Symmetries Of Discrete Objects 2020, Rotorua

MOUFANG MEETS SEVERI

Anneleen **De Schepper**

Joint work with: Jeroen **Schillewaert** and Hendrik **Van Maldeghem**

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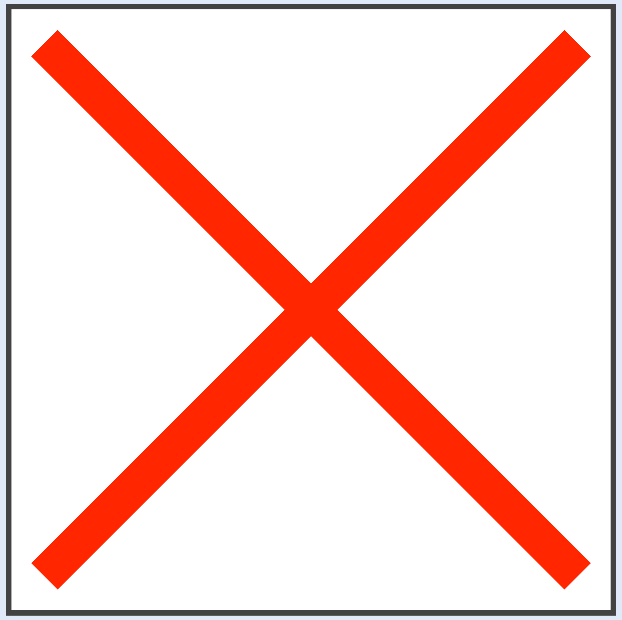
INTRODUCTION

uniform geometric description

Francesco Severi
1879 – 1961



Ruth Moufang
1905 – 1977

1. **Severi varieties**

3. an entire **spectrum** of possibly similar geometries

2. (Veronese rep. of) **Moufang planes**

long-term motivation: understand the exceptional algebraic groups (over any field)

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SEVERI VARIETIES

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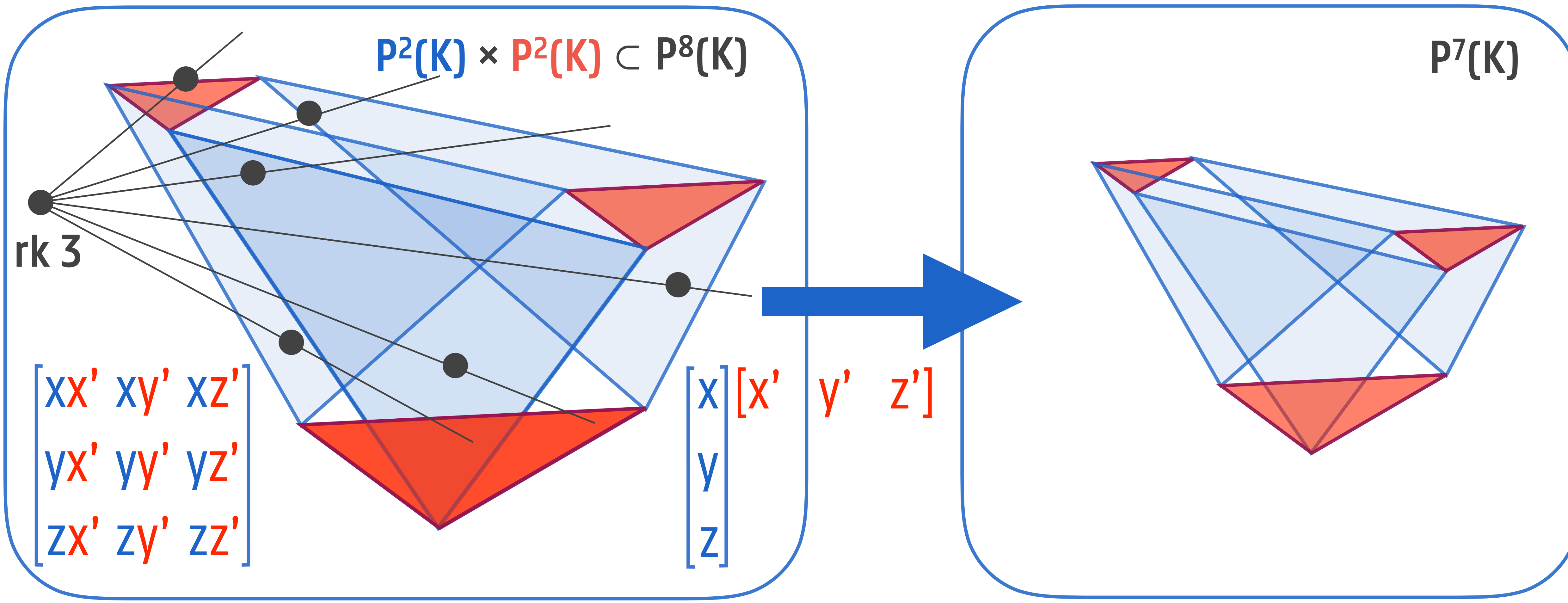
EXAMPLE: SEGRE VARIETY $S_{2,2}(K)$ (K FIELD)



The **rank 1 matrices** in the space of the **3x3 matrices** (mod scalars) ≃ $\mathbf{P^8(K)}$

