Galois actions on regular dessins of small genera

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joint work with M. Conder, G.A. Jones, M. Streit

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Outline

Compact Riemann surfaces and algebraic curves

Theorem (Riemann)

Smooth complex projective algebraic curves are compact Riemann surfaces — and conversely.

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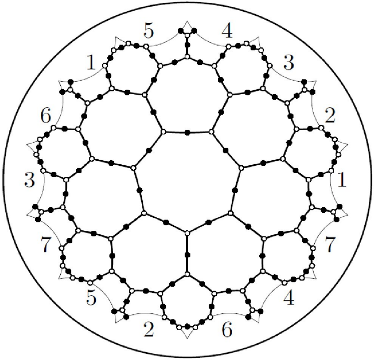
First example: Klein's quartic



Fundamental domain for the covering group (= surface group) Γ of Klein's quartic $Q : x^3y + y^3z + z^3x = 0$ in the hyperbolic plane $\mathbf{H}$.

Triangle groups and Belyĭ functions

Also visible in the picture: Γ is subgroup of a triangle group, here of the group $\Delta(2,3,7)$. The canonical projection

$$Q = \Gamma\backslash\mathbf{H} \to \Delta(2,3,7)\backslash\mathbf{H} \cong \mathbf{P}^1(\mathbf{C})$$

defines a Belyĭ function β: meromorphic, non–constant and ramified above three points only.

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Dessins d'enfants

Fact 2 :

Theorem (Belyĭ 1979)

On a compact Riemann surface X there is a Belyĭ function if and only if — as an algebraic curve — X can be defined over a number field.

We may assume that $0, 1, \infty$ are the critical values of the Belyĭ function β. Then the β–preimage of the real interval $\circ$——$\bullet$ between 0 and 1 forms a bipartite graph cutting the Riemann surface in simply connected cells, a dessin d'enfant (Grothendieck 1984).

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Dessins and conformal structures

As Grothendieck pointed out, dessins induce moreover the Riemann surface structure.

Fact 3 :

Theorem (Grothendieck, Singerman 1974)

On the other hand, every dessin on a compact oriented 2–manifold X defines a unique conformal structure on X such that the dessin belongs to some Belyǐ function on X .

Therefore, we have an equivalence between

- dessins on Riemann surfaces,
- smooth algebraic curves over number fields,
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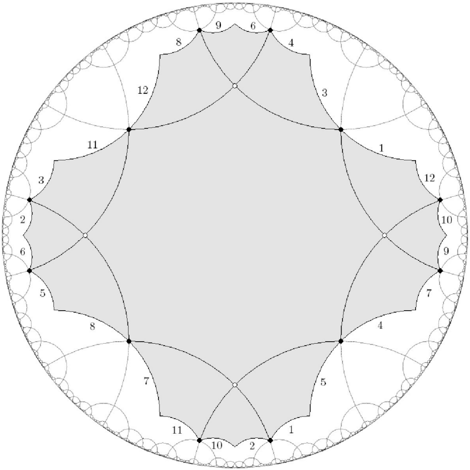
2nd example: Fermat 4

For the genus 3 Fermat curve of exponent 4 , the dessin for $\beta(x : y : z) = x^4/z^4$ can be drawn as regular embedding of the $K_{4,4}$ graph



Why so beautiful?

The dessins presented here are (orientably) regular : there is an (orientation preserving) automorphism group of the hypermap (and of the Riemann surface!) acting transitively on the set of edges.
Important because

- every dessin is a quotient of a regular one,
- their Belyĭ functions define normal coverings $X \to \mathbf{P}^1(\mathbf{C})$,
- their Riemann surfaces are quasiplatonic
- corresponding to very special points in their moduli spaces,
- and whose equations can be (a bit) easier determined than those for other Belyĭ surfaces.
- They can be classified! (Brahana . . . . . . Marston Conder)

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Good Galois behaviour

Let $\mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$ be the absolute Galois group, acting on all coefficients of the equations for the curve X and on all coefficients of the Belyĭ function. For all $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$,

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Galois invariants

Invariant under this Galois action are

- the genus of X ,
- the type of the dessin (ramification orders of β),
- the automorphism group,
- regularity.

