Combinatorial and computational group-theoretic methods in the study of graphs, maps and polytopes with maximal symmetry

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Abstract

This paper gives a brief summary of various aspects of combinatorial group theory and associated computational methods, with special reference to finitely-presented groups and their applications, found useful in the study of graphs, maps and polytopes having maximal symmetry. Recent results include the determination of all arc-transitive cubic graphs on up to 2048 vertices, and of all regular maps of genus 2 to 100, and construction of the first known examples of finite chiral 5-polytopes. Moreover, patterns in the maps data have led to new theorems about the genus spectrum of chiral maps and regular maps with simple underlying graph.

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1 Introduction

This paper is intended to give a brief summary of various aspects of combinatorial group theory and associated computational methods that have proved useful (to the author, at least) in the study of graphs, maps and polytopes having maximum possible symmetry under certain conditions. It extends (and updates) an earlier summary given in [9], but is not intended to be a comprehensive survey, by any means. Our aim is to provide examples of potential interest to students and others wishing to learn about the use of such theory and methods, together with some references to places where further details are available.

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Special focus is given to finitely-presented groups and means of investigating them (and their subgroups of finite index and quotients of finite order), with numerous applications.

We begin by giving some background on symmetries of discrete structures and their connection with certain finitely-presented groups, in Section 2. Then in Section 3 we briefly describe some ways in which Schreier coset diagrams can be used to depict and construct homomorphic images of these groups, and give some applications to exhibit the remarkable power of such an approach. We summarise a number of computational procedures for handling finitely-presented groups in Section 4, and then look at the particular case of methods for finding subgroups of small index and quotient of small order, in Section 5. Finally, we describe a theorem of Schur about centre-by-finite groups and its use in these contexts in Section 6, and complete the paper by announcing some recent results about the genus spectra of various classes of arc-transitive maps in Section 7.

2 Background: symmetries of discrete structures, and connections with finitely-presented groups

2.1 Symmetric graphs

An *automorphism* of a (combinatorial) graph $X = (V, E)$ is any permutation of its vertices that preserves adjacency. Under composition, the set of all automorphisms of X forms a group known as the *automorphism group* of X and denoted by $\mathrm{Aut}\,X$. A graph X is called *vertex-transitive*, *edge-transitive* or *arc-transitive* automorphism group $\mathrm{Aut}\,X$ has a single orbit on the set of vertices, edges or arcs (ordered edges) of X, respectively. Graphs which are arc-transitive are also called *symmetric*, and any graph that is edge-transitive but not vertex-transitive is called *semisymmetric*.

More generally, for any positive integer s, an *s-arc* in a graph X is a directed walk of length s which never includes the reverse of an arc just crossed — that is, an ordered $(s{+}1)$-tuple of vertices $(v_0, v_1, v_2, \ldots, v_{s-1}, v_s)$ such that any two consecutive v_i are adjacent in X and any three consecutive v_i are distinct. A graph X is then called s-arc-transitive if $\mathrm{Aut}\,X$ has a single orbit on the set of arcs of X. For example, circuit graphs are s-arc-transitive for all s, while the Petersen graph and all complete graphs on more than three vertices are 2-arc-transitive but not 3-arc-transitive.

The situation for arc-transitive 3-valent graphs (also called symmetric *cubic* graphs) is particularly interesting. In [32, 33] Tutte proved that if G is the automorphism group of a finite symmetric cubic graph, then G is sharply-transitive on the set of s-arcs of X for some $s \leq 5$ (in which case X is called *s-arc-regular*). The smallest example of a finite 5-arc-regular cubic graph is Tutte's 8-cage, on 30 vertices, depicted in Figure 1.

