3^4 6^{10}, 1^2 3^{12} 6^2, 3^2 6^1)$
$(1^1, 3^1, 6^1, 6^2, 6^4, 6^8, 2^1 6^{12}, 3^4 6^{10}, 3^2 6^6, 3^2 6^1)$

$|\text{mET}| = 441$, solvable

$\cdots$
$\cdots$
$X = $ **S294**
Cov
mET
$\mathbb{Z}_7$
F42

S330, family **VII**

G-Type: G_2^1 additional relations: $d^2\bar{x}yxy\bar{x}y(\bar{x}\bar{y})^3$

Diam $X = 10$,   Girth $X = 12$

$|\mathrm{Aut}\,X| = 3960 = 2^3 \cdot 3^2 \cdot 5 \cdot 11$

$|(\mathrm{Aut}\,X)_u| = 24 = 2^3 \cdot 3$

StabPart:

$(1^1, 3^1, 6^1, 12^1, 12^2, 24^2, 12^1 24^2, 12^2 24^2, 6^1 12^4, 6^1 12^2, 2^1 6^1 12^1)$
$(1^1, 3^1, 6^1, 12^1, 24^1, 24^2, 8^1 12^1 24^2, 12^2 24^2, 12^1 24^2, 6^1 12^2, 2^1 4^1)$

$|\mathrm{mET}| = 1980$, nonsolvable, $\mathbb{Z}_3 \times \mathrm{PSL}(2,11)$

. . .	
. . .	
$X =$ **S330**	
Cov	Quo
Aut X	Aut X
$\mathbb{Z}_3$	$\mathrm{PSL}(2,11)$
S110	$K_{1,3}$

S336a

G-Type: G_1 additional relations: $(\bar{x}\bar{y})^6, (x\bar{y}\bar{x}\bar{y})^3, xy(\bar{x}yx\bar{y})^2\bar{x}\bar{y}$

Diam $X = 10$,   Girth $X = 12$

$|\mathrm{Aut}\,X| = 504 = 2^3 \cdot 3^2 \cdot 7$

$|(\mathrm{Aut}\,X)_u| = 3$

StabPart:

$(1^1, 3^1, 3^2, 3^4, 3^8, 3^{16}, 3^{23}, 3^{26}, 1^2 3^{21}, 3^9, 1^3)$
$(1^1, 3^1, 3^2, 3^4, 3^8, 3^{16}, 1^2 3^{25}, 3^{26}, 3^{19}, 3^9, 1^3)$

$|\mathrm{mET}| = \mathrm{Aut}\,X$, solvable

. . .	
. . .	
$X =$ **S336a**	
Cov	Cov
Aut X	Aut X
$\mathbb{Z}_3$	$\mathbb{Z}_2^3$
S112	**F42**

S336b, family **VII**

G-Type: G_1^1 additional relations: $x\bar{y}(\bar{x}\bar{y})^2 x\bar{y}(\bar{x}\bar{y})^2, \bar{x}yx\bar{y}xy\bar{x}\bar{y}xy\bar{x}\bar{y}$

Diam $X = 10$,   Girth $X = 12$

$|\mathrm{Aut}\,X| = 1008 = 2^4 \cdot 3^2 \cdot 7$

$|(\mathrm{Aut}\,X)_u| = 6 = 2 \cdot 3$

StabPart:

$(1^1, 3^1, 6^1, 6^2, 6^4, 6^8, 2^1 3^1 6^{11}, 3^4 6^{12}, 2^1 3^3 6^9, 3^1 6^3, 1^1)$
$(1^1, 3^1, 6^1, 6^2, 6^4, 6^8, 3^3 6^9, 3^4 6^{12}, 2^1 3^1 6^{11}, 3^1 6^3, 3^1)$

$|\mathrm{mET}| = 504$, nonsolvable, $\mathbb{Z}_3 \times \mathrm{PSL}(2,7)$

S672a	
Aut X	
$\mathbb{Z}_2$	
$X =$ **S336b**	
Cov	Quo
Aut X	Aut X
$\mathbb{Z}_3$	$\mathrm{PSL}(2,7)$
F112a	$K_{1,3}$

S336c, family **VII**

G-Type: G_1^3 additional relations: $xy(\bar{x}y)^2 x\bar{y}(\bar{x}\bar{y})^2, x\bar{y}\bar{x}\bar{y}\bar{x}yx\bar{y}xy(\bar{x}\bar{y})^2$

Diam $X = 10$,   Girth $X = 12$

$|\mathrm{Aut}\,X| = 2016 = 2^5 \cdot 3^2 \cdot 7$

$|(\mathrm{Aut}\,X)_u| = 12 = 2^2 \cdot 3$

Stab-Part:

$(1^1, 3^1, 6^1, 12^1, 12^2, 12^4, 6^1 12^6, 6^3 12^6, 2^1 6^7 12^1, 3^1 6^2, 1^1 2^1)$
$(1^1, 3^1, 6^1, 12^1, 12^2, 12^4, 6^1 12^6, 6^3 12^6, 2^1 6^3 12^3, 3^1 6^2, 3^1)$

$|\mathrm{mET}| = 504$, nonsolvable, $\mathbb{Z}_3 \times \mathrm{PSL}(2,7)$

S672b	
Aut X	
$\mathbb{Z}_2$	
$X =$ **S336c**	
Cov	Quo
Aut X	Aut X
$\mathbb{Z}_3$	$PSL(2,7)$
F112b	$K_{1,3}$

S364

G-Type: G_1^3 additional relations: $c(\bar{x}\bar{y})^2(x\bar{y})^2(\bar{x}\bar{y})^2, (xyx\bar{y}\bar{x}\bar{y})^2$

  Diam $X = 12$,   Girth $X = 12$

  $|\text{Aut } X| = 2184 = 2^3 \cdot 3 \cdot 7 \cdot 13$, $\mathbb{Z}_2 \times \text{PSL}(2,13)$

