On vertex-transitive non-Cayley graphs

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Definitions

- Vertex-transitive graph: A graph is vertex-transitive if its automorphism group acts transitively on its vertices.
- Cayley graphs: Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the Cayley graph $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$.
- Every Cayley graph is vertex-transitive.

Definitions

- It is well known that a vertex-transitive graph is a Cayley graph if and only if its automorphism group contains a subgroup acting regularly on its vertex set (see, for example, [17, Lemma 4]).
- There are vertex-transitive graphs which are not Cayley graphs and the smallest one is the well-known Petersen graph. Such a graph will be called a vertex-transitive non-Cayley graph, or a VNC-graph for short.

Related problems

Marušič (1983) posed the following problem.

Problem 1

Determine the set *NC* of non-Cayley numbers, that is, those numbers for which there exists a VNC-graph of order n.

To settle this question, a lot of VNC-graphs were constructed by Marušič, McKay, Royle, Praeger, Miller, Seress etc. in 1990's.

Related problems

Feng (2002) considered the following question.

Problem 2

Determine the smallest valency for VNC-graphs of a given order.

He solved this problem for the graphs of odd prime power order.

Related problems

In [11, Table 1], for $n \leq 26$, the total number of vertex-transitive graphs of order n and the number of VNC-graphs of order n are listed. It seems that, for small orders at least, the great majority of vertex-transitive graphs are Cayley graphs. This is particularly true for small valent vertex-transitive graphs (see [14]). This may suggest the following problem.

Problem 3

Classify small valent VNC-graphs with given order.

The main purpose of this talk is to introduce some work about this problem.

Strategy in classification

Let X be a connected graph with a vertex-transitive automorphism group G. Let $\mathcal{B}$ be a G-invariant partition of $V(X)$.

- Quotient graph $X_{\mathcal{B}}$: vertex set $\mathcal{B}$, and for any two vertices $B, C \in \mathcal{B}$, B is adjacent to C if and only if there exist $u \in B$ and $v \in C$ which are adjacent in X.
- Normal quotient: when $\mathcal{B}$ is a set of orbits of some normal subgroup of G.

Strategy in classification

- The quotient theory was developed by Praeger etc., and it can be used to reduce a large vertex-transitive graph to a small one. This reduction enables us to analysis the structure of various families of vertex-transitive graphs.
- This strategy also works in the classification of small valent VNC-graphs.

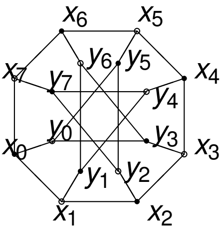
Interesting cases in classification

- Cubic VNC-graphs.
- Tetravalent VNC-graphs.

A well-known class of graphs

Generalized Petersen graphs

Let $n \geq 3$ and $1 \leq t < n/2$. The generalized Petersen graph $P(n, t)$ is the graph with vertex set $\{x_i, y_i \mid i \in \mathbb{Z}_n\}$ and edge set the union the out edges $\{\{x_i, x_{i+1}\} \mid i \in \mathbb{Z}_n\}$, the inner edges $\{y_i, y_{i+t} \mid i \in \mathbb{Z}_n\}$ and the spokes $\{\{x_i, y_i\}, \mid i \in \mathbb{Z}_n\}$.



P(8,3)

It is known that $P(n, t)$ is VNC if and only if either $t^2 \equiv -1 \pmod{n}$ or $(n, t) = (10, 2)$.

Cubic VNC-graphs of order a product of three primes

Theorem (Zhou, J. Sys. Sci. & Math. Sci., 2008)

Let p be a prime. A connected cubic graph of order $4p$ is a VNC-graph if and only if it is isomorphic to one of the following: $P(10,2)$, the Dodecahedron, the Coxeter graph, or $P(2p,k)(k^2 \equiv -1 \pmod{2p})$.

Theorem (Zhou, Adv. Math. (China), 2008)

Let p be a prime. A connected cubic graph of order $2p^2$ is a VNC-graph if and only if it is isomorphic to $P(p^2,t)(t^2 \equiv -1 (\bmod\ p^2))$.

Cubic VNC-graphs of order a product of three primes

Theorem (Zhou & Feng, J. Graph Theory, 2010)

Let $p > q$ be odd primes and X a connected cubic vertex-transitive non-Cayley graph of order $2pq$. Then

(1) X is symmetric if and only if $X \cong \mathrm{F}030$, $\mathrm{F}102$, $\mathrm{F}182C$, $\mathrm{F}182D$, $\mathrm{F}506A$, or $\mathrm{F}2162$;

(2) X is non-symmetric if and only if $X \cong \mathcal{VNC}^1_{30}$, $\mathcal{VNC}^2_{30}$, $P(pq, t)$, where $t^2 \equiv -1 \pmod{pq}$, and $\mathrm{Aut}(\mathcal{VNC}^1_{30}) \cong S_5$ and $\mathrm{Aut}(\mathcal{VNC}^2_{30}) \cong A_5$.