isomorphic image
one dimension lower

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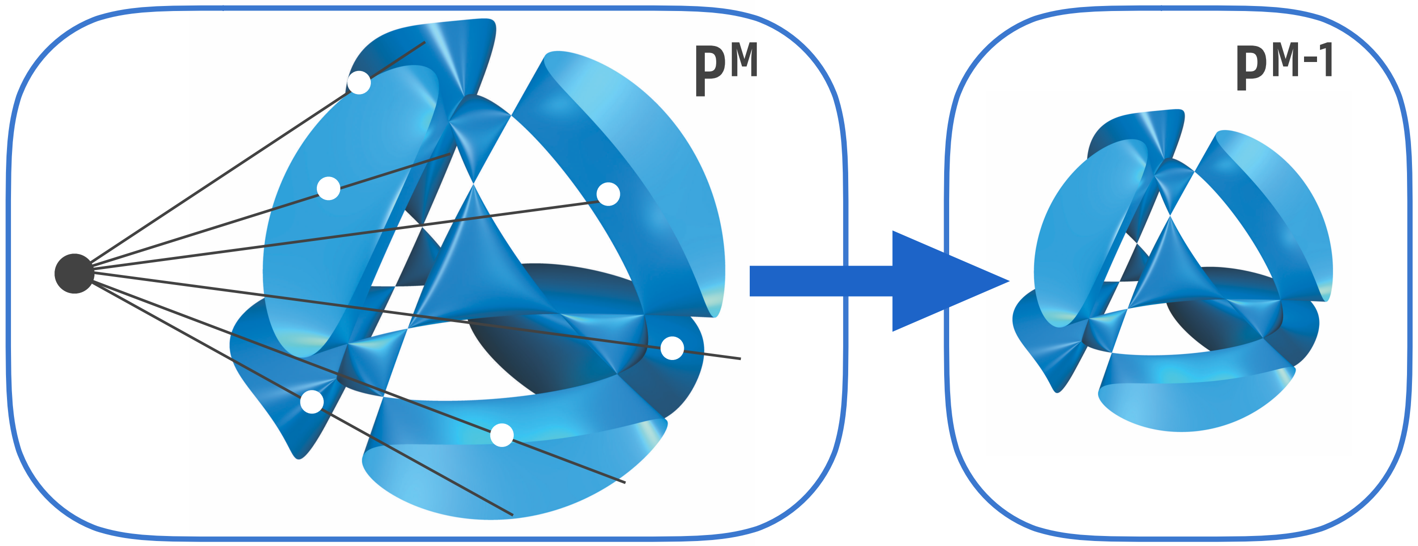
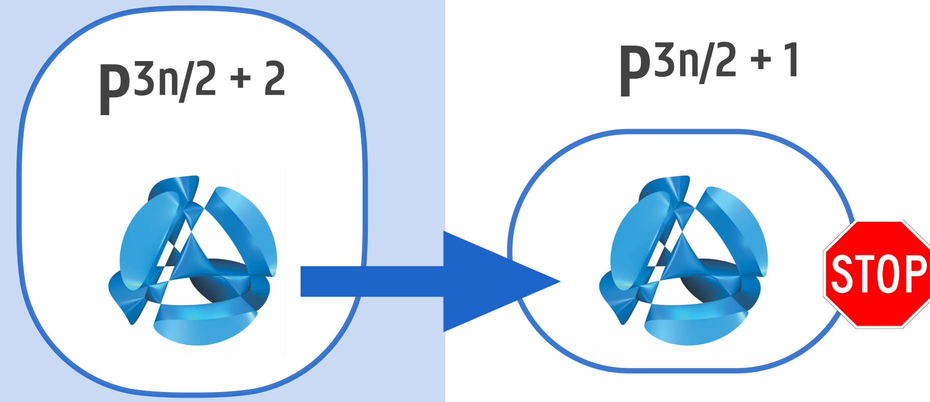
SECANT-DEFECTIVE VARIETIES



n-dim variety
(spanning P^M)

isomorphic
1 dimension lower



<u>**If**</u> projection is possible

Severi variety

- **Existence of projections depends on:**
  - intrinsic properties of the variety
  - **Hartshorne conjecture** (Zak, 1981): $M \geq 3n/2 + 2$

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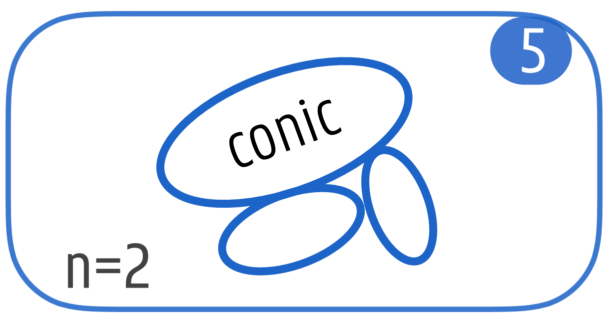
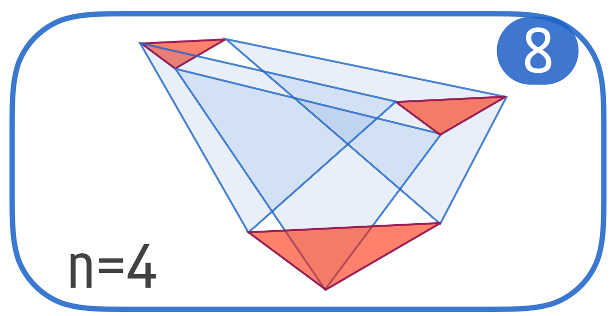
SEVERI VARIETIES

- **Classification (Zak, 1985)** Suppose **X** is an irreducible reduced **n-dim Severi variety** with $\langle X \rangle = P^{3n/2+2}$ over an **algebraically closed field K of char 0.** Then:

