Fixity of graphs

Presented by

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Based on joint work with

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and also with: Gabriel Verret, Martin Mačaj, ...

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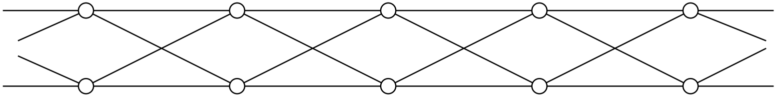
Question



Suppose $g \in \mathrm{Aut}(\Gamma)$, $g \neq \mathrm{id}$, fixes "a lot" of vertices.

What can we say about the graph?

Fixity: $\mathrm{Fx}(\Gamma) := \max\{|\mathrm{Fix}(g)| : g \in \mathrm{Aut}(\Gamma) \setminus \{\mathrm{id}\}\}$

Motion: $m(\Gamma) := \min\{|\mathrm{Supp}(g)| : g \in \mathrm{Aut}(\Gamma) \setminus \{\mathrm{id}\}\}$

$$\mathrm{Fx}(\Gamma) = |\mathrm{V}\Gamma| - m(\Gamma)$$

Historical context for permutation groups: $G \leq S_n$

$\mathrm{Fx}(G) := \max\{|\mathrm{Fix}(g)| : g \in G, g \neq \mathrm{id}\}$.

$\mathrm{Fx}(G) = n -$ "minimal degree of G".

Jordan: If G is primitive:

- $\mathrm{Fx}(G) = n - 2 \Rightarrow G = S_n$;
- $\mathrm{Fx}(G) = n - 3 \Rightarrow G = A_n$;
- for every c, there is a finite set of exceptions E_c, such that $\mathrm{Fx}(G) = n - c \Rightarrow G = A_n, S_n$ or $G \in E_c$.

Babai, Liebeck, Saxl, Guralnick, Magaard,...:

All primitive groups G with $\mathrm{Fx}(G) > \frac{1}{2}n$ are known.

Less known for imprimitive permutation groups.

Back to graphs

Motivation:

- Groups acting arc-transitively on connected (di)graphs generalise primitive permutation groups.
- "Large fixity" is related to "large automorphism group".

  $$\text{(Conder, Tucker)} \qquad G \text{ transitive } \Rightarrow |G| \geq \frac{n}{2}\, 2^{\frac{1}{1-\mathrm{RelFx}(G)}}$$

  where $\mathrm{RelFx}(G) := \mathrm{Fx}(G)/n$.
- Related to distinguishing number.
- Can be useful for "polycirculant conjecture".
- Pure curiosity!

Question

Can we somehow non-trivially bound the fixity?

NO

- $\mathrm{Fx}(K_n) = n - 2, \; \mathrm{Fx}(K_{a,b}) = (a + b) - 2.$
- $\mathrm{Fx}(\Gamma) = n - 2 \iff \exists u, v : \Gamma(u) \setminus \{v\} = \Gamma(v) \setminus \{u\}.$
- Corollary: Suppose Γ is arc-transitive and $\mathrm{Fx}(\Gamma) = n - 2$.
  - If 3-valent, then: $\Gamma \cong K_4$ or $K_{3,3}$.
  - If 4-valent, then: $\Gamma \cong K_5$ or $C_m[2K_1]$.

Problem: Suppose we are given a class of graphs $\mathcal{G}$. Find a a function $f : \mathbb{N} \to \mathbb{N}$ (as slowly growing as possible) such that all (but finitely many) graphs $\Gamma \in \mathcal{G}$ satisfy

$$\mathrm{Fx}(\Gamma) \leq f(|\mathrm{V}(\Gamma)|)$$

Look at the data

Let us look at existing datasets of VT graphs; for example:

- 3-valent arc-transitive graphs up to 10.000 vertices (Conder & Dobcsanyi, Conder);
- 4-valent arc-transitive graphs up to 640 vertices; (PSV)
- 4-valent $\frac{1}{2}$-arc-transitive graphs up to 1.000 vertices; (PSV)
- 3-valent vertex-transitive graphs up to 1.280 vertices; (PSV)