  $|(\text{Aut } X)_u| = 12 = 2^2 \cdot 3$

  StabPart:

  $(1^1, 3^1, 6^1, 12^1, 12^2, 12^4, 6^1 12^5, 6^4 12^4, 6^3 12^4, 2^1 6^1 12^3, 6^1 12^1, 3^1, 1^1)$

  $(1^1, 3^1, 6^1, 12^1, 12^2, 12^4, 6^3 12^4, 6^4 12^4, 6^5 12^3, 2^1 6^1 12^3, 6^1 12^1, 3^1, 1^1)$

  $|\text{mET}| = 1092$, simple, $\text{PSL}(2,13)$

...
...
$X = $ **S364**
Cov Aut X $\mathbb{Z}_2 = Z(\text{Aut } X)$
F182d

S378

G-Type: G_1 additional relations: $(xy)^3(\bar{x}\bar{y})^3, (x\bar{y}\bar{x}\bar{y})^3, x(y\bar{x})^2(\bar{y}x)^2(y\bar{x})^4\bar{y}$

  Diam $X = 10$,   Girth $X = 12$

  $|\text{Aut } X| = 567 = 3^4 \cdot 7$

  $|(\text{Aut } X)_u| = 3$

  StabPart:

  $(1^1, 3^1, 3^2, 3^4, 3^8, 3^{16}, 1^2 3^{22}, 3^{23}, 3^{24}, 3^{19}, 3^6)$

  $(1^1, 3^1, 3^2, 3^4, 3^8, 3^{16}, 1^2 3^{22}, 3^{23}, 3^{24}, 3^{19}, 1^6 3^4)$

  $|\text{mET}| = \text{Aut } X$, solvable

...	
...	
$X = $ **S378**	
Cov Aut X $\mathbb{Z}_3$	Cov Aut X $\mathbb{Z}_7$
F126	**S54**

S384

G-Type: G_1^3 additional relations: $(\bar{x}\bar{y})^6, xy\bar{x}yxy\bar{x}\bar{y}x\bar{y}\bar{x}\bar{y}$

  Diam $X = 10$,   Girth $X = 12$

  $|\text{Aut } X| = 2304 = 2^8 \cdot 3^2$

  $|(\text{Aut } X)_u| = 12 = 2^2 \cdot 3$

  Stab-Part:

  $(1^1, 3^1, 6^1, 12^1, 12^2, 12^4, 6^3 12^4, 6^6 12^4, 3^1 6^8 12^3, 3^1 6^1 12^3, 2^1 6^1)$

  $(1^1, 3^1, 6^1, 12^1, 12^2, 12^4, 4^1 6^3 12^4, 6^6 12^4, 2^1 6^4 12^4, 3^1 6^1 12^3, 1^1 4^1 12^1)$

  $|\text{mET}| = 576$, solvable

S768a	**S768b**
Aut X $\mathbb{Z}_2$	Aut X $\mathbb{Z}_2$
$X = $ **S384**	
Cov Aut X $\mathbb{Z}_2^2$	
F96b	

S400, family **V**

G-Type: G_1 additional relations: $(\bar{x}\bar{y})^4, (x\bar{y})^6, xyx\bar{y}\bar{x}yx(y\bar{x})^2\bar{y}x\bar{y}x\bar{y}$

  Diam $X = 10$,   Girth $X = 8$

  $|\text{Aut } X| = 600 = 2^3 \cdot 3 \cdot 5^2$

  $|(\text{Aut } X)_u| = 3$

  StabPart:

  $(1^1, 3^1, 3^2, 3^4, 3^7, 3^{12}, 3^{20}, 3^{32}, 1^1 3^{36}, 1^2 3^{17}, 3^1)$

  $(1^1, 3^1, 3^2, 3^4, 3^7, 3^{12}, 3^{20}, 3^{32}, 3^{36}, 1^2 3^{17}, 1^1 3^1)$

  $|\text{mET}| = \text{Aut } X$, solvable

...
...
$X = $ **S400**
Cov Aut X $\mathbb{Z}_5^2$
F16

S420

G-Type: G_2^4 additional relations: $cd\bar{x}y\bar{x}\bar{y}xyx\bar{y}xy\bar{x}\bar{y}, (\bar{x}\bar{y})^7$

  Diam $X = 10$,   Girth $X = 12$
  $|\text{Aut}\,X| = 10080 = 2^5 \cdot 3^2 \cdot 5 \cdot 7,\ \mathbb{Z}_2 \times (\text{mET} \rtimes \mathbb{Z}_2)$
  $|(\text{Aut}\,X)_u| = 48 = 2^4 \cdot 3$

  StabPart:
  $(1^1, 3^1, 6^1, 12^1, 12^2, 24^2, 12^1 24^3, 12^2 24^2 48^1, 4^1 6^1 12^1 24^3, 3^1 24^1, 1^1)$
  $(1^1, 3^1, 6^1, 12^1, 24^1, 24^2, 12^1 24^1 48^1, 12^2 24^2 48^1, 6^1 8^1 24^1 48^1, 3^1 24^1, 1^1 8^1)$

  $|\text{mET}| = 2520$, simple, A_7

...
...
$X = \mathbf{S420}$
Cov Aut X $\mathbb{Z}_2 = Z(\text{Aut}\,X)$
$\mathbf{S210}$

S432a

G-Type: G_1 additional relations: $(x\bar{y}\bar{x}y)^2(\bar{x}\bar{y})^3, (xy)^2\bar{x}y(x\bar{y})^2(\bar{x}\bar{y})^2$

  Diam $X = 10$,   Girth $X = 14$
  $|\text{Aut}\,X| = 648 = 2^3 \cdot 3^4$
  $|(\text{Aut}\,X)_u| = 3$