Theorem (Wolfart '97/'06)

For a quasiplatonic curve X , the minimal field of definition is its moduli field, that is the fixed field of the subgroup

$$\{\, \sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q} \mid X^\sigma \cong X \,\} \,.$$

A direct consequence of the invariants

is the following: if a regular dessin is uniquely determined up to isomorphism by its type (valencies) and automorphism group, then the curve and the dessin (its Belyĭ function) can be defined over $\mathbb{Q}$.

This applies to all quasiplatonic curves in genera $1 < g \leq 5$.

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Abelian automorphism groups

First in genus $g = 6$ there are two non–isomorphic regular dessins which cannot be distinguished by their type (both $(7, 14, 14)$) or their automorphism group (both C_{14} , the cyclic group of order 14). Here, (independent) arguments by Hidalgo, Mühlbauer, Torres–Teigell show

Theorem

If the automorphism group of a regular dessin is abelian, then the dessin and its underlying quasiplatonic curve can be defined over $\mathbb{Q}$ *.*

These two genus 6 examples can be given as

$$y^{14} = x(x-1)^3 \quad \text{and} \quad y^{14} = x(x-1)^9 \ .$$

MAGMA calculations

are needed not only for classification.

Example. In genus 16, there are two quasiplatonic curves with isomorphic automorphism groups of order 72 and regular dessins of type $(2, 6, 12)$, so both have a surface group $\Gamma \lhd \Delta(2, 6, 12)$. Both curves are however not Galois conjugate because the number of

$$N \lhd \Delta \quad \text{with} \quad N \lhd \Gamma \quad \text{whose } \Gamma/N \text{ is an elementary abelian 2–group}$$

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Up to genus 18, there are five such pairs of quasiplatonic curves which can be treated in this way.

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A survey of results

Theorem

In genera $1 < g \leq 18$ *there are 49 quasiplatonic curves not definable over* $\mathbb{Q}$ *.*

The first two correspond to a chiral pair of regular dessins in genus 7 of type $(2,6,9)$ *and automorphism group of order* 54 *, defined over* $\mathbb{Q}(\sqrt{-3})$ *.*

Most of them are defined over quadratic number fields with the exception of

- *one Galois orbit of length* 4 *in genus* 12 *of type* $(2,5,10)$ *(graph* K_{11}*), automorphism group of order* 110 *and minimal field of definition* $\mathbb{Q}(\zeta_5)$ *,*
- *one Galois orbit of length* 3 *in genus* 14 *of type* $(2,3,7)$ *and automorphism group* $\mathrm{PSL}_2\mathbf{F}_{13}$ *, defined over* $\mathbb{Q}(\cos\frac{2\pi}{7})$*.*

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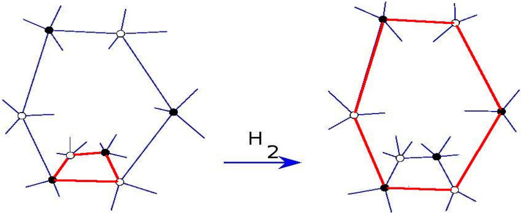
MAGMA says that for most of our low genus cases where several regular dessins of the same type and with the same automorphism group exist, these dessins are linked by (generalizations of) Wilson's (hole) operators.



And under suitable (mild) conditions one has a useful correspondence

Wilson operators $\longleftrightarrow$ **Galois conjugations**

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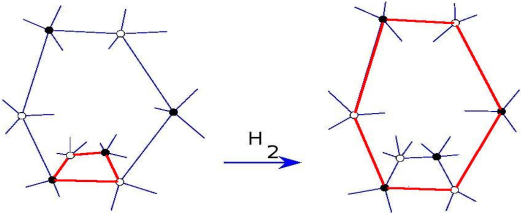

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Congruence subgroups

The remaining cases present an arithmetic mystery. There, Δ is always arithmetically defined, its quaternion algebra has a center k and the surface groups Γ are principal congruence subgroups $\Delta(\mathfrak{p})$ for (Galois conjugate) split prime ideals $\mathfrak{p}$ of k. The moduli fields are always k, and the automorphism groups are $\mathrm{PSL}_2\mathbf{F}_p$ or $\mathrm{PGL}_2\mathbf{F}_p$ for $p = N(\mathfrak{p})$.

g	8	14	16
type of Δ	2,3,8	2,3,7	3,4,6
p	7	13	5
k	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(\cos\frac{2\pi}{7})$	$\mathbb{Q}(\sqrt{6})$

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