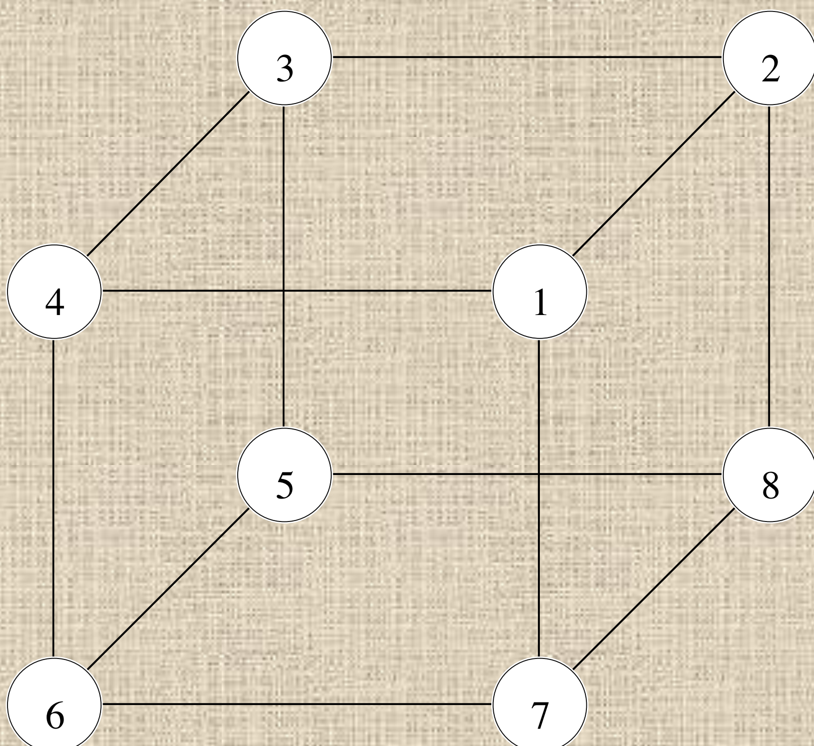
Edge-Transitive Tricirculant Tetravalent Graphs

Queenstown, Otago,
New Zealand

13 February, 2012

If Γ is a graph, a *semiregular symmetry* of Γ is a symmetry which acts on vertices as one or more cycles of the same length $n > 1$.