  StabPart:
  $(1^1, 3^1, 3^2, 3^4, 3^8, 3^{16}, 3^{32}, 1^2 3^{39}, 1^2 3^{26}, 1^1 3^{11}, 3^3)$
  $(1^1, 3^1, 3^2, 3^4, 3^8, 3^{16}, 3^{32}, 1^2 3^{39}, 1^2 3^{26}, 1^1 3^{11}, 3^3)$

  $|\text{mET}| = \text{Aut}\,X$, solvable

...
...
$X = \mathbf{S432a}$
Cov Aut X $\mathbb{Z}_3$
$\mathbf{F144a}$

S432b

G-Type: G_1^1 additional relations: $(yx)^6, xy\bar{x}yxy\bar{x}\bar{y}x\bar{y}\bar{x}\bar{y}, (\bar{x}y)^2\bar{x}\bar{y}xy\bar{x}\bar{y}xy(\bar{x}\bar{y})^2$

  Diam $X = 10$,   Girth $X = 12$
  $|\text{Aut}\,X| = 1296 = 2^4 \cdot 3^4$
  $|(\text{Aut}\,X)_u| = 6 = 2 \cdot 3$

  StabPart:
  $(1^1, 3^1, 6^1, 6^2, 6^4, 6^8, 3^1 6^{12}, 3^2 6^{19}, 2^3 3^1 6^{15}, 3^3 6^4, 1^2 3^1 6^1)$
  $(1^1, 3^1, 6^1, 6^2, 6^4, 6^8, 2^2 3^1 6^{12}, 3^2 6^{19}, 2^4 3^3 6^{14}, 3^3 6^4, 1^1 2^2)$

  $|\text{mET}| = 648$, solvable

...	
...	
$X = \mathbf{S432b}$	
Cov Aut X $\mathbb{Z}_3$	Cov Aut X $\mathbb{Z}_3^2$
$\mathbf{S144}$	$\mathbf{F48}$

S432c

G-Type: G_1^1 additional relations: $(xy)^3(\bar{x}\bar{y})^3, x(y\bar{x})^4 yx\bar{y}\bar{x}\bar{y}$

  Diam $X = 12$,   Girth $X = 12$
  $|\text{Aut}\,X| = 1296 = 2^4 \cdot 3^4$
  $|(\text{Aut}\,X)_u| = 6 = 2 \cdot 3$

  StabPart:
  $(1^1, 3^1, 6^1, 6^2, 6^4, 6^8, 2^1 6^{13}, 3^2 6^{17}, 1^2 2^4 6^{14}, 3^2 6^6, 1^2 2^1 6^1, 3^1, 1^1)$
  $(1^1, 3^1, 6^1, 6^2, 6^4, 6^8, 2^1 6^{13}, 3^2 6^{17}, 3^4 6^{14}, 3^2 6^6, 2^1 6^1, 3^1, 1^1)$

  $|\text{mET}| = 648$, solvable

...	
...	
$X = \mathbf{S432c}$	
Cov Aut X $\mathbb{Z}_2$	Cov Aut X $\mathbb{Z}_3$
$\mathbf{S216}$	$\mathbf{F144b}$

S448, family **IV**

G-Type: G_1 additional relations: $(\bar{x}\bar{y})^6, (x\bar{y}\bar{x}\bar{y})^3, (x(y\bar{x})^2\bar{y})^2$

  Diam $X = 10$,   Girth $X = 12$
  $|\text{Aut}\,X| = 672 = 2^5 \cdot 3 \cdot 7$
  $|(\text{Aut}\,X)_u| = 3$

  StabPart:
  $(1^1, 3^1, 3^2, 3^4, 3^8, 3^{16}, 3^{26}, 1^1 3^{36}, 3^{37}, 1^1 3^{17}, 1^1 3^1)$
  $(1^1, 3^1, 3^2, 3^4, 3^8, 3^{16}, 3^{26}, 1^1 3^{36}, 3^{35}, 1^1 3^{17}, 1^1 3^3)$

  $|\text{mET}| = \text{Aut}\,X$, solvable

...	
...	
$X = \mathbf{S448}$	
Cov Aut X $\mathbb{Z}_2^2$	Cov Aut X $\mathbb{Z}_2^3$
$\mathbf{S112}$	$\mathbf{F56a}$

S486
G-Type: G_1^1 additional relations: $(xy)^3(\bar{x}\bar{y})^3, (x\bar{y}\bar{x}\bar{y})^3, x(y\bar{x})^7yx\bar{y}\bar{x}\bar{y}$
  Diam $X = 12$,   Girth $X = 12$
  $|\text{Aut}\,X| = 1458 = 2\cdot 3^6$
  $|(\text{Aut}\,X)_u| = 6 = 2\cdot 3$

  StabPart:
  $(1^1, 3^1, 6^1, 6^2, 6^4, 6^8, 2^1 3^2 6^{10}, 3^3 6^{10}, 3^2 6^{11}, 3^1 6^{13}, 3^2 6^{10}, 3^4 6^3, 3^2)$
  $(1^1, 3^1, 6^1, 6^2, 6^4, 6^8, 2^1 3^2 6^{10}, 3^3 6^{10}, 3^2 6^{11}, 3^1 6^{13}, 3^2 6^{10}, 3^4 6^3, 3^2)$

  $|\text{mET}| = 729$, solvable

$\cdots$	
$\cdots$	
$X =$ **S486**	
Cov Aut X $\mathbb{Z}_3$	
F162a	

S504, family **VII**, basic
G-Type: G_1^2 additional relations: $(\bar{x}\bar{y})^6, c\bar{x}yx\bar{y}xy(x\bar{y})^2(\bar{x}\bar{y})^2$
  Diam $X = 10$,   Girth $X = 12$
  $|\text{Aut}\,X| = 1512 = 2^3\cdot 3^3\cdot 7$, $\text{PSL}(2,8)\rtimes\mathbb{Z}_3$
  $|(\text{Aut}\,X)_u| = 6 = 2\cdot 3$