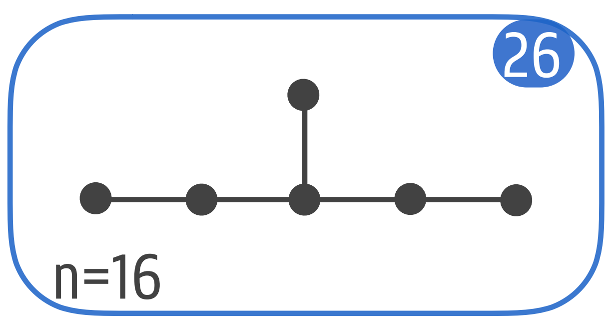


Veronese variety $V_2(K)$
Severi — 1901

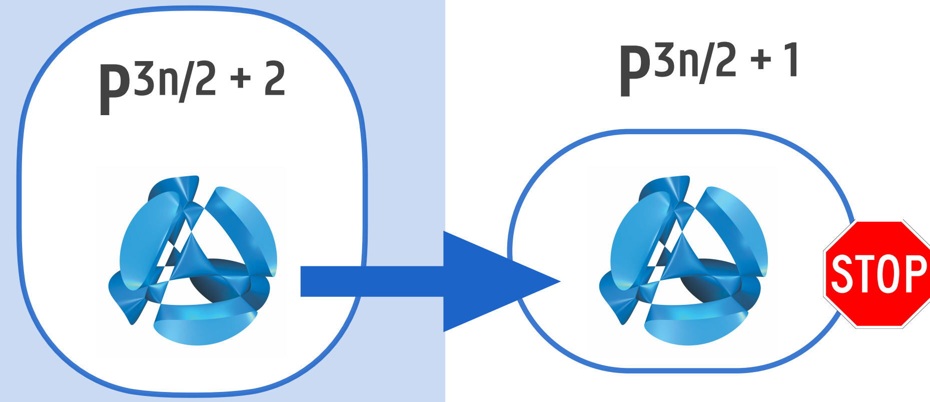


Segre variety $S_{2,2}(K)$
Scorza — 1908

Line Grassmann. $G_{5,1}(K)$
Semple — 1931



Cartan variety $E_{6,1}(K)$
Lazarsfeld — 1981



If projection is possible

Severi variety

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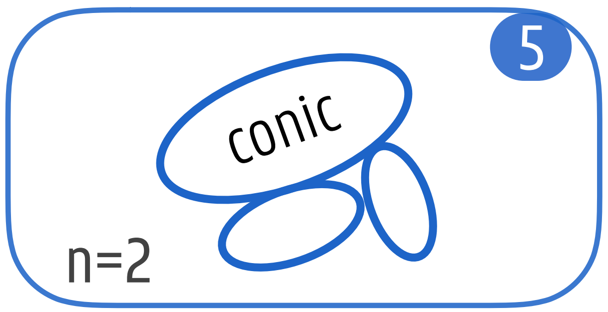
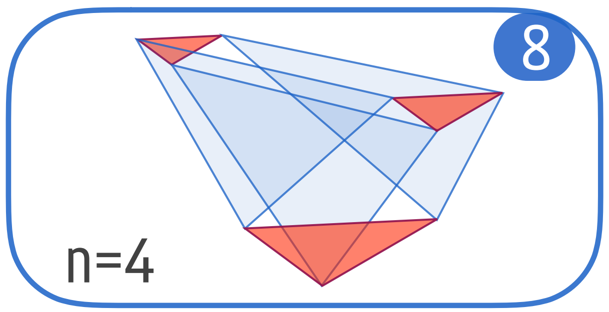
UNIFORM DESCRIPTION (OVER ARBITRARY FIELDS)



Veronese variety $V_2(K)$



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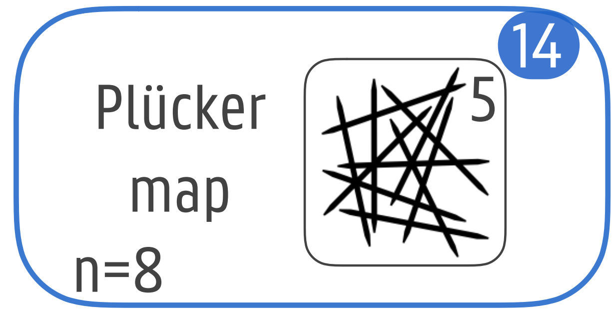
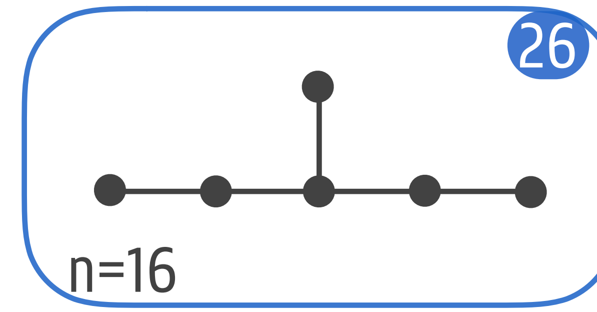


Line Grassmann. $G_{5,1}(K)$



Cartan variety $E_{6,1}(K)$

- Zak: "It is amusing that Severi varieties are in one-to-one correspondence with the **(split)** composition algebras **A over K**."
  - **K**-algebras **A** with a non-degenerate multiplicative norm form **N: A → K**
  - e.g. **complex numbers**: $N(a+bi)=(a+bi)(a-bi)=a^2+b^2$
  - comes with an involution **A → A:** $x \longmapsto \underline{x}$ such that $N(x)=x\underline{x}$

K	K × K	$Mat_{2\times 2}(K)$	split octonions
1-dim over K	2-dim over K	4-dim over K	8-dim over K

- **Veronese representation** of a "plane" over **A**

$$(x,y,z) \longmapsto (N(x),\ N(y),\ N(z);\ y\underline{z},\ z\underline{x},\ x\underline{y}) \in P^{3d+2}(K) \text{ with } d=n/2=\dim_K A$$

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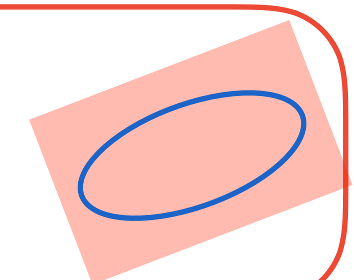
GEOMETRIC PROPERTIES

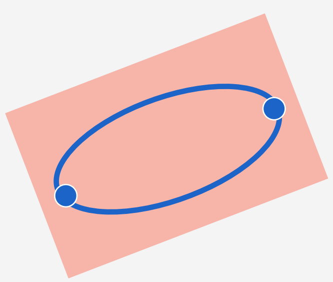
Suppose **X** is an **n**-dim **Severi variety** in $P^{3n/2+2}(K)$.

