MARKINGS OF THE GOLAY CODE

Marston Conder and *John McKay*

Abstract

The simple Mathieu group M_{24} is the automorphism group of a Steiner system $S(5,8,24)$, acting naturally on its 24 points. A *marking* of M_{24} (or the associated Golay code) is defined to be a linear ordering $x_1 < x_2 < \ldots < x_{24}$ of these points, and with any such marking one may associate a 5-tuple $(c_0, c_1, c_2, c_3, c_4)$, where c_k denotes the number of blocks of the given Steiner system containing exactly k pairs of the form $\{x_{2i-1}, x_{2i}\}$. In this paper we give a solution to the *marking problem* for M_{24}, namely the determination of the spectrum of these 5-tuples, showing that precisely 90 different 5-tuples occur.

§1. Introduction

The Steiner system $S(5,8,24)$ is a block design made up of 24 points and 759 blocks, each of size 8, with the property that every 5 points lie in exactly one block. This design is naturally associated with the Golay code, and its automorphism group is the simple Mathieu group M_{24}; see [**3**; Ch.11]. The following problem was communicated by Geoffrey Mason, and arose from a question regarding a connection between the Golay code and Virasoro algebras which act rationally on the Moonshine module:

Define a *marking* of the Golay code (or simply of M_{24}) to be a linear ordering $x_1 < x_2 < \ldots < x_{24}$ of the 24 points of the Steiner system, and for any such marking, let c_k be the number of blocks which contain exactly k pairs of the form $\{x_{2i-1}, x_{2i}\}$, for $1 \le i \le 12$. What are the possibilities for the corresponding 5-tuple $(c_0, c_1, c_2, c_3, c_4)$?

Roughly speaking, every marking determines a 48-dimensional associative subalgebra A of the Griess algebra, from which a Virasoro algebra is constructible, and markings with distinct 5-tuples give rise to distinct algebras; see [**4**] and [**5**] for example.

In this paper we provide a solution to the so-called *marking problem* for M_{24}, showing that precisely 90 different 5-tuples occur, and determining their multiplicities.

To do so we use the fact that the group M_{24} may be obtained by adjoining the following permutation to the group $L_2(23)$ acting on the projective line $\Omega = PL(23)$ over $GF(23)$:

$$x \mapsto \begin{cases} x^3/9 & \text{if } x \text{ is a square in } GF(23) \\ 9x^3 & \text{otherwise.} \end{cases}$$

A set of four generating permutations (including this one) is given in [**2**] and [**3**; Ch.10].

For the blocks of a Steiner system $S(5,8,24)$ we then take the 759 images under the action of M_{24} of the set $\{\infty, 0, 3, 5, 6, 9, 15, 19\}$, which is the *standard octad* made up of the points from the first two columns of Curtis's "Miracle Octad Generator" (or MOG array); see [**2**] or [**3**; Ch.11]. By 5-transitivity of M_{24} and a simple counting argument, every five points lie in a unique block.

In particular, 141 of these blocks contain no pairs of the form $\{\infty, 0\}$ or $\{2i-1, 2i\}$ (for $1 \le i \le 11$), while 336 blocks contain exactly one such pair, 258 contain two such pairs, 24 contain three such pairs, and 0 blocks are comprised of exactly four pairs of this form. Hence we may associate the 5-tuple $(141, 336, 258, 24, 0)$ with the canonical marking $\infty < 0 < 1 < 2 < \ldots < 21 < 22$ of the 24 points of $\Omega = PL(23)$.

Other markings (and associated 5-tuples) may be obtained by applying permutations from the full symmetric group on Ω to this canonical ordering, and indeed a random choice of a few such permutations will quickly produce a good range of different possibilities for the 5-tuple $(c_0, c_1, c_2, c_3, c_4)$.

The complete list of possibilities is given in Section 4. These were first found by a long process of random selection of markings, several thousand in total, with the 5-tuple $(165, 264, 330, 0, 0)$ proving the most elusive. Subsequently the list's completeness was verified using the analysis and enumeration of partitions described in Sections 2 and 3.

