Regular Representations of Groups

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SODO, Rotorua, NZ


Arctic Ocean
Greenland
(Denmark)
Baffin Bay
Alaska
(United States)
Resolute
Inuvik
Cambridge Bay
Dawson
Yukon
Echo Bay
Whitehorse
Northwest
Territories
Nunavut
Iqaluit
Labrador Sea
Watson Lake
Yellowknife
Rankin Inlet
Ivujivik
Hay River
Fort Nelson
Fort Smith
Hudson
Bay
Newfoundland
and
Labrador
Happy Valley
Schefferville
Gander
St. John's
Churchill
British
Columbia
Prince Rupert
Saskatchewan
Grande Prairie
Alberta
Prince George
Edmonton
Manitoba
Chisasibi
Sept-Îles
Quebec
Gulf of Saint
Lawrence
Saint Pierre
and Miquelon
(France)
Flin Flon
Prince Edward
Island
Sydney
Prince Albert
Red Deer
Chibougamau
Moosonee
Charlottetown
New
Brunswick
Fredericton
Moncton
Vancouver
Kelowna
Calgary
Saskatoon
Ontario
Quebec
Halifax
Victoria
Brandon
Winnipeg
Lethbridge
Regina
Montreal
Saint John
Nova
Scotia
Pacific Ocean
Thunder Bay
Sherbrooke
Lake Superior
Sudbury
Ottawa
Lake Ontario
Atlantic
Ocean
Toronto
Hamilton
London
United States
Lake Huron
Michigan

Overview

Group Actions

Disclaimer

Throughout this talk, assume groups are finite. Some things may apply to infinite groups also, but this hasn't been studied much to my knowledge.

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What are all the symmetries of this object?

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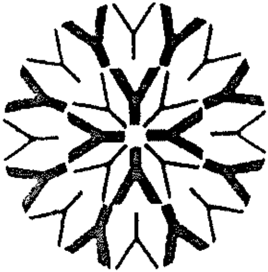

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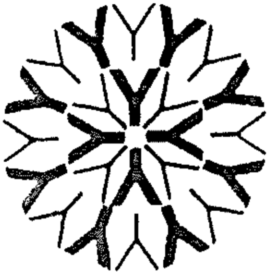

This provides us with some intuitive understanding of the group.

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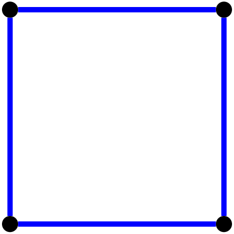

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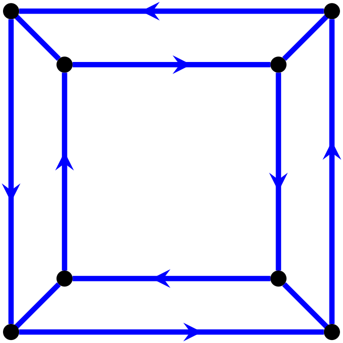

History and Definitions

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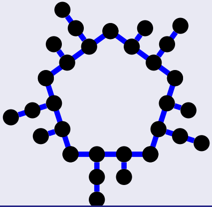
Example: $\mathbb{Z}_5$



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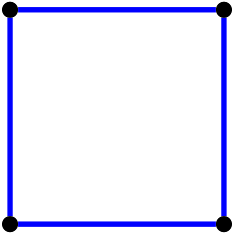
Examples



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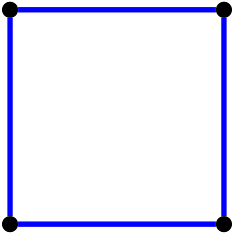
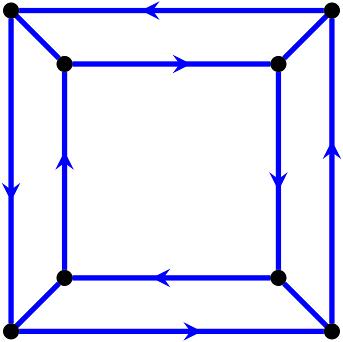


is not a DRR, GRR, or ORR for any group. Its full automorphism group is D_8, but the action of D_8 is not regular on the vertices of this graph.

Examples



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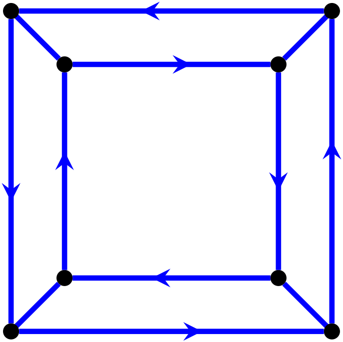

is a DRR, for D_8. Its full automorphism group is D_8, and the action of D_8 is regular on the vertices of this digraph.

