# Avoiding the Gorenstein-Walter theorem in the classification of regular maps of negative prime Euler characteristic

Jozef Širáň
STU and OU

Joint work with Marston Conder

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Although the original 160-page proof of the Gorenstein-Walter theorem was later supplanted by an alternative 25-page argument by Bender and Glauberman (1981) and Bender (1981) using Brauer characters, the shorter proof still depends on a number of substantial facts, including the Odd Order Theorem.

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In the talk we briefly outline new proofs of those three facts (and hence the entire classification) using somewhat more elementary group theory, without referring to the Gorenstein-Walter theorem.

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It is well known in our circles that a regular map $M$ of type $\{m, k\}$ may be identified with $G = \mathrm{Aut}(M)$ in its a *standard* partial presentation

$$G = \langle\, x, y, z \mid x^2, y^2, z^2, (xy)^2, (yz)^k, (zx)^m, \ldots \rangle$$

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Such a map $M$ has $|G|/(2k)$ vertices, $|G|/4$ edges and $|G|/(2m)$ faces; its Euler characteristic is $\chi = \frac{1}{2}(\frac{1}{k} + \frac{1}{m} - \frac{1}{2})|G|$, assumed now to be $-p$. By Conder and Dobcsányi (2001) it was sufficient to consider $p \geq 29$.

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Euler's formula implies $|G| = 4kmp/(km - 2k - 2m)$. By Sylow theory (note: Sylow 2-subgroups are dihedral) and a few elementary facts one concludes that $p \nmid |G|$. Hence $km - 2k - 2m = cp$ and further arguments using non-orientability criterion $G = \langle r, s\rangle$ give $|G| = tkm$ for $t \in \{1, 2, 4\}$.

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**2.** Let $|G| = 2km$ with $k$ odd, $m$ even, $k \geq 3$, $m \geq 4$, and $\gcd(k, m) = 1$. Then $G = \langle r\rangle\langle x, z\rangle \cong C_k\, D_8$, with $3 \mid k$, $m = 4$, and $G$ has a presentation

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**3.** Let $|G| = km$, $k = 2j \geq 4$, $m = 2\ell$, $(j, \ell) = 1$, $\ell$ odd and $\geq 3$. Then $G = \langle r^2, y\rangle\langle s^2, x\rangle \cong D_j \times D_\ell$, with $j$ odd, and $G$ has a presentation

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All these were proved with the help of the Gorenstein-Walter theorem:

*Let $G$ have dihedral Sylow 2-subgroups. If $O$ is the largest odd-order normal subgroup of $G$, then $G/O \cong$ either a Sylow 2-subgroup of $G$, or $A_7$, or a group $K$ such that $\mathrm{PSL}(2, q) \leq K \leq \mathrm{P\Gamma L}(2, q)$ for some $q \geq 3$.*

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Why? Let $u, v$ be two non-conjugate involutions in $G$. Then $gug^{-1} \notin \langle v\rangle$ for any $g \in G$, so $\langle v\rangle gu \neq \langle v\rangle g$ for each $g \in G$. Right mult'n by $u$ induces a fixed-point free involution on $2km$ cosets $G\!:\!\langle v\rangle$. But $G' = G$! $\Box$

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First, show that $C_G(S) = S$. Next, $N/S = N_G(S)/C_G(S)$ is isomorphic to a subgroup of $\mathrm{Aut}(S) \cong \mathrm{Aut}(C_2 \times C_2) \cong S_3$ and so $|N\,{:}\,S| \leq 6$, but $|S| = 4$ while $8 \nmid |G|$, so that $|N\,{:}\,S|$ cannot be even, $\Rightarrow$ $|N\,{:}\,S| = 3$ and $|N| = 3|S| = 12$. # of involutions in $G$ is $3|G\,{:}\,N_G(S)| = |G|/4$. $\square$

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• For every involution $u \in G \backslash N$ one has $N \cap N^u \cong C_3$; conjugation by $u$ inverts $N \cap N^u$. Elements of $N_G(S)$ of order $3$ are self-centralising in $G$.

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Hence the number of involutions of $G$ lying outside $N$ is equal to three times the number of subgroups of order $3$ in $N$, namely $3 \cdot 4 = 12$.

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Further three involutions are in $N$, so $G$ has exactly $15$ involutions. But we saw that the number of involutions in $G$ is equal to $|G|/4$, and so $|G| = 60$. Finally, since $G$ is perfect, it follows that $G \cong A_5$. $\square$

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Now $G = \langle r\rangle\langle x, z\rangle$, so that $G$ is soluble (Huppert '53). For the Fitting subgroup $F$ of $G$ we then have $C_G(F) = Z(F) \leq F$, so that conjugation of $F$ by $G$ induces a hom $G \to \mathrm{Aut}(F)$ with kernel contained in $F$. Easy: $F = F_1 \times F_2$ where $F_1$ is cyclic of odd order and $F_2$ is a 2-group or trivial.

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The fact that $F_2$ is characteristic in $G \Rightarrow F_2 = \langle x, s^2\rangle$ of order $m$, with $G/F_2 \cong \langle y, z\rangle \cong D_k$ of order $2k$. Conjugation of $F_2$ by $y \Rightarrow m = 4$, and $F_2 = \{1, x, s^2, xs^2\}$. Finally, conjugation of $F_2$ by $r \Rightarrow r^{-3}xr^3 = x$.

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Recalling $r{=}yz$, $s{=}zx$, consider a more general group $U$ with presentation

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It follows that for any positive integer $j$ we can factor out the normal subgroup generated by $r^{3j}$, to obtain a quotient of order $24j = 2km$ where $k = 3j$ (and $m = 4$), with the required presentation. $\square$

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In particular, $k/2 = j$ must be odd. The above relations and oddness + coprimality of $\ell = m/2$ and $j = k/2$ imply that $G$ is the direct product of its dihedral subgroups $\langle r^2, y\rangle \cong D_j$ and $\langle s^2, x\rangle \cong D_\ell$, as required.

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Factoring out $\langle r^j s^\ell z\rangle$ we obtain a quotient of order $4j\ell = km$. $\square$

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Thank you.