  StabPart:
  $(1^1, 3^1, 3^2, 6^2, 6^4, 6^8, 6^{15}, 6^{22}, 2^1 3^6 6^{17}, 3^3 6^8, 1^2 2^2 3)$
  $(1^1, 3^1, 6^1, 6^2, 6^4, 6^8, 6^{15}, 6^{22}, 6^{19}, 3^3 6^8, 2^1 3^1 6^2)$

  $|\text{mET}| = \text{Aut}\,X$, nonsolvable

$\cdots$	
$\cdots$	
$X =$ **S504**	
Quo Aut X $\text{PSL}(2,8)$	
$K_{1,3}$	

S506, family **II**, basic
G-Type: G_2^1 additional relations: $cd^2(\bar{x}\bar{y})^2(xy)^2(\bar{x}\bar{y})^2$
  Diam $X = 11$,   Girth $X = 12$
  $|\text{Aut}\,X| = 6072 = 2^3\cdot 3\cdot 11\cdot 23$, $\text{PSL}(2,23)$
  $|(\text{Aut}\,X)_u| = 24 = 2^4\cdot 3$

  StabPart:
  $(1^1, 3^1, 6^1, 12^1, 12^2, 24^2, 12^2 24^2, 12^2 24^3, 6^1 12^3 24^3, 4^1 6^1 12^2 24^2, 12^3, 12^1)$
  $(1^1, 3^1, 6^1, 12^1, 24^1, 24^2, 24^3, 12^2 24^3, 8^1 12^3 24^3, 4^1 6^1 12^2 24^2, 4^1 6^1 24^1, 12^1)$

  $|\text{mET}| = \text{Aut}\,X$, simple

$\cdots$	
$\cdots$	
$X =$ **S506**	

S546a, family **III**
G-Type: G_1 additional relations: $(x\bar{y}\bar{x}\bar{y})^3, (\bar{x}y\bar{x}\bar{y})^3, x\bar{y}(xy)^2(x\bar{y})^2(\bar{x}\bar{y})^2, xyx\bar{y}\bar{x}yxy\bar{x}\bar{y}xy(\bar{x}\bar{y})^2$
  Diam $X = 10$,   Girth $X = 12$
  $|\text{Aut}\,X| = 819 = 3^2\cdot 7\cdot 13$
  $|(\text{Aut}\,X)_u| = 3$

  StabPart:
  $(1^1, 3^1, 3^2, 3^4, 3^8, 3^{16}, 3^{28}, 3^{40}, 3^{44}, 3^{30}, 1^2 3^8)$
  $(1^1, 3^1, 3^2, 3^4, 3^8, 3^{16}, 3^{28}, 3^{40}, 1^2 3^{46}, 3^{30}, 3^6)$

  $|\text{mET}| = \text{Aut}\,X$, solvable

$\cdots$	
$\cdots$	
$X =$ **S546a**	
Cov Aut X $\mathbb{Z}_7$	Cov Aut X $\mathbb{Z}_{13}$
F78	**F42**

S546b, family **VII**
G-Type: G_2^1 additional relations: $(\bar{x}\bar{y})^6, cdx(y\bar{x})^2\bar{y}\bar{x}yx\bar{y}\bar{x}y(\bar{x}\bar{y})^2$
  Diam $X = 12$,   Girth $X = 12$
  $|\text{Aut}\,X| = 6552 = 2^3\cdot 3^2\cdot 7\cdot 13$
  $|(\text{Aut}\,X)_u| = 24 = 2^3\cdot 3$

  StabPart:
  $(1^1, 3^1, 6^1, 12^1, 12^2, 24^2, 12^1 24^3, 12^2 24^5, 12^5 24^3, 12^3 24^1, 6^2 12^1, 6^1, 2^1)$
  $(1^1, 3^1, 6^1, 12^1, 24^1, 24^2, 12^1 24^3, 12^2 24^5, 8^1 12^1 24^5, 12^3 24^1, 4^1 12^1, 6^1, 2^1)$

  $|\text{mET}| = 3276$, nonsolvable, $\mathbb{Z}_3\times\text{PSL}(2,13)$

$\cdots$	
$\cdots$	
$X =$ **S546b**	
Cov Aut X $\mathbb{Z}_3$	Quo Aut X $\text{PSL}(2,13)$
S182	$K_{1,3}$

S576

G-Type: G_1^1 additional relations: $(\bar{x}\bar{y})^4, (x\bar{y})^6, (xy\bar{x}\bar{y})^4, x\bar{y}(\bar{x}y)^2 x\bar{y}x(y\bar{x})^2\bar{y}x\bar{y}x\bar{y}$

Diam $X = 12$,   Girth $X = 8$

$|\text{Aut }X| = 1728 = 2^6 \cdot 3^3$

$|(\text{Aut }X)_u| = 6 = 2 \cdot 3$

StabPart:
$(1^1, 3^1, 6^1, 6^2, 3^1 6^3, 3^2 6^5, 3^2 6^9, 3^2 6^{15}, 3^3 6^{22}, 3^6 6^{19}, 1^2 3^2 6^8, 3^1 6^1, 3^1)$
$(1^1, 3^1, 6^1, 6^2, 3^1 6^3, 3^2 6^5, 3^2 6^9, 3^2 6^{15}, 3^3 6^{22}, 3^6 6^{19}, 1^1 2^2 3^3 6^7, 3^1 6^1, 3^1)$

$|\text{mET}| = 864$, solvable

...	
...	
$X = \textbf{S576}$	
Cov	Cov
Aut X	Aut X
$\mathbb{Z}_2^2$	$\mathbb{Z}_3^2$
S144	**F64**