By further theory of symmetric cubic graphs (developed by Tutte, Goldschmidt, et al), it is now known that if X is a 5-arc-regular cubic graph, then $\mathrm{Aut}\,X$ is a homomorphic image of the finitely-presented group

$$\begin{aligned} G_5 \;=\; & \langle\, h, a, p, q, r, s \mid h^3 = a^2 = p^2 = q^2 = r^2 = s^2 = 1, \\ & pq = qp,\; pr = rp,\; ps = sp,\; qr = rq,\; qs = sq,\; sr = pqrs, \\ & ap = qa,\; ar = sa,\; h^{-1}ph = p,\; h^{-1}qh = r,\; h^{-1}rh = pqr,\; shs = h^{-1} \,\rangle, \end{aligned}$$

with the subgroups $H = \langle h, p, q, r, s\rangle$, $A = \langle a, p, q, r, s\rangle$ and $H \cap A = \langle p, q, r, s\rangle$ mapping to the stabilizers of a vertex, edge and arc, respectively.

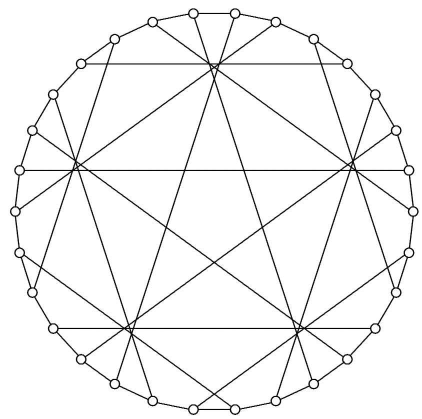


Figure 1: Tutte's 8-cage

Conversely, given any epimorphism $\theta\colon G_5 \to G$ to a finite group G, with torsion-free kernel K, a cubic graph X may be constructed on which G acts 5-arc-regularly: Take as vertices the right cosets of $V = HK$ in G_5, and join Vx to Vy by an edge whenever $xy^{-1} \in VaV$. Under right multiplication by G_5, the stabilizer of the vertex V is V, which induces S_3 on the neighbourhood $\{Ha, Hah, Hah^{-1}\}$ of H, and the group induced on X is $G_5/K \cong G$. Thus 5-arc-regular cubic graphs correspond to non-degenerate homomorphic images of the group G_5. See [18] for further details.

Tutte's work for symmetric 3-valent graphs was extended by Richard Weiss to the study of finite symmetric graphs of arbitrary valency, using the classification of doubly-transitive permutation groups. In particular, Weiss proved the following generalisation of Tutte's theorem in [34, 35]:

Theorem 1 (Weiss, 1981 & 1987) *There are no finite k-arc-transitive graphs of degree > 2 for $k \geq 8$. Moreover, if X is a finite 7-arc-transitive graph of degree $d > 2$, then $d = 3^t + 1$ for some positive integer t, and $G = \mathrm{Aut}\,X$ is obtainable as a homomorphic image of a certain finitely-presented group $R_{d,7}$.*

For example, $R_{4,7}$ has a presentation in terms of generators $p, q, r, s, t, u, v, h, b$ subject to defining relations that include $h^4 = p^3 = q^3 = r^3 = s^3 = t^3 = u^3 = v^2 = b^2 = 1$, $(hu)^3 = (uv)^2 = (huv)^2 = [h^2, u] = [h^2, v] = 1$, $[s, t] = p$, $[q, r] = 1$, and so on, and the automorphism group of every finite 7-arc-transitive 4-valent graph is a non-degenerate homomorphic image of this group $R_{4,7}$.

2.2 Regular maps

A *map* is a 2-cell embedding of a connected (multi)graph in a surface, and an automorphism of a map M is any permutation of its edges that preserves indicence. A map M is called *regular* if its automorphism group $\mathrm{Aut}\,M$ is sharply-transitive (regular) on flags, that is, on incident vertex-edge-face triples. Similarly, a map M on an orientable surface is called *rotary* (or *orientably-regular*) if the group of all its orientation-preserving automorphisms is transitive on the ordered edges of M.