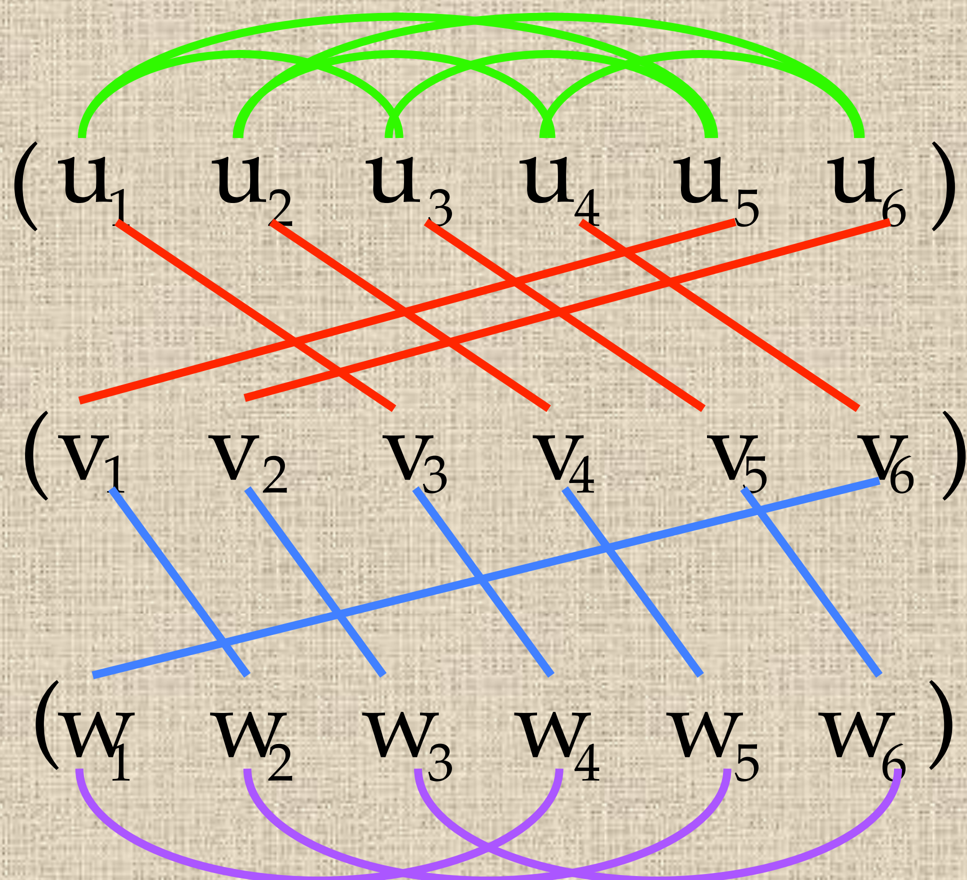


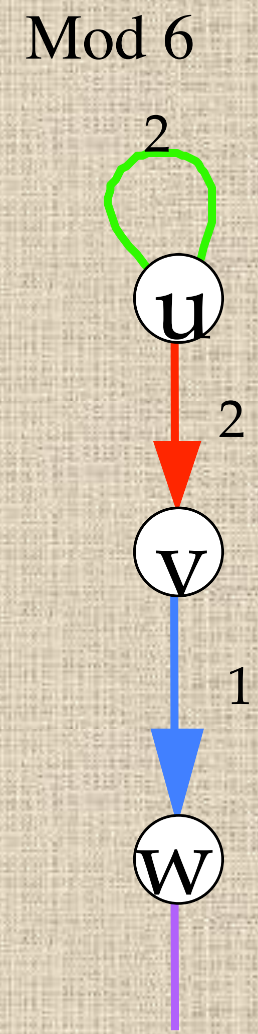
For example, in the cube, (1 2 3 4)(5 6 7 8) is a semiregular symmetry

But (1 2 3 5 6 7)(4 8) is not.