S630

G-Type: G_2^4 additional relations: $cd^2 e\bar{x}\bar{y}(\bar{x}y\bar{x}\bar{y})^2$

Diam $X = 14$,   Girth $X = 10$

$|\text{Aut }X| = 15120 = 2^4 \cdot 3^3 \cdot 5 \cdot 7$

$|(\text{Aut }X)_u| =$

StabPart:
$(1^1, 3^1, 6^1, 12^1, 12^2, 12^1 24^1, 12^2 24^1, 6^1 24^1 48^1, 6^1 48^2, 12^1 24^2 48^1,$
$24^4, 24^3, 12^1 24^1, 6^1, 2^1)$
$(1^1, 3^1, 6^1, 12^1, 24^1, 12^1 24^1, 6^1 48^1, 6^1 24^1 48^1, 12^2 24^1 48^1, 12^1 24^2 48^1,$
$12^2 48^2, 24^3, 12^1, 6^1, 2^1)$

$|\text{mET}| = 7560$, nonsolvable, $\mathbb{Z}_3 \rtimes A_7$

...
...
$X = \textbf{S630}$
Cov
Aut X
$\mathbb{Z}_3$
S210

S650, basic

G-Type: G_2^4 additional relations: $d\bar{x}\bar{y}(\bar{x}y\bar{x}\bar{y})^2$

Diam $X = 13$,   Girth $X = 10$

$|\text{Aut }X| = 15600 = 2^4 \cdot 3 \cdot 5^2 \cdot 13$, $\text{P}\Sigma\text{L}(2, 25)$

$|(\text{Aut }X)_u| = 24 = 2^4 \cdot 3$

StabPart:
$(1^1, 3^1, 6^1, 12^1, 12^2, 12^1 24^1, 12^2 24^1, 6^1 24^1 48^1, 6^1 48^2, 12^1 24^2 48^1, 12^1 24^2 48^1,$
$12^1 24^1 48^1, 12^1 24^1, 4^1)$
$(1^1, 3^1, 6^1, 12^1, 24^1, 12^1 24^1, 6^1 48^1, 6^1 24^1 48^1, 12^2 24^1 48^1, 12^1 24^2 48^1, 8^1 24^3 48^1,$
$12^1 24^1 48^1, 4^1 12^1, 4^1)$

$|\text{mET}| = 7800$, simple, $\text{PSL}(2, 25)$

...
...
$X = \textbf{S650}$

S660, family **VII**

G-Type: G_1^3 additional relations: $(\bar{x}\bar{y})^6, dx\bar{y}\bar{x}\bar{y}xy\bar{x}yx\bar{y}x\bar{y}$

Diam $X = 12$,   Girth $X = 12$

$|\text{Aut }X| = 3960 = 2^3 \cdot 3^2 \cdot 5 \cdot 11$

$|(\text{Aut }X)_u| = 12 = 2^2 \cdot 3$

StabPart:
$(1^1, 3^1, 6^1, 12^1, 12^2, 12^4, 6^3 12^5, 6^4 12^8, 6^5 12^{11}, 6^4 12^{10}, 2^1 3^1 6^1 12^4, 3^1, 0)$
$(1^1, 3^1, 6^1, 12^1, 12^2, 12^4, 6^1 12^6, 6^4 12^8, 6^5 12^{11}, 6^4 12^{10}, 2^2 6^3 12^3, 3^1, 1^1)$

$|\text{mET}| = 1980$, nonsolvable, $\mathbb{Z}_3 \times \text{PSL}(2, 11)$

...	
...	
$X = \textbf{S660}$	
Cov	Quo
Aut X	Aut X
$\mathbb{Z}_3$	$\text{PSL}(2, 11)$
S220	$K_{1,3}$

S672a

G-Type: G_1^1 additional relations: $x y \bar x y (x \bar y \bar x \bar y)^2, x \bar y (x y)^3 \bar x y (\bar x \bar y)^3$

Diam $X = 12$,   Girth $X = 12$

$|\operatorname{Aut} X| = 2016 = 2^5 \cdot 3^2 \cdot 7$

$|(\operatorname{Aut} X)_u| = 6 = 2 \cdot 3$

StabPart:

$(1^1, 3^1, 6^1, 6^2, 6^4, 6^8, 3^2 6^{11}, 3^2 6^{19}, 2^1 6^{26}, 3^2 6^{22}, 3^2 6^{11}, 3^1 6^2, 1^1 2^1)$

$(1^1, 3^1, 6^1, 6^2, 6^4, 6^8, 2^1 6^{13}, 3^2 6^{19}, 2^1 3^4 6^{25}, 3^2 6^{22}, 2^1 6^9, 3^1 6^2, 1^1 2^2)$

$|\mathrm{mET}| = 1008$, nonsolvable

$\mathbb{Z}_2 \operatorname{ext}_{\mathrm{cent}} (\mathbb{Z}_3 \times \mathrm{PSL}(2,7)) \cong \mathbb{Z}_6 \operatorname{ext}_{\mathrm{cent}} \mathrm{PSL}(2,7)$

$\cdots$	
$\cdots$	
$X = $ **S672a**	
Cov	Cov
Aut X	Aut X
$\mathbb{Z}_2$	$\mathbb{Z}_3$
S336b	**F224b**

S672b

G-Type: G_1^3 additional relations: $x y (\bar x y)^2 x \bar y (\bar x \bar y)^2$

Diam $X = 14$,   Girth $X = 12$

$|\operatorname{Aut} X| = 4032 = 2^6 \cdot 3^2 \cdot 7$

$|(\operatorname{Aut} X)_u| = 12 = 2^2 \cdot 3$

StabPart:

$(1^1, 3^1, 6^1, 12^1, 12^2, 12^4, 6^1 12^6, 6^2 12^9, 2^1 6^3 12^{11}, 6^2 12^{10}, 2^1 4^1 6^2 12^4,$
$6^1 12^1, 2^1 6^1, 3^1, 1^1)$