If M is rotary or regular then every face has the same number of edges (say p) and every vertex has the same valency (say q), and M has *type* $\{p, q\}$. Regular maps of type $\{p, q\}$ correspond to non-degenerate homomorphic images of the full $(2, p, q)$ triangle group $\Delta = \langle\, a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ac)^2 = 1\,\rangle$; the image of $\langle a, b\rangle$ gives the stabilizer of a vertex v, the image of $\langle a, c\rangle$ gives the stabilizer of an edge e, and the image of $\langle b, c\rangle$ gives the stabilizer of a face f, where (v, e, f) is a flag, and incidence corresponds to non-empty intersection of cosets. Similarly, rotary maps of type $\{p, q\}$ correspond to non-degenerate homomorphic images of the ordinary $(2, p, q)$ triangle group $\Delta^{\mathrm{o}} = \langle\, x, y, z \mid x^p = y^q = z^2 = xyz = 1\,\rangle$, which has index 2 in Δ (when x, y, z are taken as ab, bc, ca respectively). See [12] for further details.

If the rotary map M of type $\{p, q\}$ admits no orientation-reversing automorphisms, then M is said to be irreflexible, or *chiral*, and $\mathrm{Aut}\,M$ is a quotient of the ordinary $(2, p, q)$ triangle group Δ^{o} but not the full $(2, p, q)$ triangle group Δ. In that case, the kernel K of the corresponding epimorphism $\theta\colon \Delta^{\mathrm{o}} \to \mathrm{Aut}\,M$ is not normal in Δ. Indeed if M is rotary and $\theta\colon \Delta^{\mathrm{o}} \to \mathrm{Aut}\,M$ is the corresponding non-degenerate homomorphism, then M is reflexible if and only if K is normalized by any element $a \in \Delta \setminus \Delta^{\mathrm{o}}$:

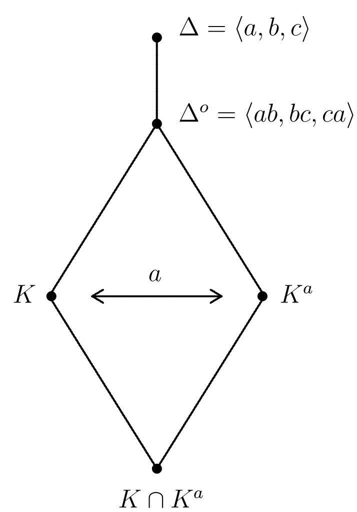


Figure 2: Chirality in terms of normal subgroups of Δ^{o}

If the subgroup K is not so normalized, then the rotary map M is chiral and K^a is the kernel of the corresponding epimorphism for the mirror image of M.

2.3 Abstract polytopes

An abstract *polytope* of rank n is a partially ordered set $\mathcal{P}$ endowed with a strictly monotone rank function having range $\{-1, \ldots, n\}$. The elements of rank $0, 1$ and $n-1$ are called the *vertices*, *edges* and *facets* of the polytope, respectively. For $-1 \leq j \leq n$, elements of $\mathcal{P}$ of rank j are called the *j-faces*, and a typical j-face is denoted by F_j. We require that $\mathcal{P}$ have a smallest (-1)-face F_{-1}, and a greatest n-face F_n, and that each maximal chain (or *flag*) of $\mathcal{P}$ has length $n+2$, and is of the form $F_{-1} - F_0 - F_1 - F_2 - \cdots - F_{n-1} - F_n$.