point set **X** with <**X**>=$P^{3n/2+2}(K)$

X contains hyperbolic **quadrics Q** inside $P^{n/2+1}(K)$ with **<Q>∩X = Q**

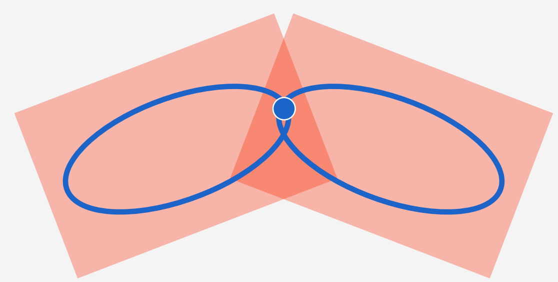


Prop 1



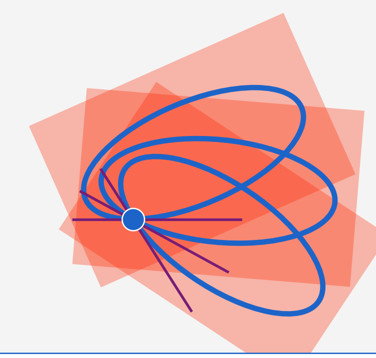
each two **points** of **X** belong to a **quadric**

Prop 2

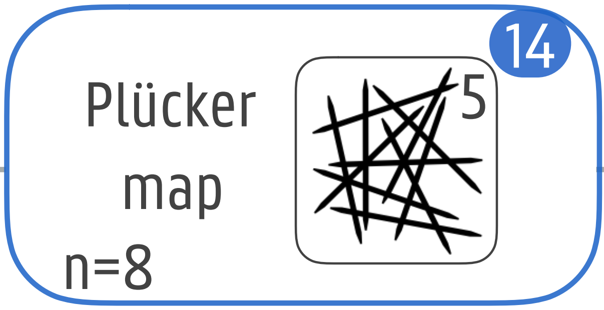


for two **quadrics** Q_1, Q_2: $\langle Q_1 \rangle \cap \langle Q_2 \rangle \subseteq X$

Prop 3



the tangent space of a **point x** has dimension **n**



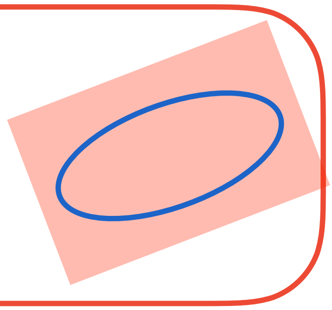
GEOMETRIC PROPERTIES

Line Grassmann. $G_{5,1}(K)$

Suppose **X** is an **n**-dim **Severi variety** in $P^{3n/2+2}(K)$.

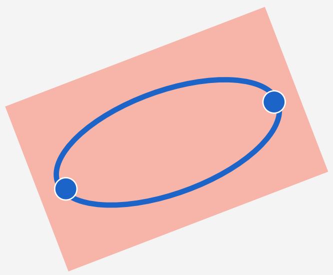
point set **X** with $<X>=P^{3n/2+2}(K)$

X contains hyperbolic **quadrics Q** inside $P^{n/2+1}(K)$ with **<Q>∩X = Q**



$Q^+(5,K)$

Prop 1



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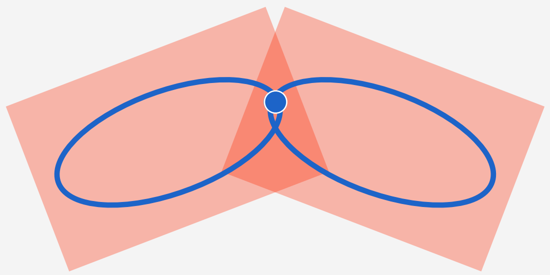
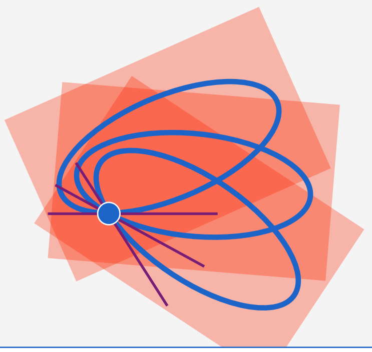
Prop 2



for two **quadrics** Q_1, Q_2: $<Q_1> \cap <Q_2> \subseteq X$

Prop 3



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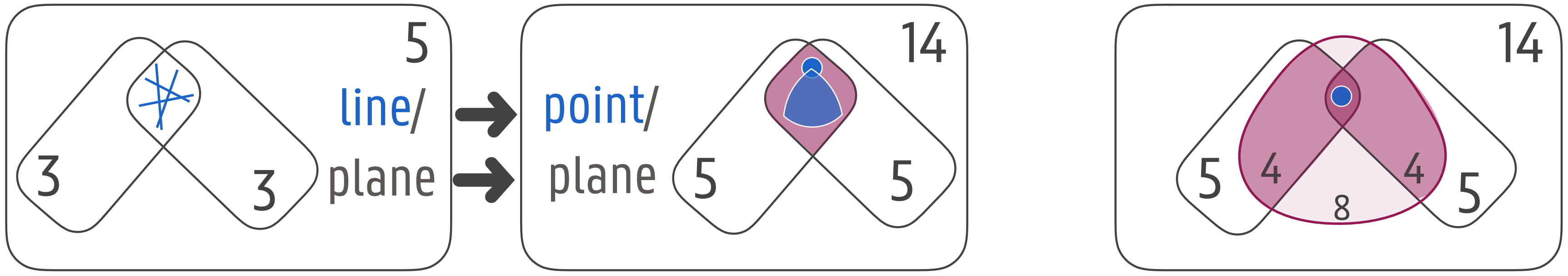


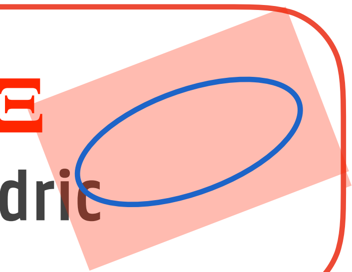
CHARACTERISATION OF SEVERI VARIETIES OVER ARBITRARY FIELDS

Suppose **X** and **Ξ** are such that:

M > **d+1**
possibly ∞

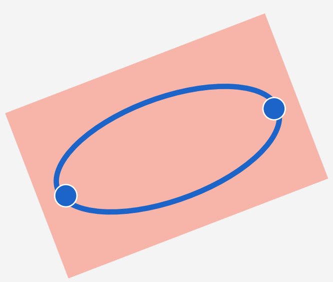
point set **X**
with <**X**>=$P^M(K)$

set **Ξ** of (d+1)-spaces with ∀ **ξ** ∈ **Ξ**
ξ ∩**X** is a **parabolic/hyperbolic quadric**