There is certainly some pattern to the list. First, by definition the components of every possible 5-tuple $(c_0, c_1, c_2, c_3, c_4)$ must sum to 759. But also they must satisfy the equations $c_1 + 2c_2 + 3c_3 + 4c_4 = 924$ and $c_2 + 3c_3 + 6c_4 = 330$, which may be seen by counting the number of blocks containing a given pair $\{x_{2i-1}, x_{2i}\}$ or two such pairs respectively. Other reasons for the spectrum of 5-tuples are much less obvious.

§2. Partitions

For the purposes of enumerating the associated 5-tuples, a marking of M_{24} is equivalent to a partition of the points of Ω into 12 pairs, that is, of type 2^{12}. Ostensibly the number of such partitions is $24!/(2^{12}12!) = 316234143225$, which is rather large, however this number may be reduced considerably by taking into account also the action of M_{24}.

Lemma 1. *In its natural representation on* 24 *points the group* M_{24} *has*

(a) *a single orbit on the* 276 *pairs* $\{x_1, x_2\}$ *of distinct points,*

(b) *a single orbit on the* 31878 *double-pairs* $\{\{x_1, x_2\}, \{x_3, x_4\}\}$ *of distinct points,*

(c) *two orbits on the* 2018940 *triple-pairs* $\{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$ *of distinct points: one of size* 318780*, and one of size* 1700160*,*

(d) *seven orbits on the* 77224455 *quadruple-pairs* $\{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8\}\}$ *of distinct points: one of size* 79695*, one of size* 637560*, two of size* 7650720*, one of size* 10200960*, one of size* 20401920*, and one of size* 30602880*.*

Proof.

The first two assertions follow from the 5-transitivity of M_{24}, while the last two can be easily verified using GAP [**6**] or MAGMA [**1**].

In particular, if the generators for M_{24} are taken as described earlier, then the triple-pairs $\{\{\infty, 0\}, \{1, 2\}, \{3, 4\}\}$ and $\{\{\infty, 0\}, \{1, 2\}, \{3, 5\}\}$ lie in different orbits, having stabilizers of orders 144 and 768 respectively. Similarly, representatives of the seven orbits on quadruple-pairs may be taken as follows:

Representative	*Order of stabilizer*	*Size of orbit*
$\{\{\infty, 0\}, \{1, 2\}, \{3, 4\}, \{5, 6\}\}$	8	30602880
$\{\{\infty, 0\}, \{1, 2\}, \{3, 4\}, \{5, 8\}\}$	12	20401920
$\{\{\infty, 0\}, \{1, 2\}, \{3, 4\}, \{5, 13\}\}$	32	7650720
$\{\{\infty, 0\}, \{1, 2\}, \{3, 4\}, \{5, 14\}\}$	24	10200960
$\{\{\infty, 0\}, \{1, 2\}, \{3, 4\}, \{5, 16\}\}$	32	7650720
$\{\{\infty, 0\}, \{1, 2\}, \{3, 5\}, \{4, 13\}\}$	384	637560
$\{\{\infty, 0\}, \{1, 2\}, \{3, 5\}, \{14, 17\}\}$	3072	79695.

Definition. *We shall say two markings of M_{24} are equivalent if they provide partitions (of type 2^{12}) which are in the same orbit under the action of M_{24}.*

Lemma 2. *Suppose the generators for M_{24} and the blocks for the Steiner system $S(5,8,24)$ are as given earlier. Then every marking of M_{24} is equivalent to (at least) one of the form $x_1 < x_2 < x_3 < \ldots < x_{24}$, where*

(a) $\{\{x_1,x_2\},\{x_3,x_4\},\{x_5,x_6\},\{x_7,x_8\}\}$ *is one of the 7 quadruple-pairs listed at the end of the proof of Lemma 1, and*

(b) $x_i = \min\{x_j : i \le j \le 24\}$ *for all* $i \in \{9,11,13,15,17,19,21,23\}$.

Proof.