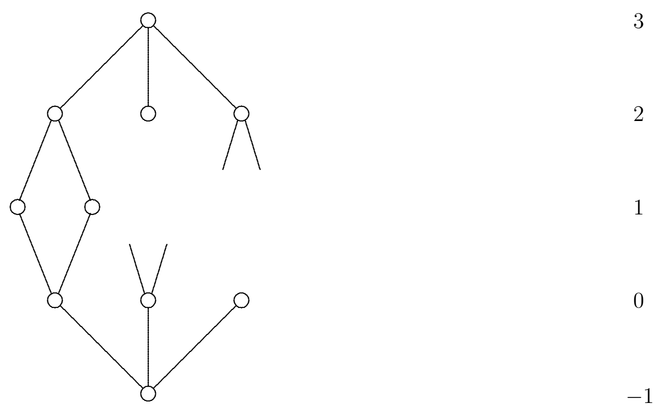


Figure 3: Partial illustration of a 3-polytope

This poset $\mathcal{P}$ must satisfy certain combinatorial conditions which generalise the properties of geometric polytopes. One requirement is a kind of homogeneity property, called the *diamond condition*: whenever $F \leq G$, with $\text{rank}(F) = j-1$ and $\text{rank}((G) = j+1$, there are exactly two j-faces H_i such that $F \leq H_i \leq G$. It is further required that $\mathcal{P}$ be *strongly flag-connected*, which means that any two flags Φ and Ψ of $\mathcal{P}$ can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$ such that each two successive faces Φ_{i-1} and Φ_i are adjacent (that is, differ in only one face), and $\Phi \cap \Psi \subseteq \Phi_i$ for all i.

An *automorphism* of an abstract polytope $\mathcal{P}$ is an order-preserving bijection $\mathcal{P} \to \mathcal{P}$. A polytope $\mathcal{P}$ is *regular* if the automorphism group $\Gamma(\mathcal{P})$ is transitive on the flags of $\mathcal{P}$.

When $\mathcal{P}$ is regular, $\Gamma(\mathcal{P})$ can be generated by n involutions $\rho_0, \rho_1, \ldots, \rho_{n-1}$, where each ρ_i maps a given *base flag* Φ to the adjacent flag Φ^i (differing from Φ only in its i-face). These generators satisfy (among others) the defining relations for the *Coxeter group* of *Schläfli type* $[p_1, \ldots, p_{n-1}]$, where $p_i = o(\rho_{i-1}\rho_i)$ for $1 \leq i < n$.

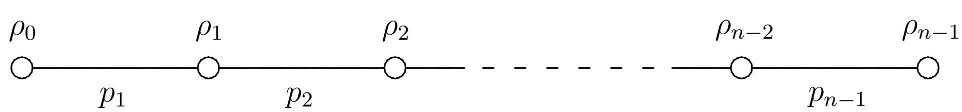


Figure 4: Dynkin diagram for the Coxeter group $[p_1, \ldots, p_{n-1}]$

The generators ρ_i for $\Gamma(\mathcal{P})$ also satisfy an extra condition known as the *intersection condition*, namely $\langle\, \rho_i : i \in I \,\rangle \cap \langle\, \rho_i : i \in J \,\rangle \;=\; \langle\, \rho_i : i \in I \cap J \,\rangle$ for every $I, J \subseteq \{0, 1, \ldots, n-1\}$.

Conversely, if Γ is a permutation group generated by n elements $\rho_0, \rho_1, \ldots, \rho_{n-1}$ which satisfy the defining relations for a Coxeter group of rank n and satisfy the intersection condition, then there exists a polytope $\mathcal{P}$ with $\Gamma(\mathcal{P}) \cong \Gamma$.

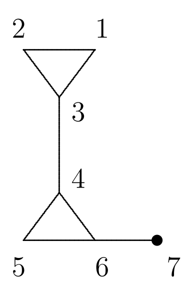
Similarly, *chiral* polytopes of rank n are obtainable from certain non-degenerate homomorphic images of the 'even-word' subgroups $\langle\, \rho_{i-1}\rho_i : 1 \le i < n \,\rangle$ of these n-generator Coxeter groups. The automorphism group of a chiral polytope has two orbits on flags, with adjacent flags always lying in different orbits.

See [15, 29] (and references therein) for further details.

3 Schreier coset diagrams

Given a transitive permutation representation of a finitely-generated group G on a set Ω, the effect of the generators of G on Ω can be depicted by a graph with Ω as vertex-set, and edges joining α to α^x for each point $\alpha \in \Omega$ and every element x in some generating set for G. Such a graph is known as a *Schreier coset graph* (or *coset diagram*) because, equivalently, given a subgroup H of G, the effect of the generators of G by right multiplication on right cosets of H can be depicted by the same graph, with Ω taken as the coset space $(G\!:\!H)$, and edges joining Hg to Hgx for every generator x. (The correspondence is obtained by letting H be the stabilizer of any point of Ω.) See [25, 8] for further details.

