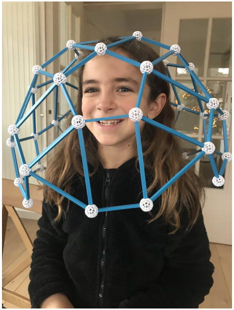
String C-group representations of finite groups: a survey

Dimitri Leemans
Université libre de Bruxelles



Coxeter groups

Coxeter groups

Definition

Let $M = (m_{ij})_{i,j=1,\dots,k}$ be a $k \times k$ matrix whose entries m_{ij} are positive integers or ∞. The matrix M is called a **Coxeter matrix** if $m_{ii} = 1$ for $i = 1, \dots, k$ and $m_{ij} = m_{ji} \geq 2$ for $1 \leq i < j \leq k$.

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Definition

Let M be a Coxeter matrix. The **Coxeter group**[a] **with Coxeter matrix** M is the group $W = W(M)$ with generators $\sigma_1, \ldots, \sigma_k$ and presentation

$$(\sigma_i\sigma_j)^{m_{ij}} = 1_W \text{ for all } i,j \text{ with } m_{ij} \neq \infty$$

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Definition

The number k of generators is called the **rank**.

Definition

If $W = W(M)$ is a Coxeter group with Coxeter matrix M, the **Coxeter diagram** $\Delta = \Delta(M)$ is a labelled graph whose vertices represent the generators of W, and, for $i, j \in K$, an edge with label m_{ij} joins the i^{th} and j^{th} vertex, omitting edges when $m_{ij} \leq 2$. Also, if $m_{ij} = 3$, we don't write the label on the corresponding edge.

Coxeter groups

Proposition (Coxeter, J. LMS 1935)

Let Δ be a Coxeter diagram without improper branches and let $\Delta_1, \ldots, \Delta_m$ be its connected components. Then

$$W(\Delta) \cong W(\Delta_1) \times \ldots \times W(\Delta_m),$$

but no component $W(\Delta_i)$ is itself a direct product of non-trivial distinguished subgroups.

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Definition

A Coxeter group is **irreducible** if its Coxeter diagram is connected. It is **reducible** otherwise.

Coxeter groups

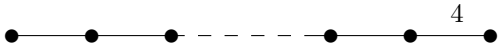
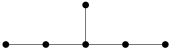
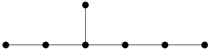
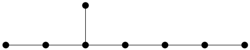
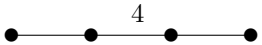
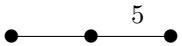
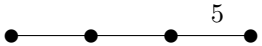
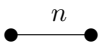
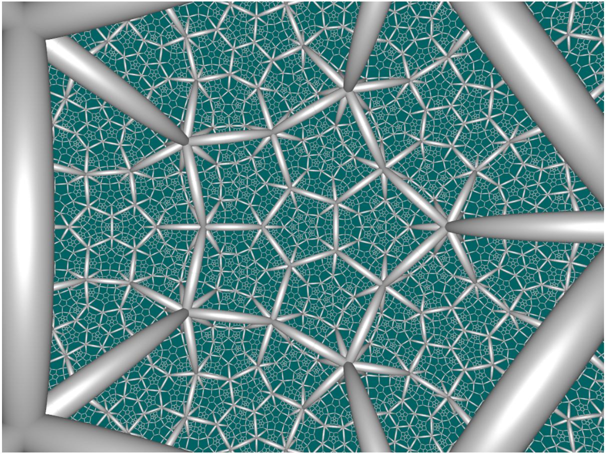
Notation	Diagram	Order
A_n with $n \geq 1$		$(n+1)!$
$B_n = C_n$ with $n \geq 2$		$2^n n!$
D_n with $n \geq 4$		$2^{n-1} n!$
E_6		$72 \cdot 6! = 51840$
E_7		$8 \cdot 9! = 2903040$
E_8		$192 \cdot 10! = 696729600$
F_4		1152
H_3		120
H_4		14400
I_n with $n \geq 1$		$2n$

TABLE 3.1: Spherical Coxeter groups

Coxeter groups

Coxeter group of type [5,3,5]



Coxeter groups

Theorem (Tits, I.H.E.S. Course 1961)

Let $W = \langle \sigma_1, \ldots, \sigma_k \rangle$ be a Coxeter group with Coxeter matrix $M = (m_{ij})_{i,j \in \{1,\ldots,k\}}$. For every $I \subseteq \{1, \ldots, k\}$, let $W_I := \langle \sigma_i : i \in I \rangle$. Then the distinguished subgroups W_I have the following property called the **intersection property**.