3-valent arc-transitive graphs

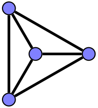
3-valent arc-transitive graphs

Let Γ be cubic arc-transitive and $G = \mathrm{Aut}(\Gamma)$. By Tutte:

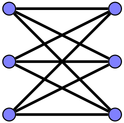
$$|G| \leq 48\,|\mathrm{V}\Gamma|.$$

This allows construction of a complete census up to a much larger order; up to order 10 000 at the moment – record holder:

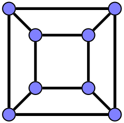
Marston Conder



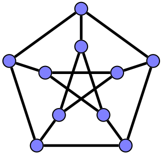
4, K_4



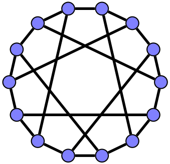
6, $K_{3,3}$



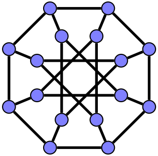
8, Q_3



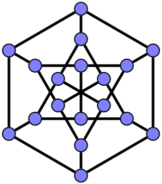
10, Petersen graph



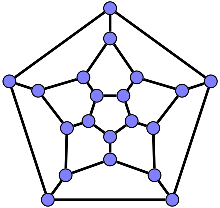
14, Heawood



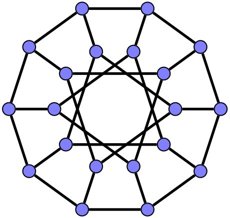
16, Mbius-Kantor



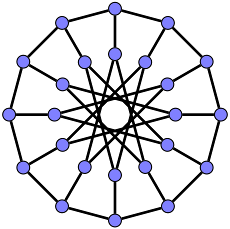
18, Pappus



20, Dodecahedron



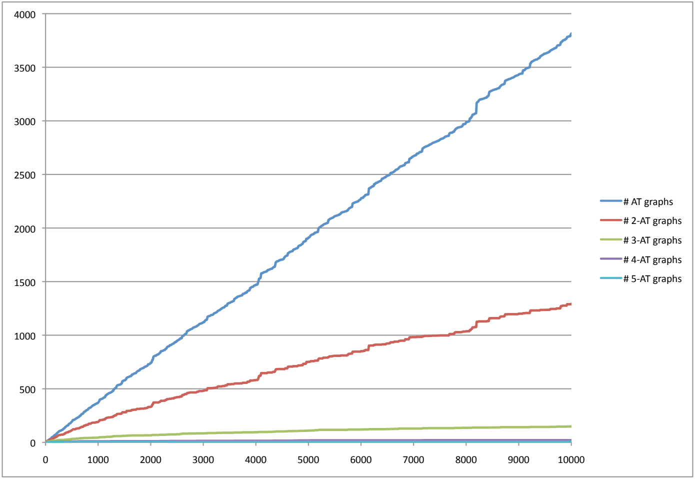
20, Desargues



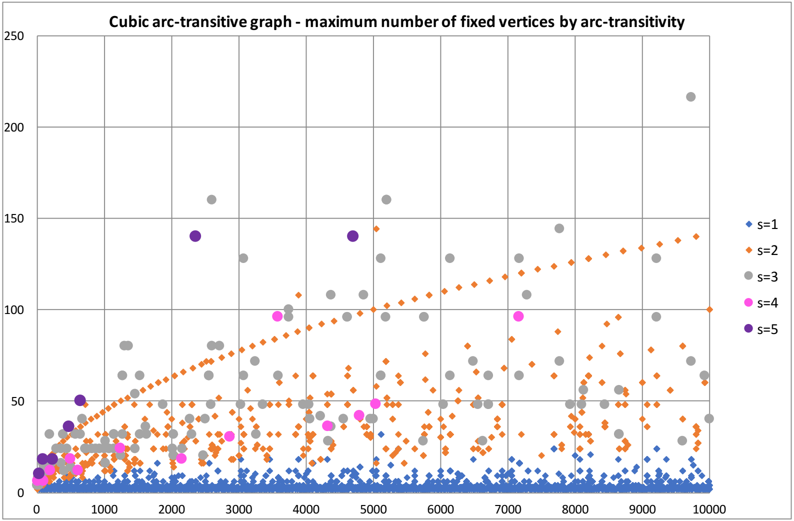
24, Nauru

Number of cubic arc-transitive graphs

There are 3 815 cubic arc-transitive graphs of order up to 10 000.