For example, the following is a coset diagram for an action of the ordinary $(2, 3, 7)$ triangle group $\langle\, x, y \mid x^2 = y^3 = (xy)^7 = 1 \,\rangle$ on 7 points, in which the triangles and heavy dot depict 3-cycles and the fixed point of the permutation induced by the generator y:



$x \mapsto (3, 4)(6, 7)$

$y \mapsto (1, 2, 3)(4, 5, 6)$

Figure 5: Example of a coset diagram

Often two Schreier coset diagrams for the same group G on (say) m and n points can be *composed* to produce a transitive permutation representation of larger degree $m+n$. This technique (attributable to Graham Higman) can be used in some instances to construct families of epimorphic images of the given group G (and interesting objects on which they act), and to prove that G is infinite. For example, one method of composition of coset graphs for the ordinary $(2, 3, 7)$ triangle group is illustrated in Figure 6 below.

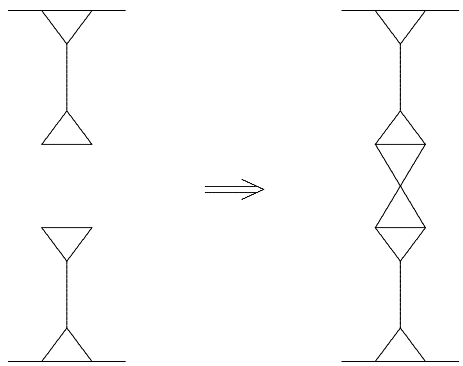


Figure 6: Composition of coset graphs for the ordinary $(2, 3, 7)$ triangle group

This method was used to prove, for example, that for every integer $m \geq 7$, all but finitely alternating and symmetric groups are epimorphic images of the $(2, 3, m)$ triangle group — and hence for all but finitely many n, there exists a rotary map M of type $\{3, m\}$ with A_n or S_n as its orientation-preserving group of automorphisms (see [5]). In fact all those maps are regular, but the same method can be adapted to prove that for all $m \geq 7$ and for all but finitely many n (for each m), there exists a chiral map M of type $\{3, m\}$ with $\mathrm{Aut}(M) \cong A_n$ (see [3]). More generally, Brent Everitt has proved that every Fuchsian group has all but finitely many A_n among its epimorphic images; see [27].

A variant of this method of composition can be used to prove that there are infinitely many 5-arc-transitive connected finite cubic graphs [6], and infinitely many 7-arc-transitive connected finite 4-valent graphs [24]. A more careful analysis shows even that there are infinitely many 5-arc-transitive 3-valent finite *Cayley graphs*, and that every such Cayley graph is a cover of one of just six examples, and that for every positive integer t, there are infinitely many 7-arc-transitive finite Cayley graphs of valency $1+3^t$ (see [10]).

4 Computational procedures

The last 40 years have seen the development of a wide range of efficient computational procedures for investigating groups with a small number of generators and defining relations. Here we give a brief description of some of those which are very useful in the kinds of contexts mentioned earlier in this paper. All of these procedures are available in the Magma package [1]. For further details and references, see the very helpful books on computational group theory by Sims [31] and Eick, Holt & O'Brien [28].

• Todd-Coxeter coset enumeration: This attempts to determine the index of a given finitely-generated subgroup H in a given finitely-presented group $G = \langle X \mid R \rangle$, by systematically enumerating the right cosets of H in G; when it succeeds, the output can given in the form of a *coset table* (in which the (i, j)th entry indicates the number of the coset obtained by multiplying the ith coset of H by the jth generator of G), or as permutations induced by the generators of G on right cosets of H.