$(1^1, 3^1, 6^1, 12^1, 12^2, 12^4, 6^1 12^6, 6^2 12^9, 2^1 12^{12}, 6^2 12^{10}, 6^2 12^5,$
$6^1 12^1, 2^1 6^1, 3^1, 1^1)$

$|\mathrm{mET}| = 1008$, nonsolvable

$\mathbb{Z}_2 \operatorname{ext}_{\mathrm{cent}} (\mathbb{Z}_3 \times \mathrm{PSL}(2,7)) \cong \mathbb{Z}_6 \operatorname{ext}_{\mathrm{cent}} \mathrm{PSL}(2,7)$

$\cdots$	
$\cdots$	
$X = $ **S672b**	
Cov	Cov
Aut X	Aut X
$\mathbb{Z}_2$	$\mathbb{Z}_3$
S336c	**F224c**

S702a

G-Type: G_1 additional relations: $(x \bar y \bar x \bar y)^3, x y \bar x \bar y \bar x y x y (\bar x \bar y)^4$

Diam $X = 10$,   Girth $X = 12$

$|\operatorname{Aut} X| = 1053 = 3^4 \cdot 13$

$|(\operatorname{Aut} X)_u| = 3$

StabPart:

$(1^1, 3^1, 3^2, 3^4, 3^8, 3^{16}, 3^{30}, 3^{56}, 1^2 3^{66}, 1^3 3^{39}, 3^{10})$

$(1^1, 3^1, 3^2, 3^4, 3^8, 3^{16}, 3^{30}, 3^{56}, 3^{64}, 1^3 3^{39}, 1^2 3^{12})$

$|\mathrm{mET}| = \operatorname{Aut} X$, solvable

$\cdots$
$\cdots$
$X = $ **S702a**
Cov
Aut X
$\mathbb{Z}_3^3$
F26

S702b

G-Type: G_1 additional relations: $(x y)^3 (\bar x \bar y)^3, (x \bar y \bar x \bar y)^3, x (y \bar x)^2 \bar y x y \bar x \bar y x (y \bar x)^6 \bar y$

Diam $X = 14$,   Girth $X = 12$

$|\operatorname{Aut} X| = 1053 = 3^4 \cdot 13$

$|(\operatorname{Aut} X)_u| = 3$

StabPart:

$(1^1, 3^1, 3^2, 3^4, 3^8, 3^{16}, 1^2 3^{22}, 3^{23}, 3^{24}, 3^{27}, 3^{30}, 3^{33}, 3^{28}, 3^{13}, 3^2)$

$(1^1, 3^1, 3^2, 3^4, 3^8, 3^{16}, 1^2 3^{22}, 3^{23}, 3^{24}, 3^{27}, 3^{30}, 3^{33}, 3^{28}, 3^{13}, 1^6)$

$|\mathrm{mET}| = \operatorname{Aut} X$, solvable

$\cdots$	
$\cdots$	
$X = $ **S702b**	
Cov	Cov
Aut X	Aut X
$\mathbb{Z}_3$	$\mathbb{Z}_{13}$
F234a	**S54**

S702c

G-Type: G_1 additional relations: $(xy)^2\bar{x}\bar{y}xy\bar{x}y(\bar{x}\bar{y})^2$

Diam $X = 11$,   Girth $X = 14$
$|\mathrm{Aut}\,X| = 1053 = 3^4 \cdot 13$
$|(\mathrm{Aut}\,X)_u| = 3$

StabPart:
$(1^1, 3^1, 3^2, 3^4, 3^8, 3^{16}, 3^{32}, 3^{57}, 3^{68}, 1^2 3^{38}, 1^2 3^6, 1^1)$
$(1^1, 3^1, 3^2, 3^4, 3^8, 3^{16}, 3^{32}, 3^{57}, 1^2 3^{66}, 1^2 3^{38}, 3^8, 1^1)$

$|\mathrm{mET}| = \mathrm{Aut}\,X$, solvable

. . .
. . .
$X = $ **S702c**
Cov   Aut X $\mathbb{Z}_3^3$
F26

S720a

G-Type: G_1 additional relations: $xy(\bar{x}y)^2 x\bar{y}(\bar{x}\bar{y})^3, x\bar{y}\bar{x}\bar{y}(xy)^2\bar{x}\bar{y}\bar{x}y(\bar{x}\bar{y})^2$

Diam $X = 10$,   Girth $X = 14$
$|\mathrm{Aut}\,X| = 1080 = 2^3 \cdot 3^3 \cdot 5$, $\mathbb{Z}_3 \, \mathrm{ext_{cent}} \, A_6$
$|(\mathrm{Aut}\,X)_u| = 3$

StabPart:
$(1^1, 3^1, 3^2, 3^4, 3^8, 3^{16}, 3^{32}, 3^{57}, 1^4 3^{64}, 3^{42}, 1^1 3^{12})$
$(1^1, 3^1, 3^2, 3^4, 3^8, 3^{16}, 3^{32}, 3^{57}, 1^2 3^{66}, 3^{42}, 1^3 3^{10})$

$|\mathrm{mET}| = \mathrm{Aut}\,X$, nonsolvable

. . .
. . .
$X = $ **S720a**
Cov   Aut X $\mathbb{Z}_3$
F240a

S720b, family **VII**

G-Type: G_1^1 additional relations: $xy(\bar{x}y)^2 x\bar{y}(\bar{x}\bar{y})^3, xyx\bar{y}xy\bar{x}\bar{y}x\bar{y}xy(\bar{x}\bar{y})^2$

Diam $X = 12$,   Girth $X = 14$
$|\mathrm{Aut}\,X| = 2160 = 2^4 \cdot 3^3 \cdot 5$
$|(\mathrm{Aut}\,X)_u| = 6 = 2 \cdot 3$