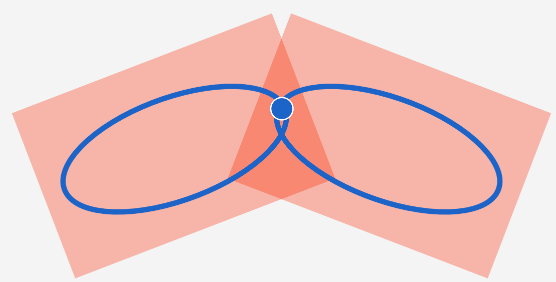
d any natural number

Prop 1



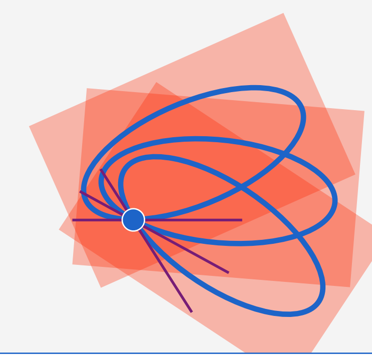
each two **points** of **X** belong to a **ξ** ∈ **Ξ**

Prop 2



for two ξ_1, ξ_2 ∈ **Ξ**:
$\xi_1 \cap \xi_2 \subseteq$ **X**

Prop 3



the tangent space of a **point x** has dimension **2d**

Theorem (Schillewaert, Van Maldeghem; 2013/2017)
X is a **2d**-dim Severi variety

VERONESE REPRESENTATIONS OF

MOUFANG PLANES

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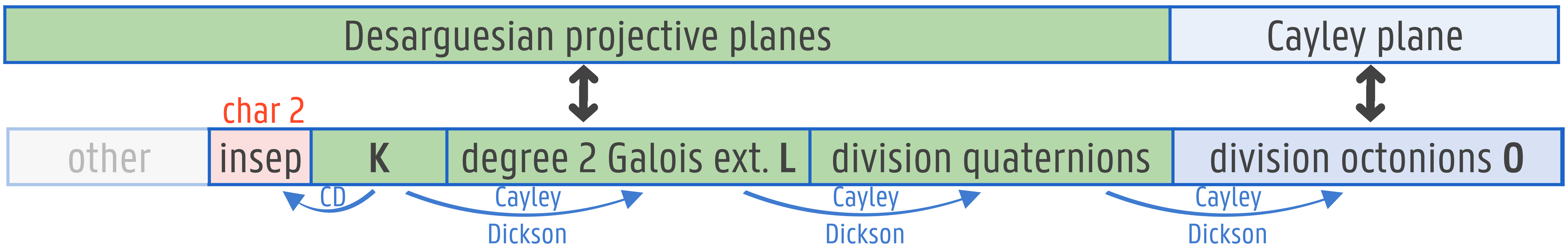
ALTERNATIVE DIVISION RINGS

$a(ab)=a^2b$ and $(ab)b=ab^2$

- **Moufang planes** ⟷ projective planes coordinatised over **alternative division rings**



- division **composition** algebras **A**
  - **norm** N: A → K and **involution** $x \longmapsto \bar{x}$ with **$N(x)=x\bar{x}$** as before
  - **division** means **$N(x)=0 \Longleftrightarrow x=0$** because $x^{-1}=\bar{x}/N(x)$
  - **remark**: existence depends on K (no quaternions over finite fields)
- consider the **Veronese representation $V^2(K,A)$** of $P^2(A)$ ($d=\dim_K A$)

$$P^2(A) \to P^{3d+2}(K) : (x,y,z) \longmapsto (N(x),N(y),N(z);\ y\bar{z},\ z\bar{x},\ x\bar{y})$$

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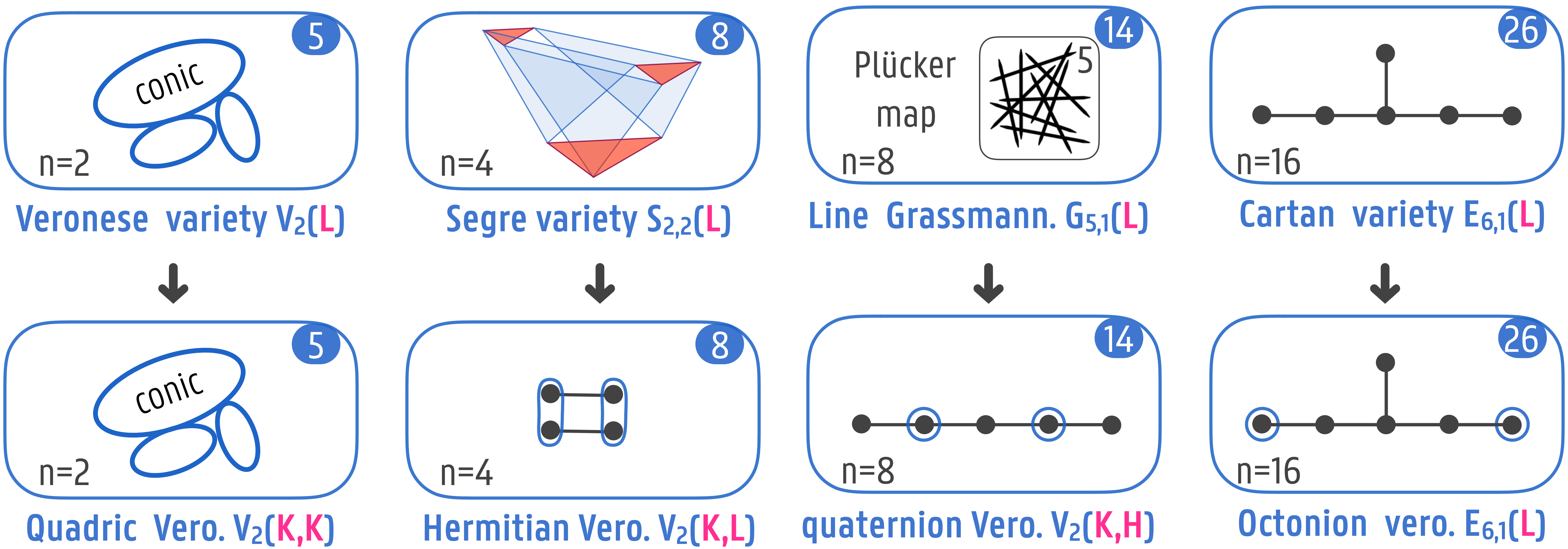
VERONESE VARIETIES $V^2(K,A)$

char 2

insep	K	degree 2 Galois ext. **L**	division quaternions	division octonions **O**

- The Veronese variety **V^2(K,A)** is contained in the (2d)-dim Severi variety over **L**