• Reidemeister-Schreier algorithm: This gives a defining presentation for a subgroup H of finite index in a finitely-presented group $G = \langle X \mid R \rangle$, when the coset table is known; the generators for H are Schreier generators (obtainable from a Schreier transversal for H in G, which can be identified with a rooted spanning tree for the corresponding coset graph for $(G\!:\!H)$), and the relations are easily derived from the coset table and the relations for G (or by 'chasing' each relation for G around the coset diagram).

• Abelian quotient algorithm: This produces the direct factors of the abelianisation $G/G' = G/[G,G]$ of a finitely-presented group $G = \langle X \mid R \rangle$, in Smith normal form. When taken together with a variant of the Reidemeister-Schreier algorithm, it can also determine the abelianisation H/H' of a subgroup H of finite index in G, when the coset table for H in G is known.

• Low index subgroups algorithm: This finds a representative of each conjugacy class of subgroups of up to a given index n in a finitely-presented group $G = \langle X \mid R \rangle$, and will be explained further in the next Section.

• p-quotient algorithm: This finds, for a given prime p and a given positive integer c, the largest possible quotient P of the finitely-presented group $G = \langle X \mid R \rangle$ with the property that P is a p-group of class at most c; for example, when $c = 1$ or 2 this is the largest abelian or metabelian p-quotient, respectively.

• Nilpotent quotient algorithm: This finds, for a given positive integer c, the largest possible nilpotent quotient of the finitely-presented group $G = \langle X \mid R \rangle$ of class at most c; for example, when $c = 1$ or 2 this is the largest abelian or metabelian quotient, respectively.

5 Low index subgroups methods

Given a finitely-presented group $G = \langle X \mid R \rangle$ and a (small) positive integer n, all subgroups of index up to n in G can be found (up to conjugacy) by a systematic enumeration of coset tables with up to n rows. In practice, this is achieved by using an extended coset table, which includes the effect of multiplying cosets of the (pseudo-)subgroup by the inverses of the elements of the generating set X for G, as depicted below:

	x_1	x_2	$\dots$	x_1^{-1}	x_2^{-1}	$\dots$
1	2	3		4		
2				1		
3					1	
4	1					
:						

Figure 7: An extended coset table

Such tables are assumed to be in *normal form*, which means that lexicographically, no coset number j appears for the first time before a coset number k less than j. The enumeration procedure usually defines more than n cosets, and then coincidences are forced between cosets. As cosets Hv and Hw of a subgroup H are equal if and only if $vw^{-1} \in H$, forcing any coincidence gives rise to a new element of the subgroup, which is then taken as an additional generator of the subgroup. The fact that every subgroup of finite index in G is finitely-generated (by Schreier's theorem) ensures that this procedure will terminate, given sufficient time and memory. See [31, 28, 14] for further details and references.

A key point about the low index subgroups algorithm is that it can be used to find 'small' finite epimorphic images of a finitely-presented group G: for each subgroup H of index n in G, the permutations induced by generators of G on right cosets of H generate the factor group G/K where K is the core of H (the intersection of all conjugates of H) in G, as a subgroup of S_n.

These images can often be used as the 'building blocks' for the construction of larger images (as in Section 3), or produce interesting examples in their own right. For instance, the first known examples of arc-transitive cubic graphs admitting no edge-reversing automorphisms of order 2, and first known 5-arc–transitive cubic graph having no s-arc-regular group of automorphisms for $s < 5$, were found in this way (see [18]). The same approach was used to help construct infinite family of 4-arc-transitive connected finite cubic graphs of girth 12, and then (unexpectedly) to a new symmetric presentation for the special linear group $\mathrm{SL}(3, \mathbb{Z})$; see [7]. Similarly, it enabled the construction of a infinite family of vertex-transitive but non-Cayley finite connected 4-valent graphs with arbitrarily large vertex-stabilizers in their automorphism groups [23], the first known example of a finite half-arc-transitive (vertex- and edge-transitive but not arc-transitive) 4-valent finite graph with non-abelian vertex-stabilizer [20], and the first known examples of finite chiral polytopes of rank 5 [15].