Fixicity of cubic arc-transitive graphs



$$f(n) = \sqrt{2n}$$

Cubic arc-transitive graphs with fixity $\sqrt{2n}$

Construction by Gabriel:

$G = \langle u, v, t \mid u^m, v^m, t^2, [u, v], u^t = u^{-1}, v^t = v^{-1} \rangle \cong \mathbb{Z}_m^2 : \mathbb{Z}_2$

$a = ut,\ b = vt,\ c = u^{-1}v^{-1}t$ (three involutions)

$\Gamma = \mathrm{Cay}(G; \{a, b, c\})$

$\sigma : u \mapsto v \mapsto u^{-1}v^{-1},\ t \mapsto t$

$\sigma \in \mathrm{Aut}(G),\ a \mapsto b \mapsto c \mapsto a$ $\quad \sigma \in Aut(\Gamma)_{1_G}$

Suppose there exists $\lambda \in \mathbb{Z}_m^*$ such that $\lambda^2 + \lambda + 1 = 0$.

Then σ fixes pointwise $\langle u^{-1}v^{\lambda}, t \rangle \cong \mathbb{Z}_m : \mathbb{Z}_2$, hence:

$$\mathrm{Fx}(\Gamma) \geq 2m = \sqrt{2n}$$

Fixity of cubic arc-transitive graphs

Conjecture: Apart from a finite set of exceptions, if Γ is a connected cubic arc-transitive graph, then

$$\mathrm{Fx}(\Gamma) \leq \sqrt{2|\mathrm{V}\Gamma|}.$$

Theorem (Spiga, Lehner, PP)

There exists a sublinear function $f(n)$, such that if Γ is a large enough connected cubic arc-transitive graph, then

$$\mathrm{Fx}(\Gamma) \leq f(|\mathrm{V}\Gamma|).$$

Essential assumptions for the proof

- G_v acts primitively on $\Gamma(v)$;
- $|G_v|$ is bounded by a constant.

We can in fact prove a more general version:

Theorem (Spiga, Lehner, PP)

For every d, there exists a sublinear function $f(n)$, such that if Γ is a large enough connected 2-arc-transitive graph of valency d, then

$$\mathrm{Fx}(\Gamma) \leq f(|\mathrm{V}\Gamma|).$$

Theorem (Spiga, Lehner, PP)

For every quasi-primitive and graph-restrictive permutation group L there exists a sublinear function $f(n)$, such that if Γ is a large enough connected G-arc-transitive graph with $G_v^{\Gamma(v)} \cong L$, then

$$\mathrm{Fx}(G) \leq f(|\mathrm{V}\Gamma|).$$

Some lemmas from the proof

L1: Let $G \leq \mathrm{Sym}(\Omega)$ be transitive and $G^+ = \langle G_\omega : \omega \in \Omega \rangle$, then

$$\exp(G) \text{ divides } |G : Z(G)|\ |\Omega/G^+|.$$

Proof: Clever use of a transfer theorem.

L2: There exists a function $f : \mathbb{N} \to \mathbb{N}$, such that: If Γ is regular G-locally-primitive not complete bipartite graph, then

$$|G| \leq f(|G : Z(G)|).$$

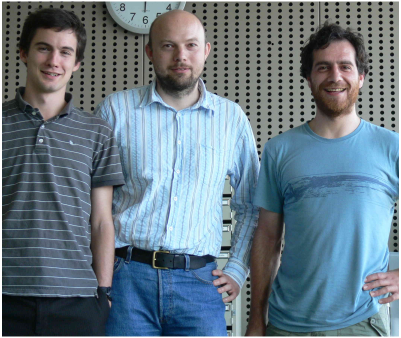
Proof: By L1 and using $\mathrm{rank}(Z) \leq \beta(\Gamma/Z)$. .