First the action of M_{24} ensures that each equivalence class contains at least one marking which satisfies condition (a) on the first four parts of the associated partition. The second condition (b) follows simply by rearrangement of the remaining eight parts. (Note: in fact the markings which satisfy these two conditions provide representatives for all double cosets HgK in the symmetric group on Ω, where $H = M_{24}$ and K is a wreath product of C_2 by S_{12}, although of course many will have more than one representative.)

§3. Enumeration

A computer program (written in PASCAL) was implemented on a SUN workstation to enumerate the representative markings described in Lemma 2, and to determine the corresponding 5-tuple in each case.

The results of this program agreed with the evidence gained by random choice of markings—as mentioned in the Introduction—and are summarized below. Starting from scratch, we give each 5-tuple $(c_0, c_1, c_2, c_3, c_4)$ together with the first representative marking which produced it, in the order they were obtained. For convenience we use the alternative notation $x_1{:}x_2{-}x_3{:}x_4{-}\ldots{-}x_{23}{:}x_{24}$ to denote a marking $x_1 < x_2 < \ldots < x_{24}$:

5-tuple	*Representative marking*
$(141, 336, 258, 24, 0)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:14–15:16–17:18–19:20–21:22
$(139, 342, 252, 26, 0)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:14–15:16–17:18–19:22–20:21
$(135, 354, 240, 30, 0)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:14–15:16–17:19–18:21–20:22

$(145, 324, 270, 20, 0)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:14–15:16–17:19–18:22–20:21
$(129, 372, 222, 36, 0)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:14–15:16–17:20–18:21–19:22
$(137, 348, 246, 28, 0)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:14–15:16–17:21–18:20–19:22
$(143, 330, 264, 22, 0)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:14–15:16–17:21–18:22–19:20
$(131, 366, 228, 34, 0)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:14–15:17–16:20–18:21–19:22
$(133, 360, 234, 32, 0)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:14–15:17–16:21–18:19–20:22
$(127, 378, 216, 38, 0)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:14–15:17–16:21–18:20–19:22
$(134, 356, 240, 28, 1)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:14–15:20–16:17–18:19–21:22
$(126, 380, 216, 36, 1)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:14–15:20–16:17–18:21–19:22
$(128, 374, 222, 34, 1)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:14–15:20–16:17–18:22–19:21
$(136, 350, 246, 26, 1)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:14–15:20–16:18–17:21–19:22
$(132, 362, 234, 30, 1)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:14–15:20–16:19–17:18–21:22
$(130, 368, 228, 32, 1)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:14–15:20–16:19–17:22–18:21
$(142, 332, 264, 20, 1)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:14–15:20–16:22–17:21–18:19
$(122, 392, 204, 40, 1)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:15–14:16–17:20–18:21–19:22
$(125, 384, 210, 40, 0)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:15–14:19–16:21–17:20–18:22
$(125, 382, 216, 34, 2)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:15–14:21–16:18–17:20–19:22
$(124, 386, 210, 38, 1)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:16–14:17–15:18–19:21–20:22
$(121, 394, 204, 38, 2)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:16–14:17–15:20–18:22–19:21
$(129, 370, 228, 30, 2)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:16–14:18–15:20–17:22–19:21
$(138, 344, 252, 24, 1)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:16–14:20–15:22–17:18–19:21
$(130, 366, 234, 26, 3)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:16–14:22–15:20–17:18–19:21
$(120, 398, 198, 42, 1)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:17–14:16–15:20–18:19–21:22
$(133, 358, 240, 26, 2)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:18–14:19–15:16–17:20–21:22
$(126, 378, 222, 30, 3)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:21–14:17–15:20–16:19–18:22
$(123, 390, 204, 42, 0)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:12–13:22–14:19–15:17–16:21–18:20
$(147, 318, 276, 18, 0)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:13–12:15–14:16–17:21–18:19–20:22
$(150, 308, 288, 12, 1)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:13–12:16–14:15–17:18–19:21–20:22
$(153, 300, 294, 12, 0)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:13–12:16–14:15–17:21–18:19–20:22
$(144, 326, 270, 18, 1)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:13–12:16–14:15–17:22–18:20–19:21