Two drawbacks of the (standard) low index subgroups algorithm are the fact that the finite quotients it produces can have large order but small minimal degree (as permutation groups), and the fact that it tends to be very slow for large index n or complicated presentations. (Also for some groups, like the modular group $\langle\, x, y \mid x^2, y^3 \,\rangle$, the number of subgroups grows exponentially, making it impossible to search very far.)

It is not difficult, however, to adapt the algorithm so that it produces only normal subgroups (of up to a given index), and this adaptation runs much more quickly (for given maximum) index, and hence can produce all quotients of up to a given order, not just those which have faithful permutation representations of small degree.

Such an adaptation was developed by the author and Peter Dobcsányi (as part of Peter's PhD thesis project), and applied to find all rotary and regular maps on orientable surfaces of genus 2 to 15, all regular maps on non-orientable surfaces of genus 2 to 30, and all arc-transitive cubic graphs on up to 768 vertices; see [12, 13]. It was also subsequently used to help find all semisymmetric cubic graphs on up to 768 vertices [19], and to assist in obtaining a refined classification of arc-transitive group actions on finite cubic graphs (by types of arc-transitive subgroups) [21].

Recently, a new method for finding normal subgroups of small index has been developed by Derek Holt and his student David Firth. This systematically enumerates the possibilities for the composition series of the factor group G/K (for any normal subgroup K of small index in G), and works much faster, for index up to $100,000$ in many groups with straightforward presentations. It too has been implemented in the MAGMA package [1].

This new method has enabled the determination of all rotary and regular maps (and hypermaps) on orientable surfaces of genus 2 to 101, all regular maps on non-orientable surfaces of genus 2 to 202, and all arc-transitive cubic graphs on up to 2048 vertices (and thereby the accidental discovery of largest known cubic graph of diameter 10); see [11]. Consequences of finding patterns in the list of maps of small genus will be described in Section 7.

6 Schur's theorem

A particularly useful (but not so well known) piece of combinatorial group theory is *Schur's theorem* on centre-by-finite groups.

Theorem 2 (Schur) *If the centre $Z(G)$ of the group G has finite index $|G:Z(G)| = m$ in G, then the order of every element of $G' = [G,G]$ is finite and divides m.*

Closely tied to the Schur-Zassenhaus theorem, this theorem can be proved easily using the transfer homomorphism $\tau\colon G \to Z(G)$ (which takes $g \mapsto g^m$ for all $g \in G$), once it is noted that $\ker\tau$ contains G'. See [30] for further background and details.

The author is grateful to Peter Neumann for pointing out the usefulness of Schur's theorem in order to obtain the following in some work with Ravi Kulkarni on families of automorphism groups of compact Riemann surfaces:

Theorem 3 ([16]) *Let p, q and d be positive integers, with $\gcd(p,q) = 1$. Then there are only finitely many finite groups which can be generated by two elements x and y of orders p and q respectively such that xy generates a subgroup of index at most d.*

In turn, the above helps disprove the possibility that certain cyclic-by-finite groups might be rotation groups of orientably-regular maps. For example, suppose the map M is a 'central cover' of the octahedral map, of type $\{3, 4t\}$ for some t, with rotation group $G = \mathrm{Aut}^o M \cong \langle\, x, y, z \mid x^3 = y^{4t} = (xy)^2 = 1,\ [x, y^4] = 1 \,\rangle$. How large can t be? Here we may note that $Z(G)$ contains $N = \langle y^4 \rangle$, with $G/N \cong S_4$, so $|G : Z(G)|$ divides 24. Also $G/G' \cong C_2$, and $G' = \langle x, y^{-1}xy, y^2 \rangle$. Hence by Schur's theorem, the order of y^2 divides 24, so t divides 12. (But furthermore, we can use Reidemeister-Schreier theory to obtain presentations (and hence the orders) of subgroups of G: the index 6 subgroup $H = \langle\, y^2, (xy^{-1})^2 \,\rangle$ has order dividing 24, so $|G| = |G\!:\!H||H|$ divides 144, so t divides 6.)