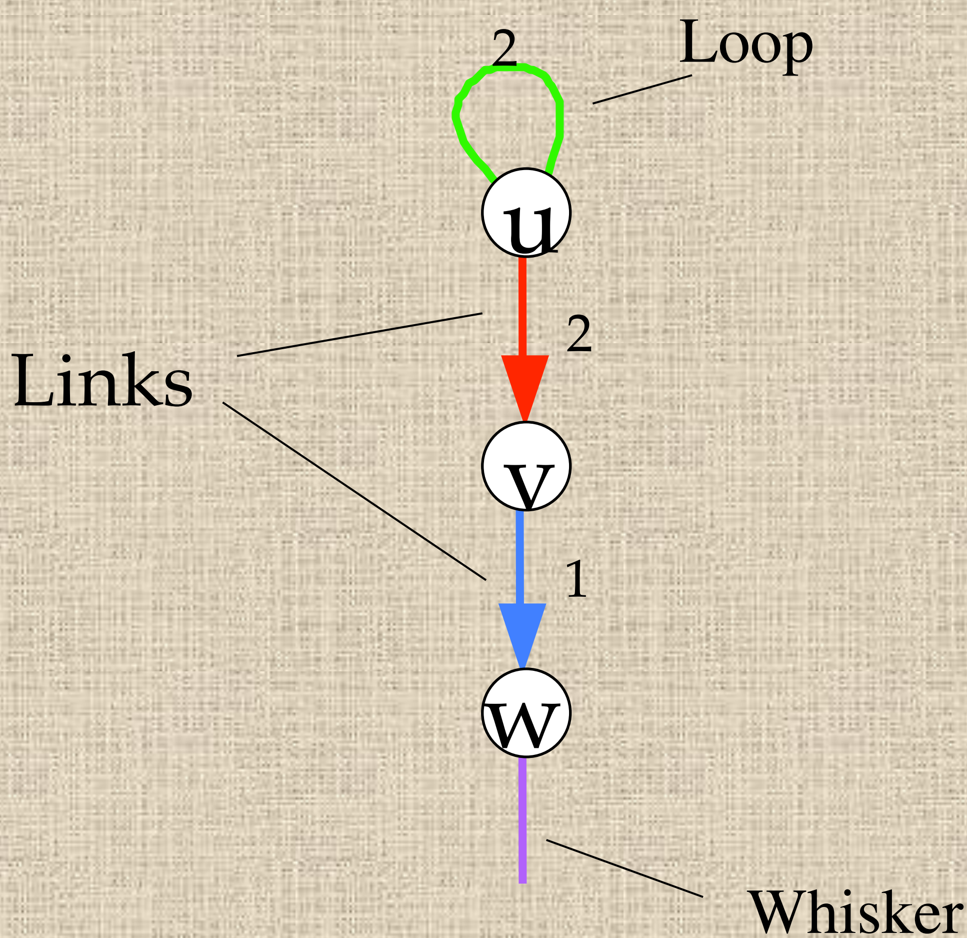
Diagrams of semiregular symmetries

Consider this SRS of order 6:





Jargon:

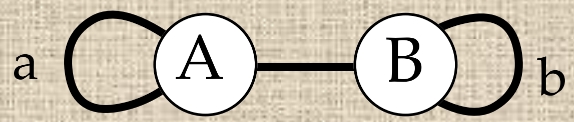


Some questions:

(1) Given a diagram, what values of the parameters give an edge-transitive graph?

Consider this diagram:

Mod N



Generalized Petersen Graphs
(Frucht, Graver, Watkins)

N= 4, a=1, b = 1: cube Q_3
N= 5, a=1, b = 2: Petersen
N= 8, a=1, b = 3 : Möbius-Kantor
N= 10, a=1, b = 2 : Dodecahedron
N= 10, a=1, b = 3 : Desargues = B(Petersen)
N= 12, a=1, b = 5 : Nauru
N= 24, a=1, b = 5 : F48

Some questions:

(1) Given a diagram, what values of the parameters give an edge-transitive graph?

(2) Given k and d, which *diagrams* on k nodes of degree d allow parameters which give an edge-transitive graph?

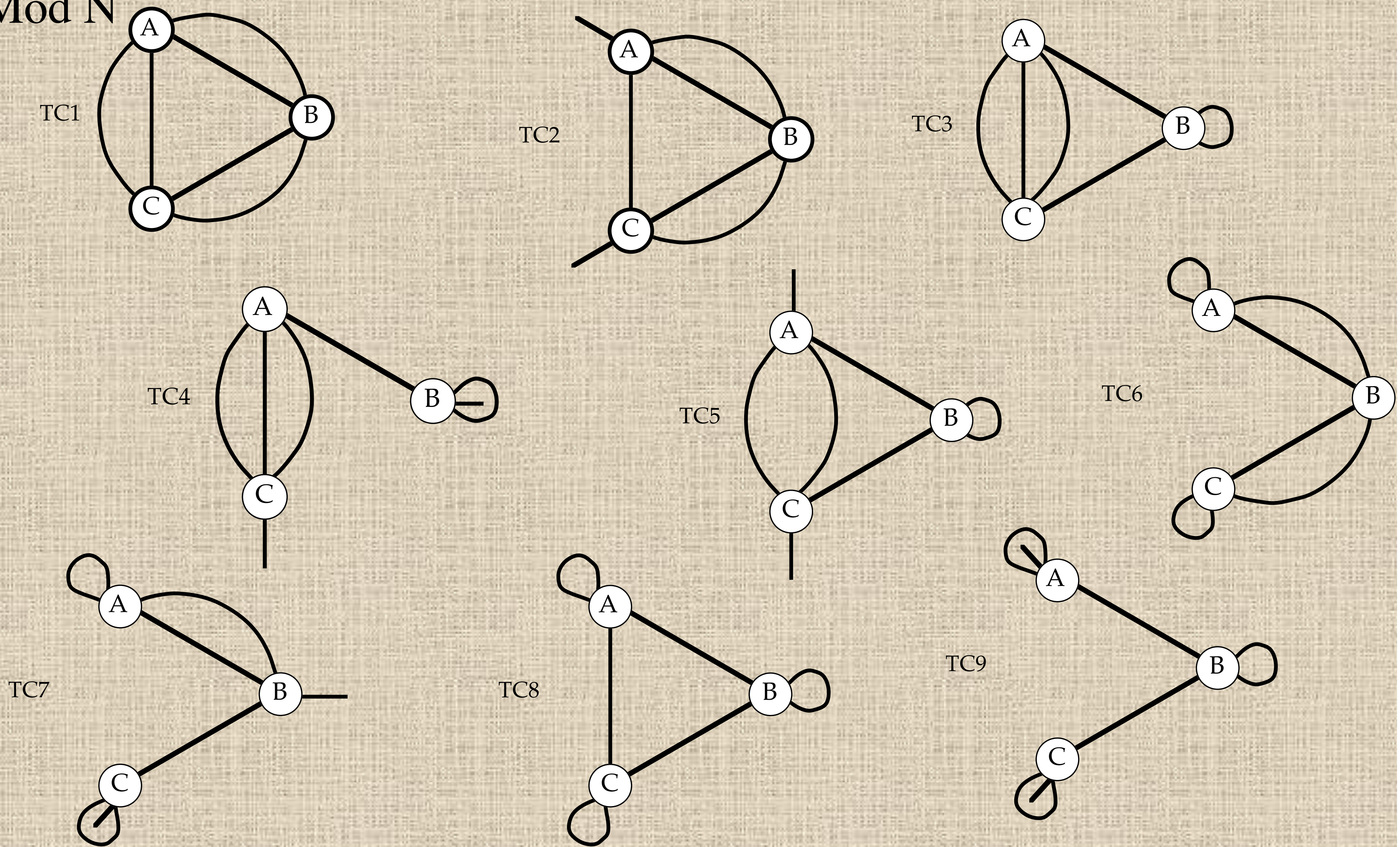
A graph is *circulant, bicirculant, tricirculant* provided that it has a SRS with exactly 1, 2, 3 cycles.