L3: If Γ is cubic G-locally-arc-transitive and $g \in G$, then

$$\frac{C_G(g)}{|G|} \leq \frac{|\mathrm{Fix}(g)|}{|\Omega|} \leq \frac{|G_\omega|\ |C_G(g)|\ |\Omega/G|}{|G|}$$

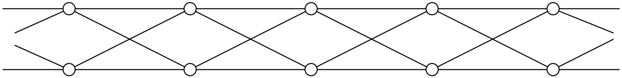
Proof: Double counting.

4-valent arc-transitive graphs



Census of 4-valent arc-transitive graphs

Difficulty: Automorphism group can be very large.

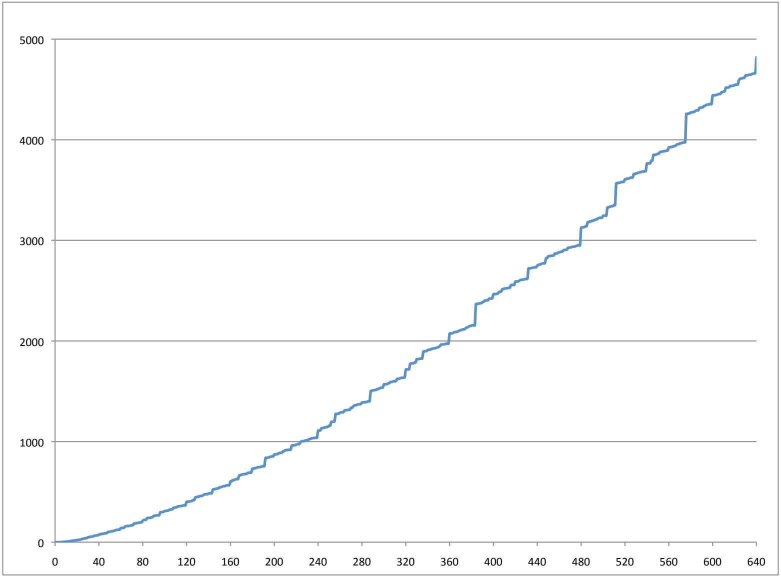


Theorem (P. Spiga, G. Verret, PP)

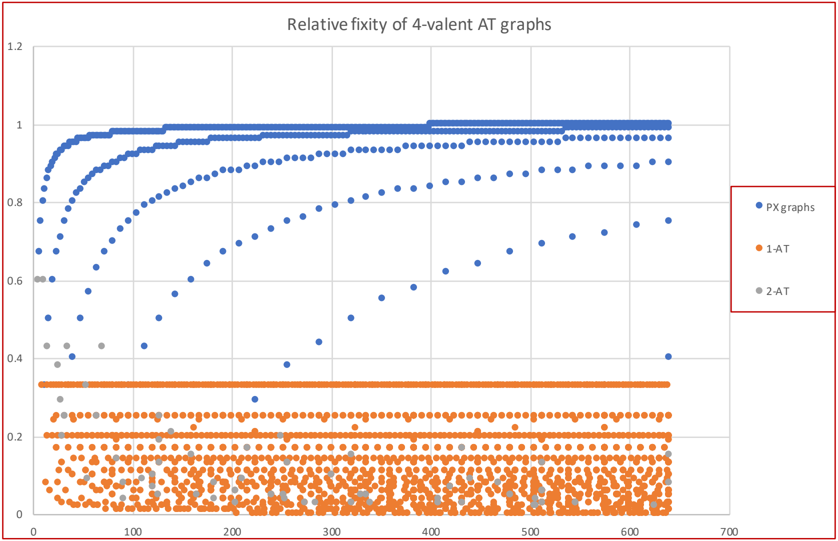
Let Γ be a connected 4-valent arc-transitive graph. Unless Γ belongs to a well-defined infinite family of graphs, or to an explicit list of small exceptions, the order of $\mathrm{Aut}(\Gamma)$ is bounded by a subquadratic function of $|V(\Gamma)|$.

All 4-valent arc-transitive graphs of order ≤ 640 are known! There are 4 820 of them. (More that cubic AT with $\leq 10\,000$ vertices.)