StabPart:
$(1^1, 3^1, 6^1, 6^2, 6^4, 6^8, 6^{16}, 3^1 6^{28}, 2^2 3^3 6^{29}, 3^5 6^{18}, 2^2 3^4 6^5, 3^1, 0)$
$(1^1, 3^1, 6^1, 6^2, 6^4, 6^8, 6^{16}, 3^1 6^{28}, 2^4 3^1 6^{30}, 3^5 6^{18}, 2^4 3^5 6^3, 3^1, 1^1)$

$|\mathrm{mET}| = 1080$, nonsolvable, $\mathbb{Z}_3 \times A_6$

. . .	
. . .	
$X = $ **S720b**	
Cov Aut X $\mathbb{Z}_3$	Quo Aut X A_6
F240a	$K_{1,3}$

S768a

G-Type: G_1^1 additional relations: $(\bar{x}\bar{y})^6, xy\bar{x}yxy\bar{x}\bar{y}x\bar{y}x\bar{y}, x\bar{y}xy(x\bar{y})^2(\bar{x}y)^3\bar{x}\bar{y}$

Diam $X = 12$,   Girth $X = 12$
$|\mathrm{Aut}\,X| = 2304 = 2^8 \cdot 3^2$
$|(\mathrm{Aut}\,X)_u| = 6 = 2 \cdot 3$

StabPart:
$(1^1, 3^1, 6^1, 6^2, 6^4, 6^8, 3^1 6^{12}, 3^2 6^{19}, 3^2 6^{27}, 3^3 6^{29}, 2^2 3^2 6^{16}, 3^2 6^2, 1^1 3^1)$
$(1^1, 3^1, 6^1, 6^2, 6^4, 6^8, 2^2 3^1 6^{12}, 3^2 6^{19}, 2^1 3^2 6^{26}, 3^3 6^{29}, 1^1 2^5 3^1 6^{15}, 3^2 6^2, 1^2 2^2)$

$|\mathrm{mET}| = 1152$, solvable

. . .	
. . .	
$X = $ **S768a**	
Cov Aut X $\mathbb{Z}_2$	Cov Aut X $\mathbb{Z}_2^2$
S384	**S192c**

S768b

G-Type: G_1^3 additional relations: $xy\bar{x}yxy\bar{x}\bar{y}x\bar{y}\bar{x}\bar{y}, (\bar{x}y\bar{x}\bar{y})^3$

  Diam $X = 14$,   Girth $X = 12$

  $|\text{Aut } X| = 4608 = 2^9 \cdot 3^2$

  $|(\text{Aut } X)_u| = 12 = 2^2 \cdot 3$

  StabPart:
  $(1^1, 3^1, 6^1, 12^1, 12^2, 12^4, 6^2 12^5, 6^4 12^7, 3^1 6^9 12^8, 3^1 6^5 12^{11}, 2^2 6^9 12^4,$
  $3^1 12^3, 1^1 6^1 12^1, 3^1 6^1, 3^1)$
  $(1^1, 3^1, 6^1, 12^1, 12^2, 12^4, 4^1 6^2 12^5, 6^4 12^7, 2^1 6^4 12^{11}, 3^1 6^5 12^{11}, 1^1 4^3 6^2 12^6,$
  $3^1 12^3, 1^1 6^3, 3^1 6^1, 1^1 2^1)$

  $|\text{mET}| = 1152$, solvable

$\cdots$	
$\cdots$	
$X = $ **S768b**	
Cov	Cov
Aut X	Aut X
$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
S384	**F192a**

8 Voltage rules generating S54, S112 and S120

In this final section we present five diagrams (see Figures 7,8,9,10,11) describing a voltage rule by which the three semisymmetric graphs **S54**, **S112** and **S120** are generated from their respective basic (and symmetric) counterparts: **F2**, **F2**, and the Petersen graph **F10**, via a sequence of direct elementary abelian covers; see also Figures 5, 6a and 6b. In the last case, two different routes may be taken, one leading through the dodecahedron **F20a** and the other through the Desarguesian graph **F20b**. Furthermore, a voltage rule is also described that gives the graph **S120** as a regular (but indirect) D_3-cover of **F20b**.
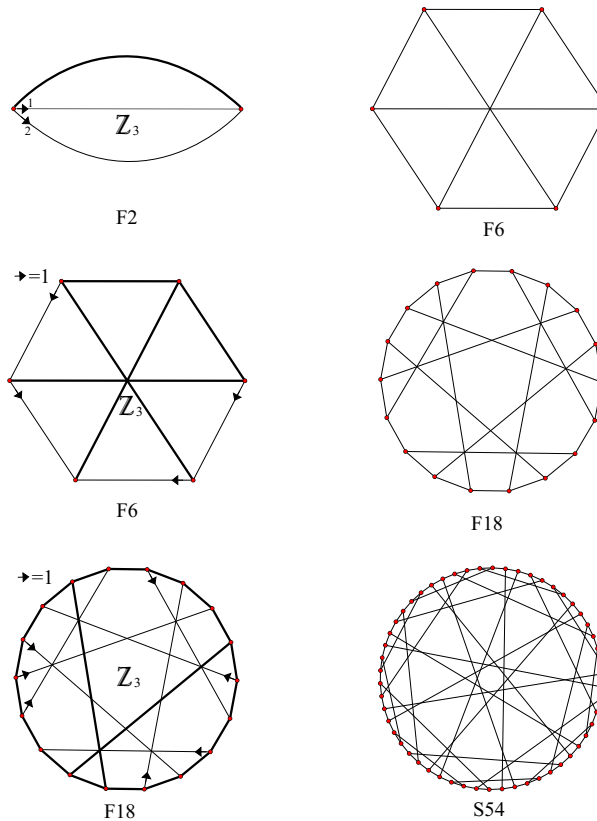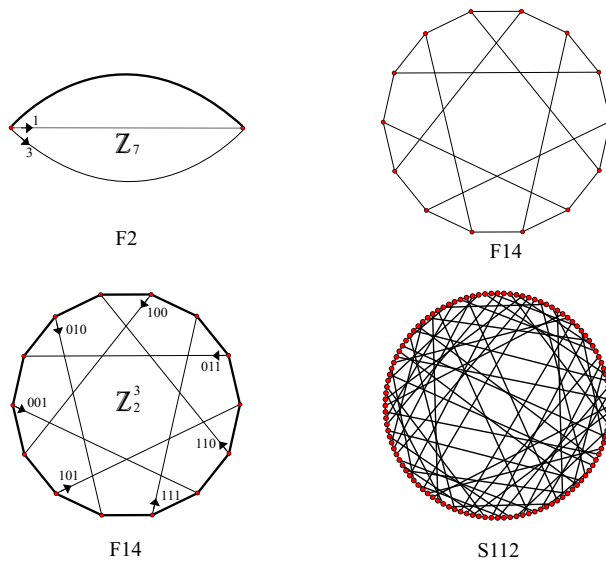