- *For every $I, J \subseteq K$, the group $W_I \cap W_J = W_{I \cap J}$;*

C-groups

(String) C-groups are smooth quotients of Coxeter groups, i.e.

- The orders of the products of generators are preserved;
- The intersection property is preserved.

Finite string C-groups

Start from the [5,3,5] that is infinite.

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$+\ (\sigma_1\sigma_2\sigma_3)^5 = 1_W$ gives $L_2(19) \times J_1$.
(Hartley, L., Math. Z. 2004)

Abstract regular polytopes and string C-groups

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Given an ARP and a base chamber, one can construct a string C-group as we will see later.

Conversely, given a string C-group one can construct an ARP using an algorithm described by Jacques Tits in 1956.

Different approaches to classify ARPs

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Fix the genus (rank three)

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Fix the diagram (or Schläfli type)

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Fix the genus (rank three)

Fix the diagram (or Schläfli type)

Fix the automorphism group

Abstract regular polytopes

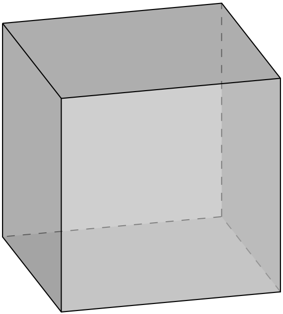


Figure: A Cube

Abstract regular polytopes

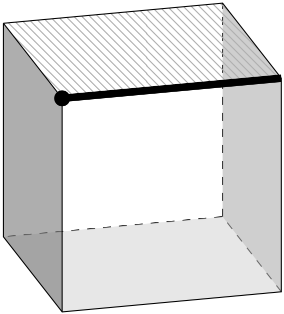
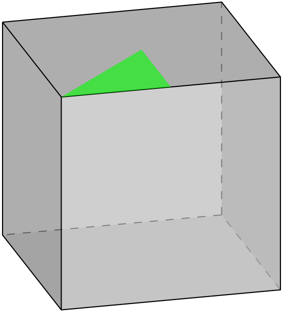


Figure: A chain on the Cube consisting of a vertex, an edge containing that vertex and a face containing the edge

Abstract regular polytopes and String C-groups

There is a natural one-to-one correspondence between abstract regular polytopes and string C-groups.

String C-groups



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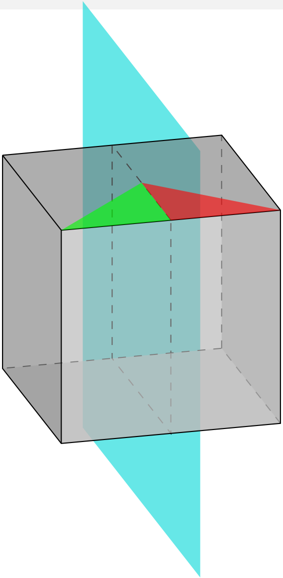

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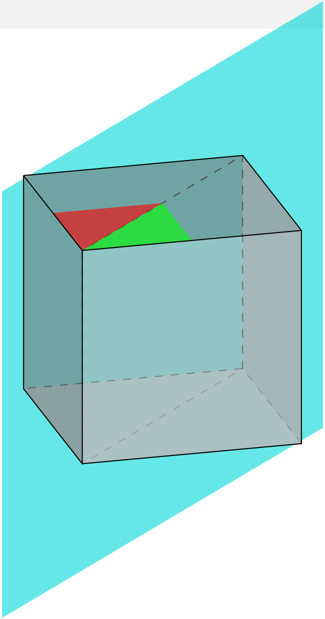