The number of 4-valent arc-transitive graphs

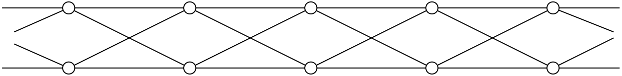


Relative fixity of tetravalent arc-transitive graphs



Praeger-Xu graphs $\mathrm{PX}(r, s)$

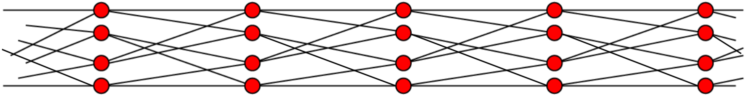

For $s = 1$: $\mathrm{PX}(r, 1) := C_r[2K_1]$.



For $s \geq 2$:

- Vertices:= "traversing" $(s-1)$-paths;
- Adjacency:= "maximal overlap of paths".

$\mathrm{PX}(r, 2)$:



Praeger-Xu graphs $\mathrm{PX}(r, s)$

- $|\mathrm{V}(\mathrm{PX}(r,s))| = r2^s$;
- $|\mathrm{Aut}(\mathrm{PX}(r,s)| = |\mathrm{Aut}(\mathrm{PX}(r,1)| = r2^{r+1}$; (unles $r = 4$)
- $\mathrm{Fx}(\mathrm{PX}(r,s)) = (r-s)2^s$;
- $\mathrm{RelFx}(\mathrm{PX}(r,s)) = \frac{r-s}{r} = 1 - \frac{s}{r}$.

Characterisation:

Theorem [Praeger, Xu; (PSV)] Let Γ be a connected 4-valent G-arc-transitive graph (but not G-arc-regular). Then

$$\Gamma \cong \mathrm{PX}(r,s) \iff \exists N \lhd G,\ N_v \neq 1,\ N \text{ abelian}.$$

Fixity of 4-valent arc-transitive graphs

Theorem [Spiga, PP; 2019] Let Γ be a connected 4-valent arc-transitive graph. If $\mathrm{Fx}(\Gamma) > \frac{1}{3}$, then either:

- $\Gamma \cong \mathrm{PX}(r, s)$ with $1 \leq s < 2r/3$; or
- Γ is one of six 2-arc-transitive exceptions.

Remarks:

- The theorem holds also for half-arc-transitive graphs;
- Due to relationship between 4-valent arc-transitive graphs and 3-valent vertex- but not arc-transitive graphs, we get:

Theorem Let Γ be a connected 3-valent vertex-transitive graph. If $\mathrm{Fx}(\Gamma) > \frac{1}{3}$, then either:

- $\Gamma \cong \mathrm{Split}(\mathrm{PX}(r, s))$; or
- Γ is one of six arc-transitive exceptions.

A few words about the proof

- If $G \leq \mathrm{Sym}(\Omega)$, $N \lhd G$, $N_\omega = 1$, $g \in G$, then
$$\mathrm{RelFx}_{\Omega/N}(g) \geq \mathrm{RelFx}_{\Omega}(g)$$

This allows inductive approach by considering quotients.

- If G_ω is a 2-group and $\mathrm{O}_2(G) = 1$, then $\mathrm{Fx}(G) \leq 1/3$.
- If Γ not 2-arc-transitive, there exists a minimal normal subgroup $N \lhd G$ with $N \cong \mathbb{Z}_2^d$.
  - If $N_\omega \neq 1$, then $\Gamma \cong \mathrm{PX}(r, s)$;
  - If $N_\omega = 1$, then $\Gamma \to \Gamma/N$ is a covering projection, $\mathrm{Fx}(\Gamma/N) > 1/3$, and by induction, $\Gamma/N \cong \mathrm{PX}(r, s)$. The proof then follows by careful consideration of elementary abelian covers of $\mathrm{PX}(r, s)$.
- If Γ is 2-arc-transitive, we use the fact that $|\mathrm{Aut}(\Gamma)_v|$ is bounded above by a constant. Plus subtle examination of almost simple groups.

Conclusion

- The proofs were almost entirely algebraic.

  Can we use more graph theoretical approaches?

- Datasets of graphs were essential in posing the conjectures!
- What happens with larger valence?
- Explore the relationship between $|\mathrm{Aut}(\Gamma)|$ and $\mathrm{Fx}(\Gamma)$.
- What can be said about graphs with small fixity? (GRRS, FGR, ...)