Cubic VNC-graphs of order 4 times a prime power

Theorem (Kutnar, Marušič & Zhang, J. Graph Theory, 2012)

Every cubic VNC-graphs of order $4p^2$, $p > 7$ a prime, is a generalized Petersen graph.

A question

Are there infinite family of cubic VNC-graphs different from generalized Petersen graph?

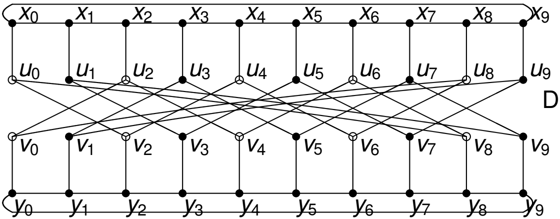
Two families of cubic vertex-transitive graphs

We modify the generalized Petersen graph construction slightly so that the subgraph induced by the out edges is a union of two n-cycles.

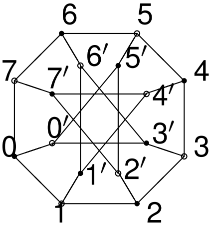
Double generalized Petersen graphs

Let $n \geq 3$ and $t \in \mathbb{Z}_n - \{0\}$. The *double generalized Petersen graph* $DP(n,t)$ (DGPG for short) is defined to have vertex set $\{x_i, y_i, u_i, v_i \mid i \in \mathbb{Z}_n\}$ and edge set the union of the *out edges* $\{\{x_i, x_{i+1}\}, \{y_i, y_{i+1}\} \mid i \in \mathbb{Z}_n\}$, the *inner edges* $\{\{u_i, v_{i+t}\}, \{v_i, u_{i+t}\} \mid i \in \mathbb{Z}_n\}$ and the *spokes* $\{\{x_i, u_i\}, \{y_i, v_i\} \mid i \in \mathbb{Z}_n\}$.

DP(10,2) and P(8,3)



DP(10,2)



P(8,3)

An interesting problem

Problem

Determining all vertex-transitive graphs and all VNC-graphs among DGPGs.

The complete solution of this problem may be a topic for our future effort. Here, we just give a sufficient condition for a DGPG being vertex-transitive non-Cayley.

A sufficient condition

Sufficient condition

Let p be a prime such that $p \equiv 1 \pmod 4$. Then $DP(2p, \lambda)$ is a connected cubic VNC-graph of order $8p$, where λ is a solution of $x^2 \equiv -1 \pmod p$ in $\mathbb{Z}_{2p}$..

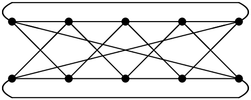
The second family

Definition 1

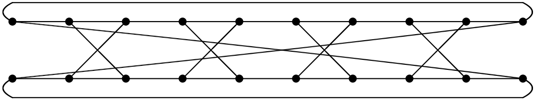
For integer $n \geq 2$, let $X(n,2)$ be the graph of order $4n$ and valency 3 with vertex set $V_0 \cup V_1 \cup \ldots V_{2n-2} \cup V_{2n-1}$, where $V_i = \{x_i^0, x_i^1\}$, and adjacencies $x_{2i}^r \sim x_{2i+1}^r (i \in \mathbb{Z}_n, r \in \mathbb{Z}_2)$ and $x_{2i+1}^r \sim x_{2i+2}^s (i \in \mathbb{Z}_n; r, s \in \mathbb{Z}_2)$.

Remark Note that $X(n,2)$ is obtained from $C_n[2K_1]$ by expending each vertex into an edge, in a natural way, so that each of the two blown-up endvertices inherits half of the neighbors of the original vertex.

$C_5[2K_1]$



$X(5,2)$



The second family

The second family

Let $EX(n,2)$ be the graph obtained from $X(n,2)$ by blowing up each vertex x_i^r into two vertices $x_i^{r,0}$ and $x_i^{r,1}$. The adjacencies are as the following: $x_{2i}^{r,s} \sim x_{2i+1}^{r,t}$ and $x_{2i+1}^{r,s} \sim x_{2i+2}^{s,r}$, where $i \in \mathbb{Z}_n$ and $r, s, t \in \mathbb{Z}_2$ (see Fig. 1 for EX(5,2)).

A picture of EX(5,2)

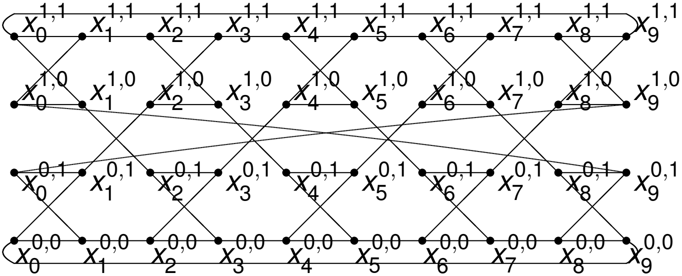


Figure: The graph $EX(5,2)$

Dobson et al. (2007) showed that $EX(n, 2)$ is vertex-transitive for each $n \geq 2$. However, $EX(n, 2)$ is not necessarily a Cayley graph.