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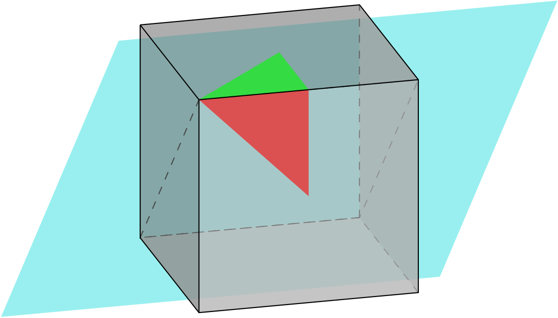

String C-groups

Definition

A **group generated by involutions (or ggi)** is a pair (G, S) such that G is a group and $S := \{\rho_0, \ldots, \rho_{r-1}\}$ is a generating set of involutions of G.

Definition

A subgroup $\langle \rho_i : i \neq j \rangle$ of G is called a **maximal parabolic subgroup**.

String C-groups

Definition

A **C-group of rank** r is a ggi (G, S) that satisfy the following property.

$$\forall I, J \subseteq \{0, \ldots, r-1\}, \langle \rho_i \mid i \in I \rangle \cap \langle \rho_j \mid j \in J \rangle = \langle \rho_k \mid k \in I \cap J \rangle$$

This property is called the **intersection property** and denoted by (IP). We call any subgroup of G generated by a subset of S a *parabolic subgroup* of the C-group (G, S).

String C-groups

Definition

A **C-group** (G, S) **of rank** r is a **string C-group** if its set of generating involutions S can be ordered in such a way that $S := \{\rho_0, \ldots, \rho_{r-1}\}$ satisfies

$$\forall i, j \in \{0, \ldots, r-1\}, o(\rho_i \rho_j) = 2 \text{ if } |i-j| > 1$$

This property is called the **string property**.

Definition

For a given group G, we will call (G, S) a **string C-group representation** of G.

What groups to look at?

"Small" groups
Soluble groups
Nilpotent groups
2-groups
Simple groups
Sporadic groups
Almost simple groups
Non-solvable groups
etc.

Small groups

Groups of even order ≤ 2000

[1]See Marston's website for both soluble and non-solvable

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Ratio 0.000009% (for soluble) VS 85% (for non-solvable)

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String C-group representations of 2-groups

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Most groups are 2-groups.
49,910,526,325 - 412,607,930 $=$ 49,497,918,395 2-groups
(99,17% of 2-groups among the groups of even order less than 2001)

String C-group representations of 2-groups

2-groups are also important for abstract regular polytopes as they give the smallest examples of a given rank $n > 8$.

String C-group representations of 2-groups

Theorem (Conder, Adv. Math., 2013)

Let F_n be the number of flags in a regular polytope of rank n. Then a lower bound for F_n is given by $F_n \geq 2 \cdot 4^{n-1}$ for all $n \geq 9$, and this bound is attained by a family of tight polytopes of type $\{4| \ldots |4\}$, one for each n. For rank $n \leq 8$, the fewest flags occur for regular n-polytopes as follows:

n	$min(F_n)$	Type(s) attaining the lower bound
2	6	$\{3\}$
3	24	$\{3\|3\}$, $\{3\|4\}$ (and dual $\{4\|3\}$)
4	96	$\{4\|3\|4\}$
5	432	$\{3\|6\|3\|4\}$ (and dual $\{4\|3\|6\|3\}$)
6	1 728	$\{4\|3\|6\|3\|4\}$
7	7 776	$\{3\|6\|3\|6\|3\|4\}$ (and dual $\{4\|3\|6\|3\|6\|3\}$)
8	31 104	$\{4\|3\|6\|3\|6\|3\|4\}$

String C-group representations of 2-groups

The **Frattini subgroup** $\Phi(G)$, of a finite group G is the intersection of all maximal subgroups of G.

Let G be a finite p-group for a prime p, and set $\mho_1(G) = \{g^p \mid g \in G\}$.

Theorem (Burnside Basis Theorem)

Let G be a p-group and $|G : \Phi(G)| = p^d$.