$(134, 354, 246, 22, 3)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:13–12:16–14:17–15:20–18:22–19:21
$(138, 342, 258, 18, 3)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:13–12:16–14:18–15:20–17:22–19:21
$(146, 320, 276, 16, 1)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:13–12:16–14:20–15:22–17:18–19:21
$(140, 334, 270, 10, 5)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:13–12:16–14:22–15:20–17:18–19:21
$(137, 346, 252, 22, 2)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:13–12:22–14:21–15:20–16:19–17:18
$(118, 404, 192, 44, 1)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:14–12:13–15:18–16:21–17:19–20:22
$(117, 406, 192, 42, 2)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:14–12:15–13:18–16:20–17:19–21:22
$(140, 338, 258, 22, 1)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:14–12:16–13:15–17:19–18:22–20:21
$(149, 312, 282, 16, 0)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:14–12:16–13:22–15:18–17:21–19:20
$(116, 408, 192, 40, 3)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:15–12:14–13:21–16:17–18:22–19:20
$(132, 360, 240, 24, 3)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:15–12:16–13:21–14:17–18:22–19:20
$(118, 402, 198, 38, 3)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:15–12:17–13:21–14:16–18:22–19:20
$(154, 296, 300, 8, 1)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:16–12:22–13:14–15:20–17:21–18:19
$(122, 390, 210, 34, 3)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:18–12:13–14:16–15:20–17:21–19:22
$(126, 376, 228, 24, 5)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:18–12:13–14:21–15:20–16:17–19:22
$(114, 414, 186, 42, 3)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:18–12:14–13:15–16:21–17:20–19:22
$(122, 388, 216, 28, 5)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:18–12:14–13:17–15:20–16:21–19:22
$(120, 396, 204, 36, 3)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:18–12:15–13:16–14:21–17:20–19:22
$(136, 348, 252, 20, 3)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:18–12:16–13:14–15:20–17:21–19:22
$(114, 412, 192, 36, 5)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:18–12:16–13:15–14:21–17:20–19:22
$(128, 368, 240, 16, 7)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:18–12:16–13:17–14:21–15:20–19:22
$(124, 384, 216, 32, 3)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:18–12:17–13:20–14:21–15:16–19:22
$(112, 420, 180, 44, 3)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:18–12:21–13:15–14:16–17:20–19:22
$(110, 424, 180, 40, 5)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:18–12:21–13:17–14:16–15:20–19:22
$(118, 400, 204, 32, 5)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:19–12:13–14:17–15:20–16:21–18:22
$(156, 288, 312, 0, 3)$	∞:0–1:2–3:4–5:6–7:8–9:10–11:22–12:16–13:14–15:20–17:21–18:19
$(121, 396, 198, 44, 0)$	∞:0–1:2–3:4–5:6–7:8–9:11–10:12–13:17–14:21–15:18–16:19–20:22
$(119, 402, 192, 46, 0)$	∞:0–1:2–3:4–5:6–7:8–9:11–10:15–12:18–13:16–14:21–17:19–20:22
$(116, 410, 186, 46, 1)$	∞:0–1:2–3:4–5:6–7:8–9:11–10:15–12:18–13:17–14:22–16:19–20:21
$(117, 408, 186, 48, 0)$	∞:0–1:2–3:4–5:6–7:8–9:12–10:15–11:18–13:20–14:16–17:22–19:21