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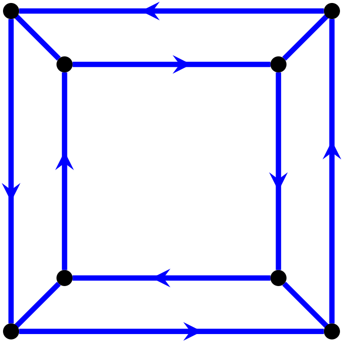

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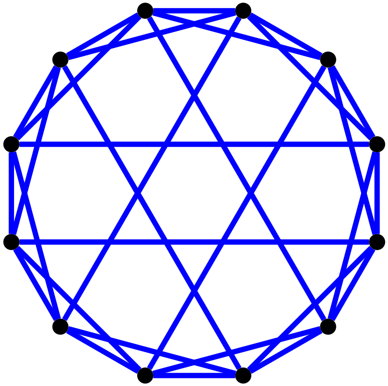

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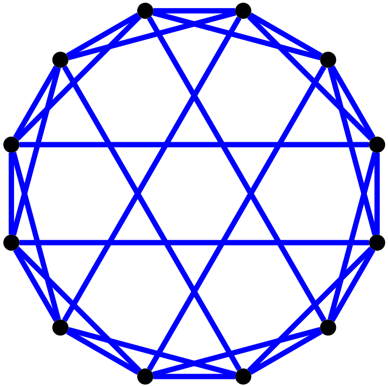

is a GRR, for D_{12}. Its full automorphism group is D_{12}, and the action of D_{12} is regular on the vertices of this graph.

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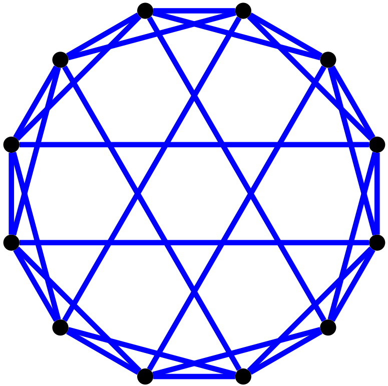

is a GRR, for D_{12}. Its full automorphism group is D_{12}, and the action of D_{12} is regular on the vertices of this graph. It is also a digraph, so it is a DRR, but it is not an oriented graph, so is not an ORR.

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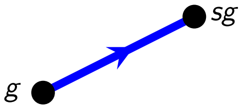
The *Cayley (di)graph* $\Gamma = \mathrm{Cay(G, S)}$ *is the (di)graph whose vertices are the elements of* G*, with an arc from* g *to* sg *if and only if* $s \in S$*.*



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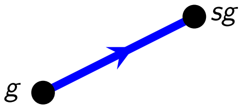

Notice

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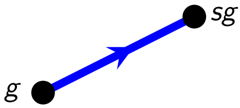

Notice

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- right-multiplication by any element of G is necessarily an automorphism of this (di)graph (there is an arc from gh to sgh).

Cayley graphs and Regular Representations

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So, a DRR/GRR/ORR must be a Cayley digraph that happens to not have any extra automorphisms.

Obstructions

Obstructions for GRRs

Observation

For any Cayley (di)graph $\Gamma = \mathrm{Cay}(\mathrm{G}, \mathrm{S})$, if α is an automorphism of the group G that fixes S setwise, then the map defined by α on the vertices of Γ is an automorphism of Γ.

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Proof.

Suppose that there is an arc from g to sg, so $s \in S$. Since α is a group automorphism, $(sg)^\alpha = s^\alpha g^\alpha$. Since α fixes S, $s^\alpha \in S$, so there is an arc from g^α to $s^\alpha g^\alpha = (sg)^\alpha$. $\square$

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The generalised dicyclic group $\mathrm{Dic}(A, y)$ where A is an abelian group of even order and y has order 2 in A,

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In the generalised dicyclic group $D =\mathrm{Dic}(A, y)$, the map ι defined by $a^\iota = a$ for $a \in A$, and $g^\iota = g^{-1}$ for $g \in D - A$ is a group automorphism.

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Define α by $g^\alpha = g$ if $g \in \langle S \rangle$, and $g^\alpha = gs$ otherwise. Since right-multiplication by s fixes $\langle S \rangle$, this is an automorphism of the graph.