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VERONESE VARIETIES $V^2(K,A)$

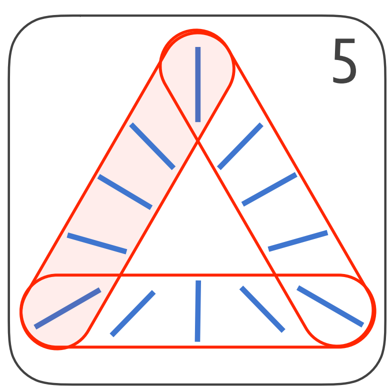
char 2

insep	K	degree 2 Galois ext. **L**	division quaternions	division octonions **O**

- The Veronese variety **V^2(K,A)** is contained in the (2d)-dim Severi variety over **L**

- semi-linear **involution** σ on P^5(**L**), no fixpoints
  - spread of **fixed lines** <p, σ(p)>
  - **projective plane** with lines and 3-spaces

fixed ⊇ elts.



↓ Plucker ↓ Plucker

Line Grassmann. $G_{5,1}$(**L**) 14 ⊇ quaternion Veronesean V_2(**K**,**H**) 14

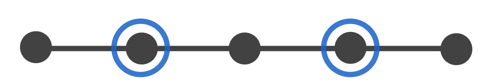
line-free quadric $Q^-(5,K)$ on $Q^+(5,L)$

Plücker map

n=8 14

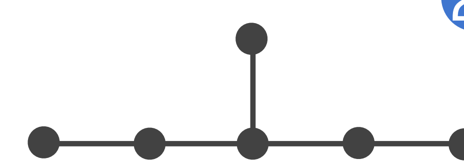
Line Grassmann. $G_{5,1}$(L)

↓



n=8 14

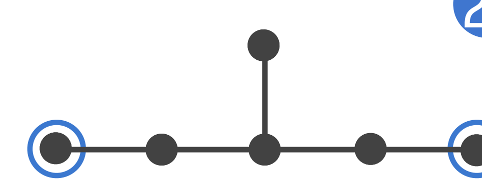
quaternion Vero. V_2(K,H)



n=16 26

Cartan variety $E_{6,1}$(L)

↓



n=16 26

Octonion vero. $E_{6,1}$(L)

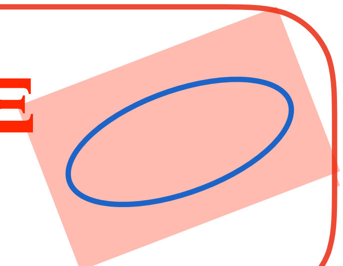
CHARACTERISATION OF MOUFANG VERONESE VARIETIES OVER ARBITRARY FIELDS

Suppose **X** and **Ξ** are such that:

M > **d+1**
possibly ∞

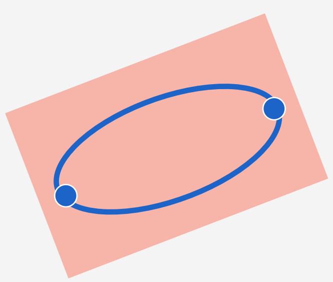
point set **X**
with <**X**>=P^M(K)

set **Ξ** of (d+1)-spaces with ∀ ξ ∈ **Ξ**
ξ ∩**X** is a line-free **quadric**



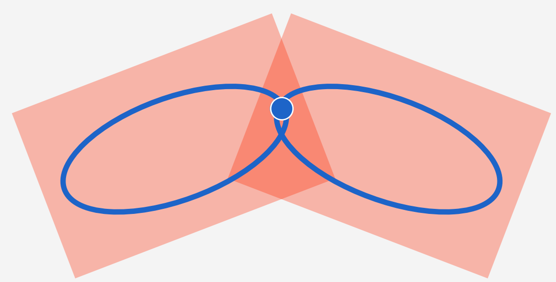
d any natural number or ∞

Prop 1



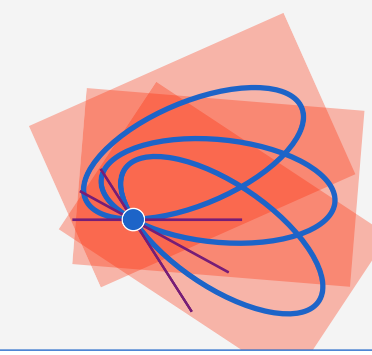
each two **points** of **X** belong to a ξ ∈ **Ξ**

Prop 2



for two $\xi_1, \xi_2 \in \Xi$:
$\xi_1 \cap \xi_2 \subseteq X$

Prop 3



the tangent space of a **point x** is spanned by $T_x(\xi_1)$ and $T_x(\xi_2)$

Theorem (Krauss, Schillewaert, Van Maldeghem; 2013/2015)
(**X**, **Ξ**) arises as the **Veronese** representation of **P^2(A)** where **A** is a **division** composition algebra over **K**

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THE MOUFANG-SEVERI DICHOTOMY

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OTHER VARIETIES?

point set X with $\langle X \rangle = P^M(K)$

d any natural number

$M > d+1$ possibly ∞

Prop 1+2+3

Ξ (d+1)-spaces with $\forall \xi \in \Xi$: $\xi \cap X$ is parabolic/hyperbolic d is either 1,2,4,8	Ξ (d+1)-spaces with $\forall \xi \in \Xi$: $\xi \cap X$ is a non-degenerate quadric not necessarily isomorphic!	Ξ (d+1)-spaces with $\forall \xi \in \Xi$: $\xi \cap X$ is line-free d is either 1,2,4,8 or 2^a if char 2
Severi varieties	~~an entire **spectrum** of possibly similar objects~~	(Veronese rep. of) **Moufang planes** over div. comp. alg.