Circulancies in graphs

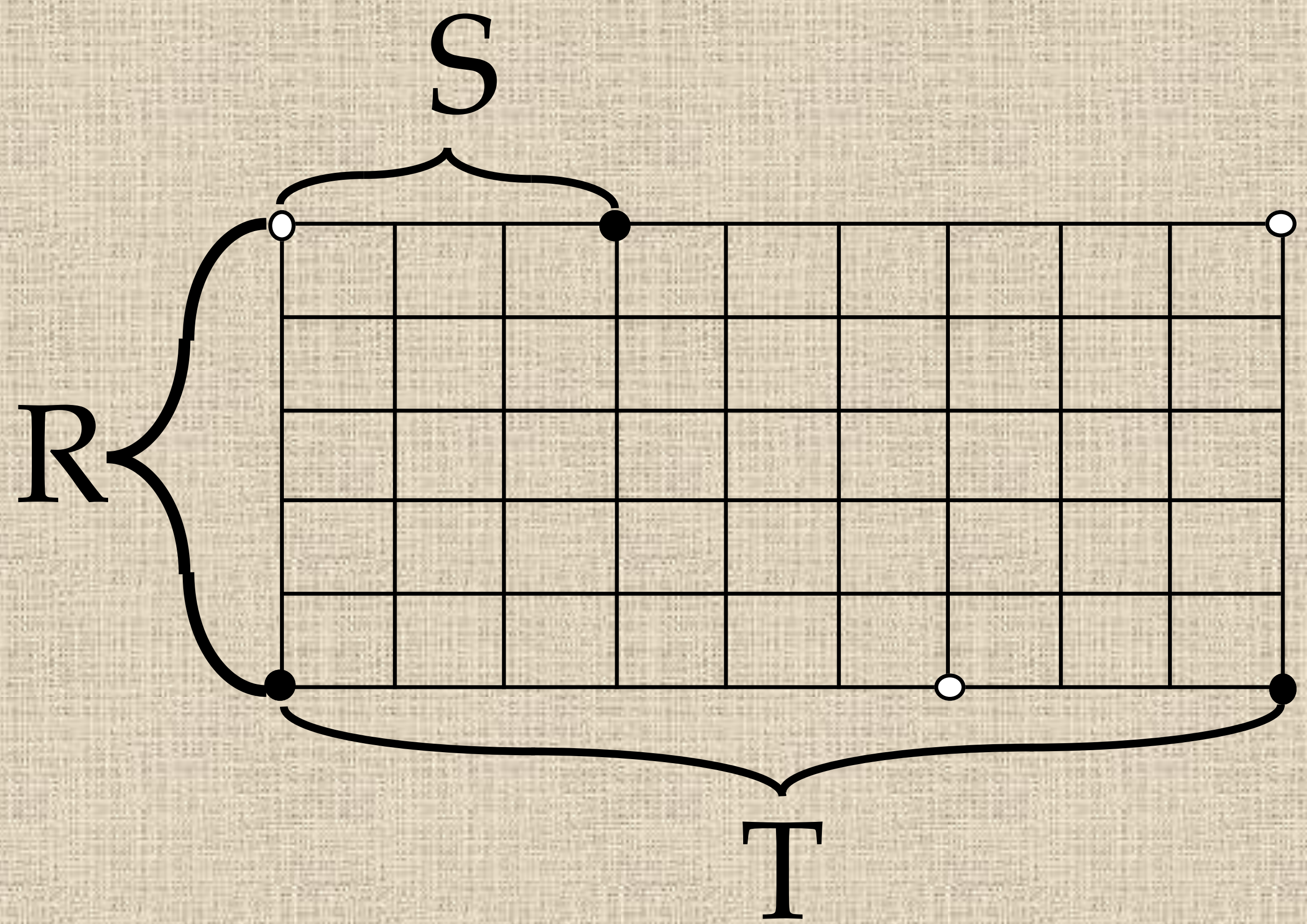
	Trivalent	Tetravalent
Circulant	K_4, $K_{3,3}$	Two families
Bicirculant	Generalized Petersens and {6, 3} maps	Kovacs, Kuzman, Malnic, Wilson
Tricirculant	Kovacs, Kutnar, Marusic, Wilson	