(1) $G/\Phi(G) \cong \mathbb{Z}_p^d$. *Moreover, if $N \lhd G$ and G/N is elementary abelian, then $\Phi(G) \leq N$.*

(2) **Every minimal generating set of G contains exactly d elements**[a].

(3) $\Phi(G) = G'\mho_1(G)$. *In particular, if $p = 2$, then $\Phi(G) = \mho_1(G)$.*

[a] d is called the *rank* of G and denoted by $d(G)$.

String C-group representations of 2-groups

Corollary (Hou, Feng, L., J. Group Theory 2019)

A given 2-group has only string C-group representations with a fixed rank, that is, the rank of the 2-group.

String C-group representations of 2-groups

Theorem (Hou, Feng, L., J. Group Theory 2019)

Let $n \geq 10$, $s, t \geq 2$ *and* $n - s - t \geq 1$. *Set*
$R = \{\rho_0^2, \rho_1^2, \rho_2^2, (\rho_0\rho_1)^{2^s}, (\rho_1\rho_2)^{2^t}, (\rho_0\rho_2)^2, [(\rho_0\rho_1)^4, \rho_2], [\rho_0, (\rho_1\rho_2)^4]\}$ *and define*

$$H = \begin{cases} \langle \rho_0, \rho_1, \rho_2 \mid R, [(\rho_0\rho_1)^2, \rho_2]^{2^{\frac{n-s-t-1}{2}}} \rangle & \text{when } n-s-t \text{ is odd,} \\ \langle \rho_0, \rho_1, \rho_2 \mid R, [(\rho_0\rho_1)^2, (\rho_1\rho_2)^2]^{2^{\frac{n-s-t-2}{2}}} \rangle & \text{when } n-s-t \text{ is even.} \end{cases}$$

Then $(H, \{\rho_0, \rho_1, \rho_2\})$ *is a string C-group of order* 2^n *and type* $\{2^s, 2^t\}$.

String C-group representations of 2-groups

Theorem (Hou, Feng, L., Disc. Comput. Geom., to appear)

For any integers $d, n, k_1, k_2, \ldots, k_{d-1}$ *such that* $d \geq 3$, $n \geq 5$, $k_1, k_2, \ldots, k_{d-1} \geq 2$ *and* $k_1 + k_2 + \ldots + k_{d-1} \leq n - 1$, *there exists a string C-group* $(G, \{\rho_0, \rho_1, \ldots, \rho_{d-1}\})$ *of order* 2^n *and type* $\{2^{k_1}, 2^{k_2}, \ldots, 2^{k_{d-1}}\}$.

Simple groups

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1980 – Kourovka Notebook Problem 7.30:

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Which finite simple groups can be generated by three involutions, two of which commute?

Nuzhin and Mazurov: every non-abelian finite simple group with the following exceptions:

$$PSL_3(q),\ PSU_3(q),\ PSL_4(2^n),\ PSU_4(2^n),$$
$$A_6,\ A_7,\ M_{11},\ M_{22},\ M_{23},\ McL.$$

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$PSU_4(3)$ and $PSU_5(2)$, have recently been discovered not to have such generating sets by Martin Mačaj and Gareth Jones.

Simple groups

The exceptions for polyhedra remain exceptions for polytopes.

Theorem (Vandenschrick, preprint)

Every non-abelian finite simple group is the automorphism group of at least one abstract regular polytope with the following exceptions:

$$PSL_3(q),\ PSU_3(q),\ PSL_4(2^n),\ PSU_4(2^n),$$
$$A_6,\ A_7,\ PSU_4(3),\ PSU_5(2)\ M_{11},\ M_{22},\ M_{23},\ McL.$$

String C-group representations of sporadic groups

G	Order of G	Rank 3	Rank 4	Rank 5
M_{11}	7,920	0	0	0
M_{12}	95,040	23	14	0
M_{22}	443,510	0	0	0
M_{23}	10,200,960	0	0	0
M_{24}	244,823,040	490	155	2
J_1	175,560	148	2	0
J_2	604,800	137	17	0
J_3	50,232,960	303	2	0
HS	44,352,000	252	57	2
McL	898,128,000	0	0	0
He	4,030,387,200	1188	76	0
$O'N$	460,815,505,920	Unknown	31	0
Co_3	495,766,656,000	21118	1746	44

String C-group representations of symmetric groups

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E. H. Moore (1896) : $(n-1)$-simplex.