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Define α by $g^\alpha = g$ if $g \in \langle S \rangle$, and $g^\alpha = gs$ otherwise. Since right-multiplication by s fixes $\langle S \rangle$, this is an automorphism of the graph.
The automorphism group of the graph has more than one element fixing e, so is not regular. □

Obstructions for ORRs

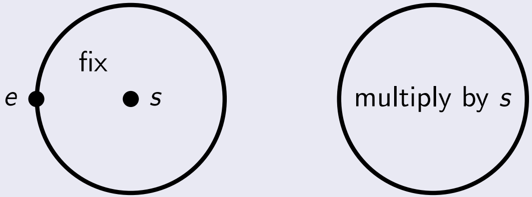
Observation

The Cayley (di)graph $\mathrm{Cay}(G, S)$ is connected if and only if $\langle S \rangle = G$.

Observation

A disconnected Cayley (di)graph on G with more than 2 vertices is never a regular representation.

Proof by picture.



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If $|G| > 2$ and G cannot be generated without elements of order 2, it cannot have an ORR. (In fact, it has no connected oriented Cayley digraph.)

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Note...

... that in Dih(A), every element ax of Ax has $(ax)^2 = axax = aa^{-1}x^2 = e$. Thus, generalised dihedral groups cannot be generated without an element of order 2.

So generalised dihedral groups do not admit ORRs. [Babai, 1980]

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There are none.

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If the group G has a subgroup M of index 2 and there is a non-identity automorphism φ of G that maps every element g of $G - M$ to either g or g^{-1}, then G cannot admit a bipartite GRR with the cosets of M as the bipartition sets.

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The groups G and subgroups M that have such an automorphism satisfy one of:

- *M is abelian and G is not generalised dihedral over M;*
- *M contains an abelian subgroup Z of index 2, and there is some $g \in G - M$ such that $g^2 \neq 1$, $g^2 \in Z \cap Z(G)$, and $z^g = z^{-1}$ for every $z \in Z$; or*
- *$Z(M)$ has index 4 in M; there is some $g \in G - M$ such that:*
    - *g has order 4;*
    - *g inverts every element of $Z(M)$;*
    - *there is some $m \in M - Z(M)$ such that gm does not have order 2; and*
    - *the commutator subgroup of M is $\langle g^2 \rangle$.*

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Conjecture (Du, Feng, Spiga, 2020+)

With 59 exceptions of order at most 64, the groups classified by the obvious obstruction are the only groups not admitting bipartite GRRs.

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Conjecture (Du, Feng, Spiga, 2020+)

"abelian" is not a necessary hypothesis.

Theme/Lesson

With the exception of some "small noise", regular representations exist as long as obvious structural obstructions are avoided.

Asymptotics

Theorem (Erdös–Rényi, 1963)

Almost every (di)graph is asymmetric.

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Question

If we force a (di)graph to have some symmetry (automorphisms), is it still true that almost every such (di)graph has no symmetry beyond what we force?

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- Does every group of order r have this property, or is it only true over all groups of order r, as $r \to \infty$;
- We can look at labelled digraphs, or digraphs up to isomorphism;
- They made a similar conjecture about GRRs and Cayley graphs.

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Within the class of nilpotent groups R of odd order r, as $r \to \infty$ the proportion of DRRs on R from all Cayley digraphs on R tends to 1.

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Let R be a group of order r, where r is sufficiently large. The number of subsets S of R such that $Cay(R, S)$ is not a DRR is at most

$$2^{r - \frac{br^{0.499}}{(4\log_2(r))^3} + 2},$$

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Corollary

As $r \to \infty$, if R is a group of order r, the proportion of DRRs on R (up to isomorphism) out of all Cayley digraphs on R (up to isomorphism) tends to 1.

Main tools - Blocks

Definition

Let G be a permutation group acting transitively on a set X. Then $B \subseteq X$ is a block under the action of G if $\forall g \in G$, $g(B) \cap B \neq \emptyset$ implies $g(B) = B$.

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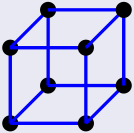

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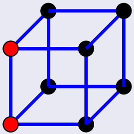

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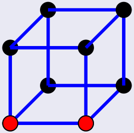

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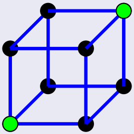

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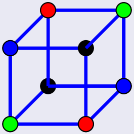

The set of all blocks partitions X.

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A permutation group that does not admit blocks is primitive. The O'Nan-Scott Classification of primitive groups can be applied.

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This has no obvious useful generalisation to the case of undirected graphs, and is the main reason our proofs do not generalise to that situation.

Related Work and Open Problems

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Bipartite DRRs

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Graphical Frobenius Representations

A number of researchers including Watkins, Tucker, Conder, and Spiga have proved results about Frobenius representations of permutation groups.

Open problems

Is the Babai-Godsil Conjecture true for GRRs?

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Is the Babai-Godsil Conjecture true for GRRs?
Are the Du-Feng-Spiga Conjectures true for bipartite GRRs and bipartite DRRs?

Thank you!

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