7 More recent results

Some new discoveries have been made (and proved) very recently as a result of observations made about the data produced from the computations described at the end of Section 5.

Two major breakthroughs in the study of rotary and regular maps were made possible by noticing that there is no orientably-regular but chiral map of genus 2, 3, 4, 5, 6, 9, 13, 23, 24, 30, 36, 47, 48, 54, 60, 66, 84 or 95, and similarly that there is no regular orientable map of genus 20, 32, 38, 44, 62, 68, 74, 80 or 98 with simple underlying graph. A lot of these exceptional genera are of the form $p+1$ where p is prime — a phenomenon that was not so easy to observe until the rotary maps of genus 2 to 100 were known — and this observation led to the following (proved in joint work with Jozef Siráň and Tom Tucker):

Theorem 4 [22] *If M is an irreflexible (chiral) orientably-regular map of genus $p+1$ where p is prime, then*
either $p \equiv 1$ *mod* 3 *and* M *has type* $\{6,6\}$,
or $p \equiv 1$ *mod* 5 *and* M *has type* $\{5,10\}$,
or $p \equiv 1$ *mod* 8 *and* M *has type* $\{8,8\}$.
In particular, there are no such maps of genus $p+1$ whenever p is a prime such that $p-1$ is not divisible by 3, 5 *or* 8.

Theorem 5 [22] *There is no regular map M with simple underlying graph on an orientable surface of genus $p+1$ whenever p is a prime congruent to* 1 *mod* 6, *for $p > 13$.*

In fact, what was achieved in [22] is a complete classification of all regular and orientably-regular maps M for which $|\mathrm{Aut}\, M|$ is coprime to the map's Euler characteristic χ (if χ is odd) or to $\chi/2$ (if χ is even), and that leads not only to the above two theorems, but also to a simpler proof of the following theorem of Breda, Nedela & Siráň:

Theorem 6 [2] *There is no regular map M on a non-orientable surface of genus $p+1$ whenever p is a prime congruent to* 1 *mod* 12, *for $p > 13$.*

Here is a sketch proof of the classification leading to these three results. First, let M be a rotary map on an orientable surface of genus g, let $G = \mathrm{Aut}^o M$ be its group of orientation-preserving automorphisms, and suppose $|G|$ is coprime to $g-1$. Then by the Euler-Poincaré formula, the type $\{k,m\}$ of M is restricted (by arithmetic) to one of five different families. Moreover, the group $G = \mathrm{Aut}^o M$ is almost Sylow-cyclic, meaning that every Sylow subgroup of odd order in G is cyclic, and every Sylow 2-subgroup of G contains a cyclic subgroup of index 2. The Suzuki-Wong classification of non-solvable almost Sylow-cyclic groups can be used to deduce that $G = \mathrm{Aut}^o M$ is solvable, except in the case of one of the five families. It is then possible to classify those cases where the vertex-stabilizer and face-stabilizer intersect trivially, and use Ito's theorem and Schur's transfer theory to deal with the more general case. What is remarkable is that the map M turns out to be reflexible whenever the coprime condition is satisfied.

Another outcome concerns reflexibility of Cayley maps. Briefly, a Cayley map for a group G is an embedding of a Cayley graph for G in a surface as a rotary map — or equivalently, a rotary map which admits the action of G as a group of automorphisms acting regularly (sharply-transitively) on vertices. From an inspection of the rotary maps of genus 2 to 100 in [11], it was noticed that for small genus, a rotary Cayley map for a cyclic group is reflexible if and only if it is anti-balanced (that is, if and only if the embedding of the Cayley map sees the neighbours of the identity element ordered in a way that is reversed by their inversion), and then this was proved in general in a piece of joint work with Jozef Siráň and Young Soo Kwon [17] just a few months before this paper was written.

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