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Theorem (Moore, Proc. LMS 1896)

For every $n \geq 3$, there is a string C-group representation of $Sym(n)$ in its natural permutation representation, of rank $n-1$ whose generating involutions are the transpositions $(i, i+1)$ with $i = 1, \ldots, n-1$.

String C-group representations of symmetric groups

Proposition (Whiston, J. Algebra, 2000)

The size of an independent set in S_n is at most $n-1$, with equality only if the set generates the whole group S_n.

String C-group representations of symmetric groups

Sjerve and Cherkassoff (1993) (see also Conder 1980): S_n is a group generated by three involutions, two of which commute, provided that $n \geq 4$.

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Theorem ("Moore, Sjerve, Cherkassoff, Conder")

Every group S_n with $n \geq 4$ has a string C-group representation of rank three and one of rank $n - 1$.

String C-group representations of symmetric groups

Theorem (Fernandes, L., Adv. Math., 2011)

Let $n \geq 4$. For every $r \in \{3, \ldots, n-1\}$, there exists at least one string C-group representation of rank r for S_n.

String C-group representations of symmetric groups

Let $\{\rho_0, \ldots, \rho_{r-1}\}$ be a set of involutions of a permutation group G of degree n. We define the **permutation representation graph** $\mathcal{G}$ as the r-edge-labeled multigraph with n vertices and with a single i-edge $\{a, b\}$ whenever $a\rho_i = b$ with $a \neq b$.

String C-group representations of symmetric groups

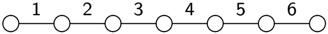
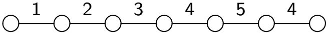
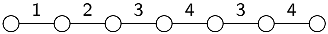
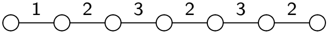
Generators	Permutation representation	Schläfli type
(1,2),(2,3),(3,4),(4,5),(5,6),(6,7)		{3,3,3,3,3}
(1,2),(2,3),(3,4),(4,5)(6,7),(5,6)		{3,3,6,4}
(1,2),(2,3),(3,4)(5,6),(4,5)(6,7)		{3,6,5}
(1,2),(2,3)(4,5)(6,7),(3,4)(5,6)		{6,6}

Table: The induction process used on S_7

String C-group representations of symmetric groups

Number of representations, up to duality, for S_n $(5 \leq n \leq 14)$

G\r	3	4	5	6	7	8	9	10	11	12	13
S_5	4	1	0	0	0	0	0	0	0	0	0
S_6	2	4	1	0	0	0	0	0	0	0	0
S_7	35	7	1	1	0	0	0	0	0	0	0
S_8	68	36	11	1	1	0	0	0	0	0	0
S_9	129	37	7	7	1	1	0	0	0	0	0
S_{10}	413	203	52	13	7	1	1	0	0	0	0
S_{11}	1221	189	43	25	9	7	1	1	0	0	0
S_{12}	3346	940	183	75	40	9	7	1	1	0	0
S_{13}	7163	863	171	123	41	35	9	7	1	1	0
S_{14}	23126	3945	978	303	163	54	35	9	7	1	1

String C-group representations of symmetric groups

Number of representations, up to duality, for S_n $(5 \leq n \leq 14)$

G\r	3	4	5	6	7	8	9	10	11	12	13
S_5	4	**1**	0	0	0	0	0	0	0	0	0
S_6	2	4	**1**	0	0	0	0	0	0	0	0
S_7	35	7	**1**	**1**	0	0	0	0	0	0	0
S_8	68	36	11	**1**	**1**	0	0	0	0	0	0
S_9	129	37	7	7	**1**	**1**	0	0	0	0	0
S_{10}	413	203	52	13	7	**1**	**1**	0	0	0	0
S_{11}	1221	189	43	25	9	7	**1**	**1**	0	0	0
S_{12}	3346	940	183	75	40	9	7	**1**	**1**	0	0
S_{13}	7163	863	171	123	41	35	9	7	**1**	**1**	0
S_{14}	23126	3945	978	303	163	54	35	9	7	**1**	**1**