$(112, 414, 198, 26, 9)$	∞:0–1:2–3:4–5:6–7:8–9:12–10:16–11:18–13:20–14:21–15:17–19:22
$(120, 392, 216, 24, 7)$	∞:0–1:2–3:4–5:6–7:8–9:12–10:16–11:20–13:18–14:19–15:17–21:22
$(128, 372, 228, 28, 3)$	∞:0–1:2–3:4–5:6–7:8–9:12–10:17–11:21–13:20–14:18–15:16–19:22
$(108, 426, 186, 30, 9)$	∞:0–1:2–3:4–5:6–7:8–9:16–10:12–11:13–14:22–15:18–17:20–19:21
$(110, 426, 174, 46, 3)$	∞:0–1:2–3:4–5:6–7:8–9:17–10:18–11:20–12:15–13:16–14:21–19:22
$(100, 452, 156, 44, 7)$	∞:0–1:2–3:4–5:6–7:8–9:19–10:11–12:13–14:17–15:18–16:21–20:22
$(120, 390, 222, 18, 9)$	∞:0–1:2–3:4–5:6–7:8–9:20–10:15–11:18–12:13–14:21–16:17–19:22
$(115, 414, 180, 50, 0)$	∞:0–1:2–3:4–5:6–7:8–9:21–10:15–11:18–12:19–13:20–14:16–17:22
$(155, 294, 300, 10, 0)$	∞:0–1:2–3:4–5:6–7:8–9:22–10:12–11:21–13:14–15:20–16:18–17:19
$(114, 416, 180, 48, 1)$	∞:0–1:2–3:4–5:6–7:8–9:22–10:15–11:18–12:14–13:20–16:17–19:21
$(140, 336, 264, 16, 3)$	∞:0–1:2–3:4–5:6–7:8–9:22–10:17–11:21–12:19–13:20–14:18–15:16
$(120, 384, 240, 0, 15)$	∞:0–1:2–3:4–5:6–7:9–8:12–10:16–11:18–13:20–14:21–15:17–19:22
$(165, 264, 330, 0, 0)$	∞:0–1:2–3:4–5:6–7:9–8:22–10:18–11:15–12:16–13:14–17:19–20:21
$(108, 428, 180, 36, 7)$	∞:0–1:2–3:4–5:6–7:10–8:9–11:18–12:13–14:21–15:16–17:20–19:22
$(104, 440, 168, 40, 7)$	∞:0–1:2–3:4–5:6–7:10–8:9–11:18–12:15–13:16–14:21–17:20–19:22
$(96, 460, 156, 36, 11)$	∞:0–1:2–3:4–5:6–7:10–8:9–11:18–12:16–13:15–14:21–17:20–19:22
$(108, 432, 168, 48, 3)$	∞:0–1:2–3:4–5:6–7:10–8:9–11:18–12:21–13:15–14:16–17:22–19:20
$(96, 464, 144, 48, 7)$	∞:0–1:2–3:4–5:6–7:10–8:11–9:16–12:18–13:15–14:21–17:19–20:22
$(92, 472, 144, 40, 11)$	∞:0–1:2–3:4–5:6–7:10–8:11–9:21–12:18–13:15–14:16–17:22–19:20
$(108, 424, 192, 24, 11)$	∞:0–1:2–3:4–5:6–7:10–8:12–9:13–11:18–14:21–15:16–17:20–19:22
$(112, 416, 192, 32, 7)$	∞:0–1:2–3:4–5:6–7:10–8:12–9:13–11:18–14:21–15:16–17:22–19:20
$(100, 448, 168, 32, 11)$	∞:0–1:2–3:4–5:6–7:10–8:12–9:15–11:18–13:16–14:21–17:20–19:22
$(88, 476, 156, 20, 19)$	∞:0–1:2–3:4–5:6–7:10–8:12–9:16–11:18–13:15–14:21–17:20–19:22
$(72, 528, 96, 48, 15)$	∞:0–1:2–3:4–5:8–6:10–7:15–9:21–11:13–12:18–14:17–16:22–19:20
$(80, 504, 120, 40, 15)$	∞:0–1:2–3:4–5:13–6:7–8:9–10:12–11:20–14:22–15:18–16:17–19:21
$(64, 544, 96, 32, 23)$	∞:0–1:2–3:4–5:13–6:7–8:9–10:15–11:20–12:18–14:22–16:17–19:21
$(48, 576, 96, 0, 39)$	∞:0–1:2–3:4–5:13–6:7–8:11–9:20–10:15–12:18–14:22–16:17–19:21

The program also counted the multiplicities of all 5-tuples, separately for each of the 7 initial quadruple-pairs of the marking, and the overall relative multiplicities were then calculated by taking a linear combination of these totals with the orbit-sizes in part (d) of

Lemma 1 as coefficients. The results of this calculation are given in the next Section.