Tricirculant tetravalent graphs

Mod N

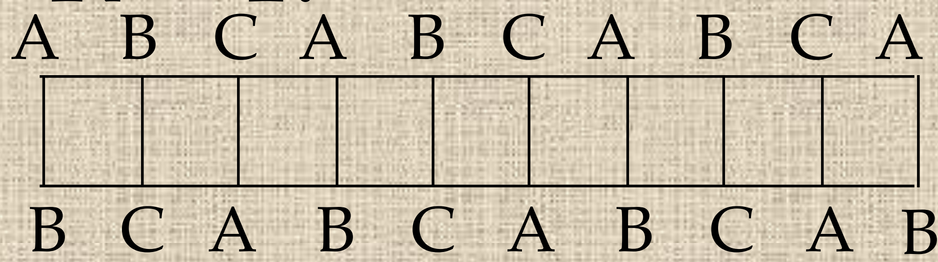


Toroidal maps of type {4,4}

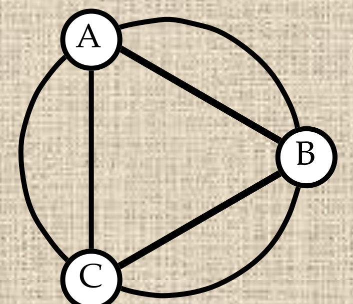


If a {4, 4} map has a *translation* that is a tricirculant:

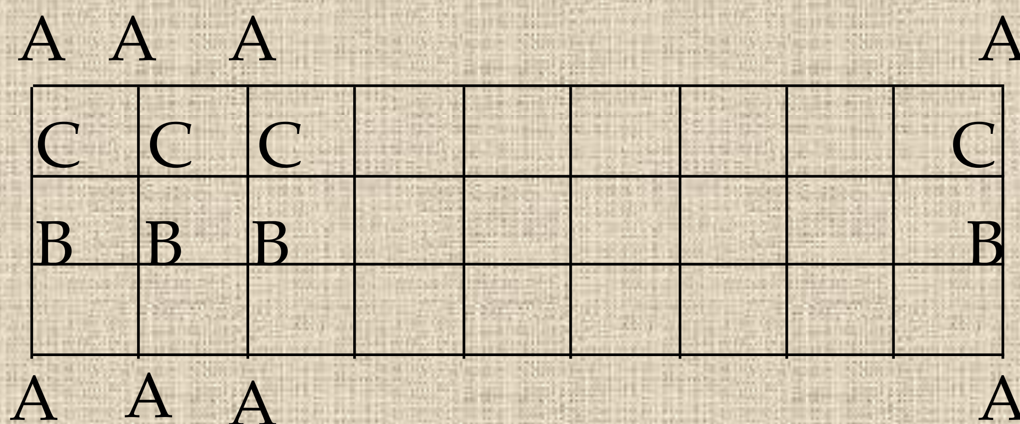
I: R = 1.