String C-group representations of symmetric groups

Number of representations, up to duality, for S_n $(5 \leq n \leq 14)$

G\r	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
S_5	4	**1**	0	0	0	0	0	0	0	0	0
S_6	2	4	**1**	0	0	0	0	0	0	0	0
S_7	35	7	**1**	**1**	0	0	0	0	0	0	0
S_8	68	36	11	**1**	**1**	0	0	0	0	0	0
S_9	129	37	7	**7**	**1**	**1**	0	0	0	0	0
S_{10}	413	203	52	13	**7**	**1**	**1**	0	0	0	0
S_{11}	1221	189	43	25	**9**	**7**	**1**	**1**	0	0	0
S_{12}	3346	940	183	75	40	**9**	**7**	**1**	**1**	0	0
S_{13}	7163	863	171	123	41	35	**9**	**7**	**1**	**1**	0
S_{14}	23126	3945	978	303	163	54	35	**9**	**7**	**1**	**1**

String C-group representations of symmetric groups

Theorem (Fernandes, L., Adv. Math., 2011)

For $n \geq 5$ or $n = 3$, Moore's generators give, up to isomorphism, the unique string C-group representation of rank $n-1$ for S_n. For $n = 4$, there are, up to isomorphism and duality, two representations, namely the ones corresponding to the hemicube and the tetrahedron.

Theorem (Fernandes, L., Adv. Math., 2011)

For $n \geq 7$, there exists, up to isomorphism and duality, a unique string C-group representation of rank $(n-2)$ for S_n.

String C-group representations of symmetric groups

Theorem (Fernandes, L., Mixer, Transactions of the AMS 2018)

For $n \geq 9$, there exists, up to isomorphism and duality, seven string C-group representation of rank $(n-3)$ for S_n.
For $n \geq 11$, there exists, up to isomorphism and duality, nine string C-group representation of rank $(n-4)$ for S_n.

Conjecture

The number of string C-group representations of rank $n-i$ for S_n with $1 \leq i \leq (n-3)/2$ is a constant independent on n.

The sequence looks like 1, 1, 7, 9, 35, 48, ...

String C-group representations of alternating groups

What about alternating groups ?

String C-group representations of alternating groups

G	Rank 3	Rank 4	Rank 5	Rank 6	Rank 7	Rank 8
A_5	2	0	0	0	0	0
A_6	0	0	0	0	0	0
A_7	0	0	0	0	0	0
A_8	0	0	0	0	0	0
A_9	41	6	0	0	0	0
A_{10}	94	2	4	0	0	0
A_{11}	64	0	0	3	0	0
A_{12}	194	90	22	0	0	0
A_{13}	1558	102	25	10	0	0
A_{14}	4347	128	45	9	0	0
A_{15}	5820	158	20	42	6	0

Source: http://homepages.ulb.ac.be/∼dleemans/polytopes

String C-group representations of alternating groups

Theorem (Fernandes, L., Mixer, JCTA 2012)

For each $n \notin \{3, 4, 5, 6, 7, 8, 11\}$, there is a rank $\lfloor \frac{n-1}{2} \rfloor$ string C-group representation of the alternating group A_n.

We found a striking example! A_{11} has string C-group representations of rank 3 and 6, but not 4 nor 5!

String C-group representations of alternating groups

Theorem (Cameron, Fernandes, L., Mixer, Proceedings of the LMS 2017)

The rank of A_n is 3 if $n = 5$; 4 if $n = 9$; 5 if $n = 10$; 6 if $n = 11$ and $\lfloor \frac{n-1}{2} \rfloor$ if $n \geq 12$. Moreover, if $n = 3, 4, 6, 7$ or 8, the group A_n is not a string C-group.

String C-group representations of alternating groups

Theorem (Fernandes, L., Ars Math. Contemp. 2019)

Let $n \geq 12$. For every $r \in \{3, \ldots, \lfloor (n-1)/2 \rfloor\}$, there exists at least one string C-group representation of rank r for A_n.