§4. Conclusion

Our findings may be summarized as follows:

Theorem. *Suppose* $x_1 < x_2 < \ldots < x_{24}$ *is any linear ordering of the 24 points of a Steiner system* $S(5,8,24)$*. If* c_k *denotes the number of blocks of this system containing exactly* k *pairs of the form* $\{x_{2i-1}, x_{2i}\}$*, then the corresponding 5-tuple* $(c_0, c_1, c_2, c_3, c_4)$ *must be one of the following:*

5-tuple	Multiplicity	5-tuple	Multiplicity
$(165, 264, 330, 0, 0)$	185472	$(156, 288, 312, 0, 3)$	2975280
$(155, 294, 300, 10, 0)$	12241152	$(140, 336, 264, 16, 3)$	3825360
$(153, 300, 294, 12, 0)$	44204160	$(138, 342, 258, 18, 3)$	88408320
$(149, 312, 282, 16, 0)$	91808640	$(136, 348, 252, 20, 3)$	91808640
$(147, 318, 276, 18, 0)$	122411520	$(134, 354, 246, 22, 3)$	734469120
$(145, 324, 270, 20, 0)$	1346526720	$(132, 360, 240, 24, 3)$	425890080
$(143, 330, 264, 22, 0)$	1786095360	$(130, 366, 234, 26, 3)$	1040497920
$(141, 336, 258, 24, 0)$	7181475840	$(128, 372, 228, 28, 3)$	45904320
$(139, 342, 252, 26, 0)$	11812711680	$(126, 378, 222, 30, 3)$	795674880
$(137, 348, 246, 28, 0)$	22799145600	$(124, 384, 216, 32, 3)$	1040497920
$(135, 354, 240, 30, 0)$	29741918976	$(122, 390, 210, 34, 3)$	734469120
$(133, 360, 234, 32, 0)$	36754058880	$(120, 396, 204, 36, 3)$	989493120
$(131, 366, 228, 34, 0)$	34336431360	$(118, 402, 198, 38, 3)$	1224115200
$(129, 372, 222, 36, 0)$	26787720960	$(116, 408, 192, 40, 3)$	703866240
$(127, 378, 216, 38, 0)$	15729880320	$(114, 414, 186, 42, 3)$	734469120
$(125, 384, 210, 40, 0)$	8862594048	$(112, 420, 180, 44, 3)$	489646080
$(123, 390, 204, 42, 0)$	2733857280	$(110, 426, 174, 46, 3)$	61205760
$(121, 396, 198, 44, 0)$	1112832000	$(108, 432, 168, 48, 3)$	45904320
$(119, 402, 192, 46, 0)$	306028800	$(140, 334, 270, 10, 5)$	24482304
$(117, 408, 186, 48, 0)$	23802240	$(126, 376, 228, 24, 5)$	91808640
$(115, 414, 180, 50, 0)$	12241152	$(122, 388, 216, 28, 5)$	459043200