Sufficient condition

Let $p > 3$ be a prime. Then the graph $EX(p, 2)$ is a connected cubic VNC-graph of order $8p$.

Cubic VNC-graphs of order $8p$

Recently, we classified cubic VNC-graphs of order $8p$.

Theorem (Zhou & Feng, 2011, submitted to Elec. J. Combin.)

A connected cubic graph of order $8p$ for a prime p is a VNC-graph if and only if it is isomorphic to $\mathrm{F56B}$, $\mathrm{F56C}$, $DG(2p, \lambda)$ or $EX(p, 2)$.

Cubic VNC-graphs of square-free order

Li, Lu & Wang (2012) classified cubic vertex-transitive graphs of square-free order. From their result, one may pick out all cubic VNC-graphs of square-free order.

Tetravalent VNC-graphs

Tan (1996) constructed three families of tetravalent vertex-transitive non-Cayley graphs which are metacirculant graphs.

Tetravalent VNC-graphs of order $4p$

Recently, we classified all tetravalent VNC-graphs of order $4p$ for each prime p.

Theorem (Zhou, to appear in J. Graph Theory)

There are one sporadic and five infinite families of tetravalent VNC-graphs, of which the sporadic one has order 20, and one infinite family exists for every prime $p > 3$, two families exist if and only if $p \equiv 1 \pmod 8$ and the other two families exist if and only if $p \equiv 1 \pmod 4$. For each family there is a unique graph for a given order. (Examples 1-6).

Tetravalent VNC-graphs of order $4p$

Example 1

Let $p > 3$ be a prime. The graph VNC^1_{4p} has vertex set $\mathbb{Z}_p \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and its edges are defined by $\{(i,(x,y)),(i+1,(y,z))\} \in E(VNC^1_{4p})$ for all $i \in \mathbb{Z}_p$ and $x, y, z \in \mathbb{Z}_2$.

Example 2

Let p be a prime and let $r \in \mathbb{Z}_p^*$ satisfy $r^4 = -1$. The graph VNC^2_{4p} is defined to have vertex set $\{x_i^j \mid i \in \mathbb{Z}_4, j \in \mathbb{Z}_p\}$ and edge set $\{\{v_i^j, v_{i+1}^{j+r^i}\}, \{v_i^j, v_{i+1}^{j-r^i}\} \mid i \in \mathbb{Z}_4, j \in \mathbb{Z}_p\}$.

Tetravalent VNC-graphs of order $4p$

Example 3

Let p be a prime and let $r \in \mathbb{Z}_p^*$ satisfy $r^4 = -1$. The graph VNC_{4p}^3 is defined to have vertex set $\{x_i^j \mid i \in \mathbb{Z}_4, j \in \mathbb{Z}_p\}$ and edge set $\{\{x_{i-1}^j, x_i^j\}, \{x_0^{j-1}, x_0^j\}, \{x_1^j, x_1^{j+r}\}, \{x_2^j, x_2^{j+r^2}\}, \{x_3^j, x_3^{j+r^3}\} \mid i \in \mathbb{Z}_4, j \in \mathbb{Z}_p\}$.

Example 4

Let p be a prime and let $t \in \mathbb{Z}_{2p}^*$ satisfy $t^2 = -1$. The graph VNC_{4p}^4 is defined to have vertex set $\{x_i, y_i \mid i \in \mathbb{Z}_{2p}\}$ and edge set $\{\{x_i, x_{i+1}\}, \{x_i, x_{i+p}\}, \{x_i, y_i\}, \{y_i, y_{i+t}\}, \{y_i, y_{i+p}\} \mid i \in \mathbb{Z}_{2p}\}$.

Tetravalent VNC-graphs of order $4p$

Example 5

Let p be a prime and let $t \in \mathbb{Z}_{2p}^*$ satisfy $t^2 = -1$. The graph VNC_{4p}^5 is defined to have vertex set $\{x_i, y_i \mid i \in \mathbb{Z}_{2p}\}$ and edge set $E = \{\{x_i, x_{i+2}\}, \{x_i, x_{i+p}\}, \{x_i, y_i\}, \{y_i, y_{i+2t}\}, \{y_i, y_{i+p}\} \mid i \in \mathbb{Z}_{2p}\}$.

Example 6

Let $G = A_5$ be the alternating group of degree 5. Let $H = \langle(1\ 2\ 3)\rangle$, $d_1 = (1\ 4)(2\ 5)$ and $d_2 = (1\ 2)(4\ 5)$. Then $\mathrm{Cos}(G, H, Hd_1H \cup Hd_2)$ is a connected tetravalent VNC-graph of order 20, denoted by VNC_{20}^6, and $\mathrm{Aut}(VNC_{20}^6) \cong S_5$.

Thanks!

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