This makes A_{11} very special!

The Rank Reduction Theorem

Theorem (Brooksbank, L., Proc. AMS 2019)

Let $(G; \{\rho_0, \ldots, \rho_{n-1}\})$ *be an irreducible string C-group of rank* $n \geq 4$. *If* $\rho_0 \in \langle \rho_0\rho_2, \rho_3 \rangle$, *then* $(G; \{\rho_1, \rho_0\rho_2, \rho_3, \ldots, \rho_{n-1}\})$ *is a string C-group of rank* $n-1$.

The Rank Reduction Theorem

The condition $\rho_0 \in \langle \rho_0\rho_2, \rho_3 \rangle$ is an easy one to verify, making the Rank Reduction Theorem a powerful tool in the search for new polytopes. For example, suppose that $\rho_2\rho_3$ has odd order $2k+1$. Then

$$((\rho_0\rho_2)\rho_3)^{2k+1} = (\rho_0(\rho_2\rho_3))^{2k+1} = \rho_0 \in \langle \rho_0\rho_2, \rho_3 \rangle,$$

so we obtain the following immediate and useful consequence of the Rank Reduction Theorem.

Corollary

Let $(G; \{\rho_0, \ldots, \rho_{n-1}\})$ be an irreducible string C-group of rank $n \geq 4$. If $\rho_2\rho_3$ has odd order, then $(G; \{\rho_1, \rho_0\rho_2, \rho_3, \ldots, \rho_{n-1}\})$ is a string C-group of rank $n-1$.

The Rank Reduction Theorem

Theorem (Brooksbank, L., Proc. AMS 2019)

Let $k \geq 2$ and $m \geq 2$ be integers.

(a) *The symplectic group* $\mathrm{Sp}(2m, \mathbb{F}_{2^k})$ *is a string C-group of rank* n *for each* $3 \leq n \leq 2m+1$.

(b) *The orthogonal groups* $\mathrm{O}^+(2m, \mathbb{F}_{2^k})$ *and* $\mathrm{O}^-(2m, \mathbb{F}_{2^k})$ *are string C-groups of rank* n *for each* $3 \leq n \leq 2m$.

The Rank Reduction Theorem

Conjecture (Brookbank-Leemans)

The group A_{11} is the only finite simple group whose set of ranks of string C-group representations is not an interval in the set of integers.

Symplectic groups

Theorem (Brooksbank, Contemp. Math., to appear)

Let F_q be the finite field with q elements and $G = PSp(4, F_q)$. Then $rk(G) = 0$ if $q = 3$, and $rk(G) = \{3, 4, 5\}$ if $q \neq 3$.

String C-groups - theoretical results

G	Max rank	Enum	Reference
${}^2B_2(q)$	3	yes	L. - Kiefer-L.
$L_2(q)$	4 for $q = 11, 19$ 0 for 2, 3, 7,9 3 for all others	yes	L.-Schulte Conder et al.
$L_3(q)$	0	yes	Brooksbank-Vicinsky
$U_3(q)$	0	yes	Brooksbank-Vicinsky
$L_4(q)$	0 if q is even 4 if q is odd	no	Brooksbank-L.
${}^2G_2(q)$	3	no	L.-Schulte-Van Maldeghem
$Sp_{2m}(2^k)$ $Sp_4(q)$	$\geq 2m+1$ 5	no no	Brooksbank, Ferrara. L. Brooksbank
A_n	$\lfloor \frac{n-1}{2} \rfloor$ if $n \geq 12$ or $n = 9$ $\lfloor \frac{n+1}{2} \rfloor$ if $n = 5, 10, 11$ 0 if $n = 6, 7, 8$	no	Cameron et al.

Table: Highest rank reps and simple groups

D. Leemans, *String C-group representations of almost simple groups: a survey*. arXiv:1910.08843. To appear in Contemp. Math.

Polygons, Buildings
and Related Geometries

A conference in honour of Hendrik Van Maldeghem's 60th birthday
Ghent, June 24–26, 2020

https://algebra.ugent.be/pbrg/

KIA MIHI!