$(154, 296, 300, 8, 1)$	5100480	$(118, 400, 204, 32, 5)$	795674880
$(150, 308, 288, 12, 1)$	61205760	$(114, 412, 192, 36, 5)$	459043200
$(146, 320, 276, 16, 1)$	153014400	$(110, 424, 180, 40, 5)$	367234560
$(144, 326, 270, 18, 1)$	183617280	$(128, 368, 240, 16, 7)$	97546680
$(142, 332, 264, 20, 1)$	326430720	$(120, 392, 216, 24, 7)$	15301440
$(140, 338, 258, 22, 1)$	367234560	$(112, 416, 192, 32, 7)$	6375600
$(138, 344, 252, 24, 1)$	2478833280	$(108, 428, 180, 36, 7)$	22952160
$(136, 350, 246, 26, 1)$	4100785920	$(104, 440, 168, 40, 7)$	30602880
$(134, 356, 240, 28, 1)$	7222279680	$(100, 452, 156, 44, 7)$	73956960
$(132, 362, 234, 30, 1)$	10527390720	$(96, 464, 144, 48, 7)$	15301440
$(130, 368, 228, 32, 1)$	12669592320	$(120, 390, 222, 18, 9)$	20401920
$(128, 374, 222, 34, 1)$	15607468800	$(112, 414, 198, 26, 9)$	61205760
$(126, 380, 216, 36, 1)$	12914415360	$(108, 426, 186, 30, 9)$	13601280
$(124, 386, 210, 38, 1)$	9221667840	$(108, 424, 192, 24, 11)$	3825360
$(122, 392, 204, 40, 1)$	5845150080	$(100, 448, 168, 32, 11)$	9563400
$(120, 398, 198, 42, 1)$	2509436160	$(96, 460, 156, 36, 11)$	30602880
$(118, 404, 192, 44, 1)$	1652555520	$(92, 472, 144, 40, 11)$	7650720
$(116, 410, 186, 46, 1)$	244823040	$(120, 384, 240, 0, 15)$	2709630
$(114, 416, 180, 48, 1)$	30602880	$(80, 504, 120, 40, 15)$	382536
$(137, 346, 252, 22, 2)$	122411520	$(72, 528, 96, 48, 15)$	956340
$(133, 358, 240, 26, 2)$	1468938240	$(88, 476, 156, 20, 19)$	2550240
$(129, 370, 228, 30, 2)$	5875752960	$(64, 544, 96, 32, 23)$	478170
$(125, 382, 216, 34, 2)$	6120576000	$(48, 576, 96, 0, 39)$	26565
$(121, 394, 204, 38, 2)$	2325818880		
$(117, 406, 192, 42, 2)$	734469120		

Note: these multiplicities are relative multiplicities, and sum to 316234143225, which is the number of partitions of $\Omega = PL(23)$ into 12 pairs; they may be multiplied by $2^{12}12!$ to obtain the true multiplicities in terms of linear orderings $x_1 < x_2 < \ldots < x_{24}$.

Finally we report an observation (kindly made to us by Simon Blackburn) on the extreme case of the 5-tuple $(48, 576, 96, 0, 39)$. This arises for example from the marking 0:19–15:5–∞:3–6:9–1:20–14:21–11:4–16:13–2:10–17:7–22:18–8:12, which is not listed but is a natural choice from the MOG array, in which the 24 points of Ω are arranged as follows:

0	∞	1	11	2	22
19	3	20	4	10	18
15	6	14	16	17	8
5	9	21	13	7	12

Each column of this grid consists of two pairs from the corresponding partition, and the union of any two columns is a block of the given Steiner system. This accounts for 15 of the 39 blocks contributing to c_4; the other 24 blocks may be obtained by deleting either the first two or the middle two or the last two columns, and then taking the top halves of an even number of the remaining columns together with the bottom halves of the rest.

Acknowledgment

The first author would like to thank the Mathematics Faculty of the University of Waterloo for its hospitality (when this work was initiated), and also the Auckland University Research Committee for its support.

References

[**1**] W. Bosma & J. Cannon, *Handbook of MAGMA Functions*, Department of Pure Mathematics, University of Sydney, 1993.

[**2**] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker & R.A. Wilson, *ATLAS of Finite Groups* (Oxford University Press, 1985).

[**3**] J.H. Conway & N.J.A. Sloane, *Sphere Packings, Lattices and Groups* (Springer-Verlag, New York, 1988).

[**4**] C. Dong, H. Li and G. Mason, Some twisted sectors for the Moonshine Module, *Contemporary Mathematics*, **193** (1996), 25–43.

[**5**] C. Dong, G. Mason & Y. Zhu, Discrete series of the Virasoro algebra and the moonshine module, *Proc. Symposia in Pure Mathematics (American Math. Society)* **56** II (1994), 295–316.

[**6**] M. Schönert et al, *GAP: Groups, Algorithms and Programming.* Lehrstuhl D für Mathematik, RWTH Aachen (Aachen, Germany, 1992).

Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand

Department of Computer Science, Concordia University, 1455 de Maisonneuve Blvd West, Montréal H3G 1M8, Canada