Figure 7.

$\mathbb{Z}_7$

F2

F14

010 100
011
001 $\mathbb{Z}_2^3$
110
101
111

F14

S112

Figure 8.

----- =1

$\mathbb{Z}_2$

F10

F20A

----- =1

$\mathbb{Z}_2$

F20A

F40
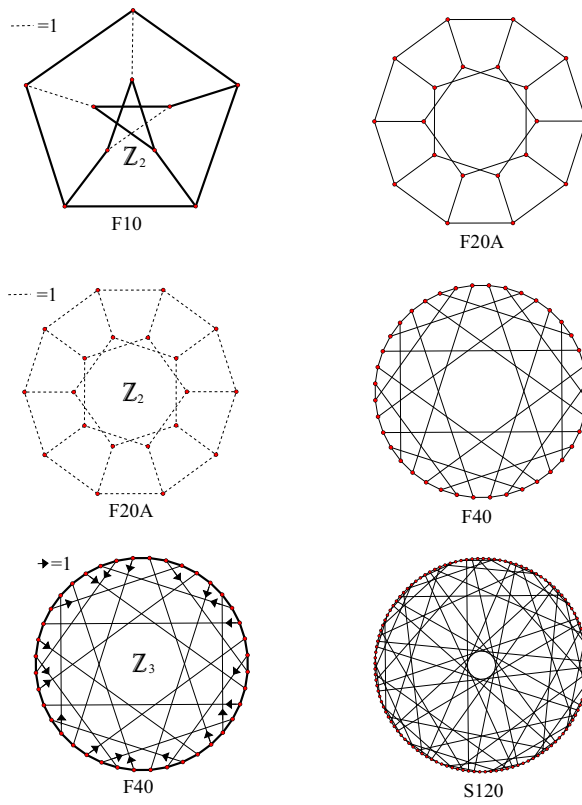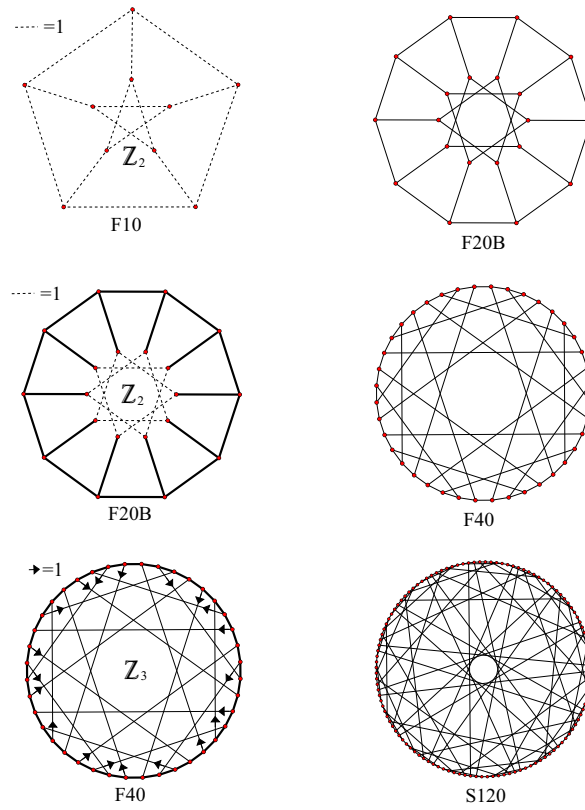

=1

$\mathbb{Z}_3$

F40

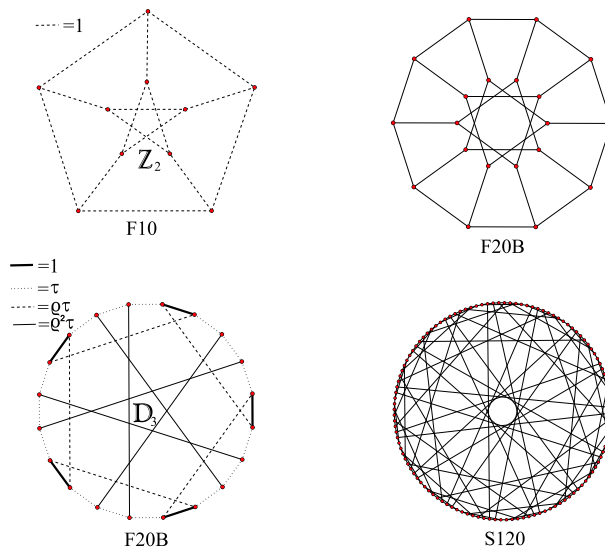S120

Figure 9.

Figure 10.

Figure 11.

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