- **Conjecture** (HVM, JS; 2012): there are **no other such geometries**
    - **true!** (ADS, JS, HVM; 2019+)
- **Remark** this reflects the situation for composition algebras

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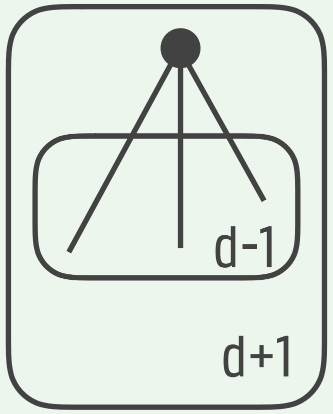
NO OTHER SUCH GEOMETRIES

- **inductive** approach, reducing the size of the quadrics **through a point**
- small index: cut off high d's
- remaining cases are quite hard as they are a mix of real cases

d+1	max dim of subspaces on quadric							
2	0							
3	0	1						
4	0	1						
5	0	1	2					
6	0	1	2					
7	0	1	2	3				
8	0	1	2	3				
9	0	1	2	3	4			
10	0	1	2	3	4			
11	0	1	2	3	4	5		
12	0	1	2	3	4	5		
13	0	1	2	3	4	5	6	
14	0	1	2	3	4	5	6	
15	0	1	2	3	4	5	6	7

d ➔ d-2

max dim ➔ max dim -1



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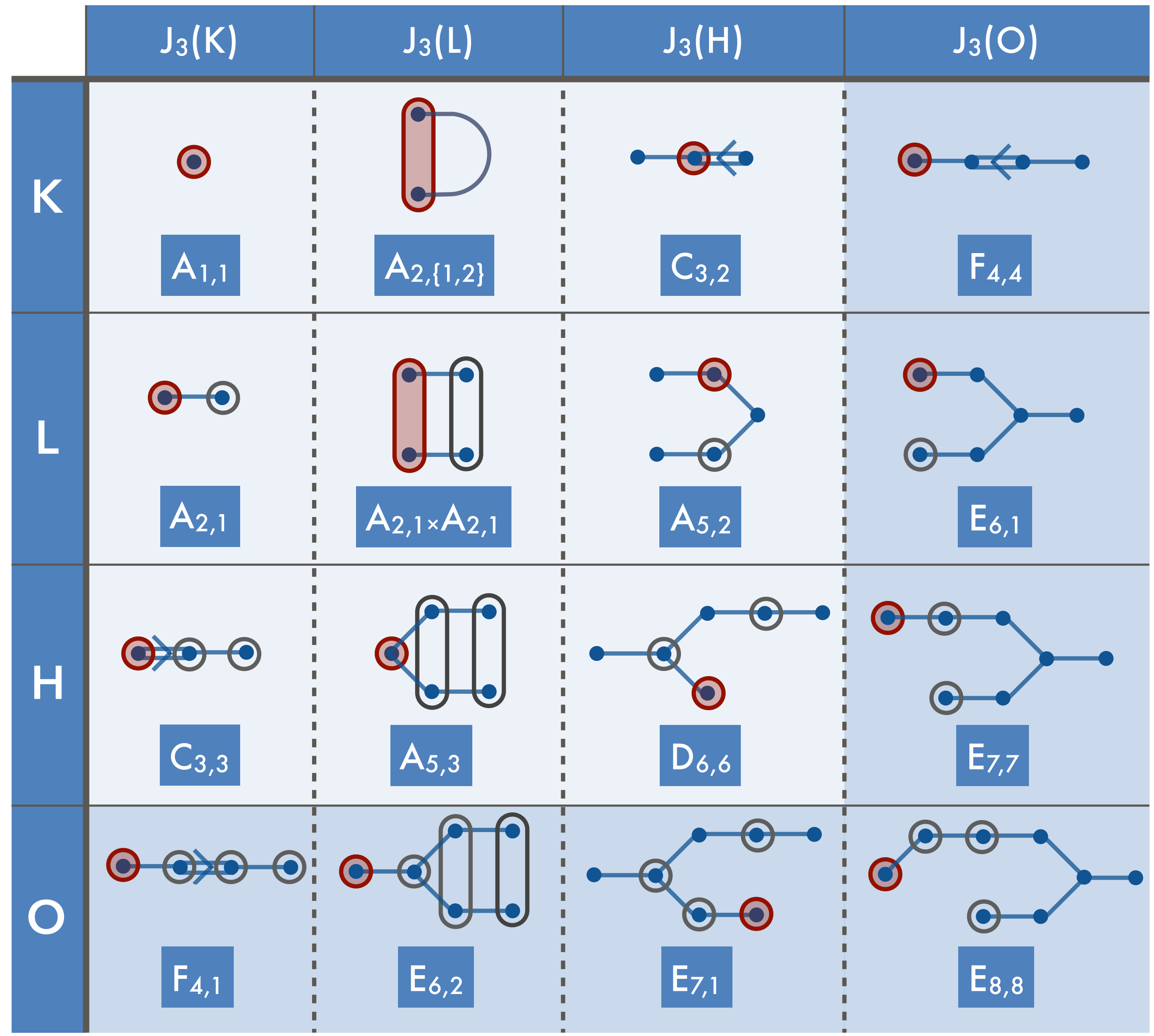
MOTIVATION

- The **Freudenthal-Tits magic square (FTMS)** is based on pairs of composition algebras
- Its **second row** consists of the Moufang/Severi varieties
- **uniform characterisation** of the second row
- **goal**: extend this to other rows



THE DEGENERATE CASE

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HJELMSLEV PLANES

point set X with $\langle X \rangle = P^M(K)$

d any natural number

$M > d+1$ possibly ∞

Prop 1+2+3

Ξ

$(d+1)$-spaces with $\forall \xi \in \Xi$:

$\xi \cap X$ is

~~non-degenerate~~

not necessarily isomorphic

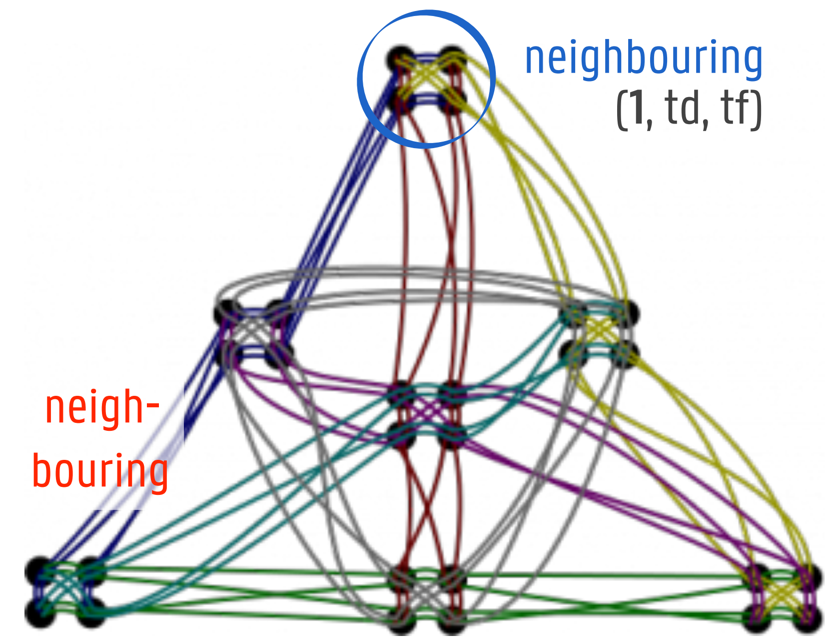
Severi varieties

there actually are **other examples** satisfying Prop 1, 2 and 3

(Veronese rep. of) **Moufang planes** over div. comp. alg.

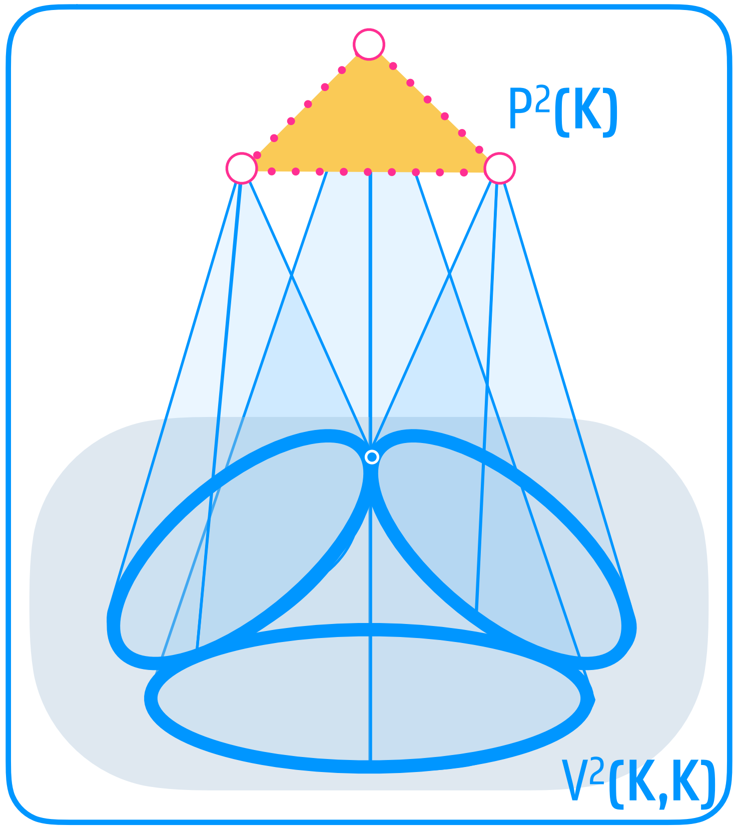
Hjelsmlev planes and other ring geometries

HJELSMLEV PLANES OF LEVEL 2



- A **Hjelmslev plane (HP) of level 2** is a point-line geometry with a **neighbouring relation** in which
  - through two **points** at least one **line**
  - two **lines** meet in at least one **point**
  - there is a canonical **epimorphism** π to a **projective plane** with $\pi(x) = \pi(y) \iff$ x and y are **neighbouring** points/lines
- **Example**: HP over the **dual numbers** over K: given by **K⊕tK** with $t^2=0$
  - points (**a**+tb, **c**+td, **e**+tf) with (a,c,e) ≠ (0,0,0) and π(**a**+tb, **c**+td, **e**+tf)=(a,c,e); lines similarly
  - **Veronese representation** ($\underline{a+tb}$=a-tb) satisfies Prop 1, 2, 3!

$$P^2(A) \to P^{3d+2}(K) : (x,y,z) \longmapsto (x\underline{x}, y\underline{y}, z\underline{z};\ y\underline{z},\ z\underline{x},\ x\underline{y})$$



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“DEGENERATE” COMPOSITION ALGEBRAS

- “dual numbers” over division composition algebras

K	degree 2 Galois ext. L	division quaternions H	char 2 insep I
K⊕tK	L⊕tL	H⊕tH	I⊕tI

  - (Hjelsmlev) Veronese variety has **degenerate quadrics** whose basis is **line-free**
  - similar behaviour: also satisfies our axioms

- “dual numbers” over split composition algebras

L’=K × K		H’=$Mat_{2\times 2}(K)$	
ternions (tL’=K)	L’⊕tL’	sextonions (tH’=L’)	H’⊕tH’

  - Veronese variety has two types of **degenerate quadrics** whose basis is **hyperbolic**
  - similar behaviour: also satisfies Prop 1, 2,3 ; except **H’⊕tH’**

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Anneleen De Schepper

Symmetries Of Discrete Objects 2020, Rotorua

THANK YOU

all kinds of
non-degenerate
quadrics

line-free
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