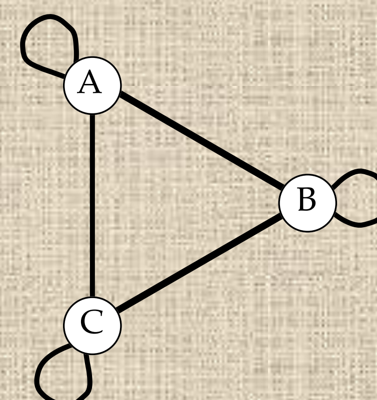
TC1



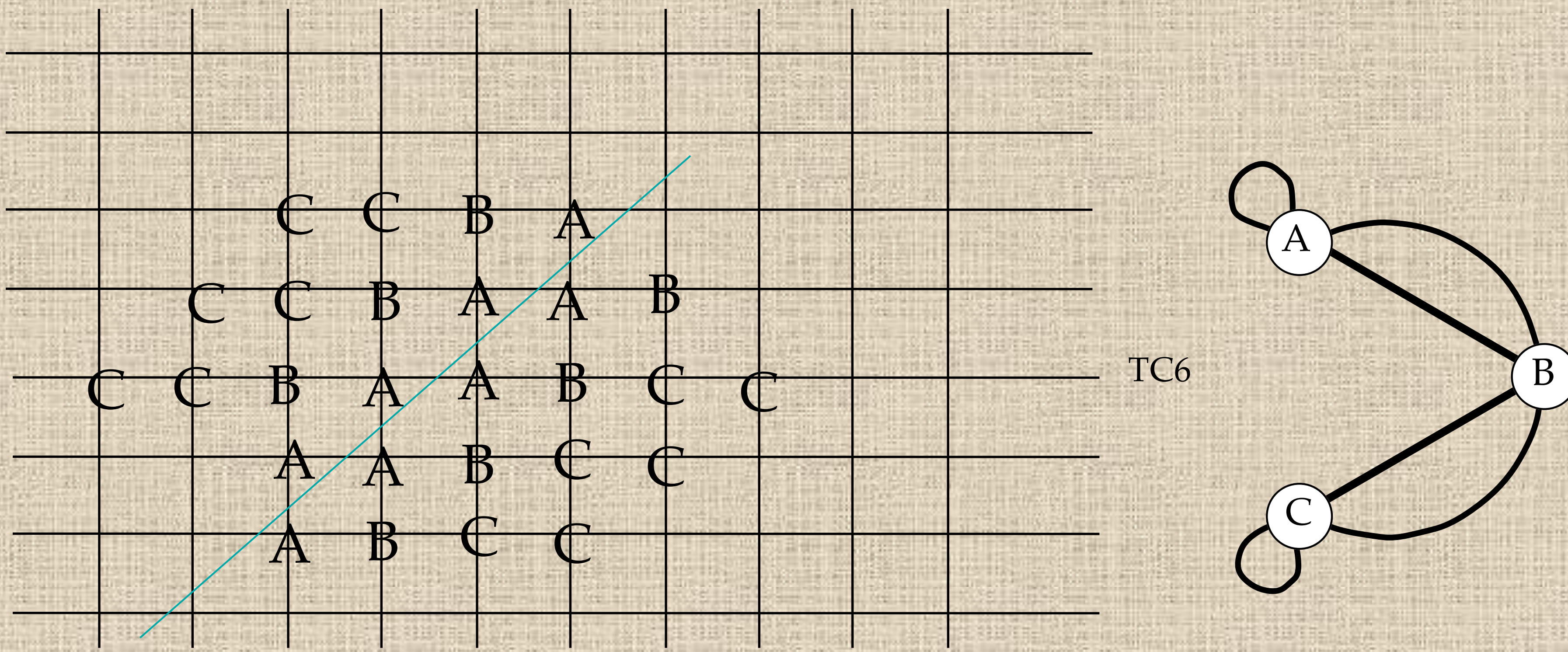
II: R = 3.



TC8

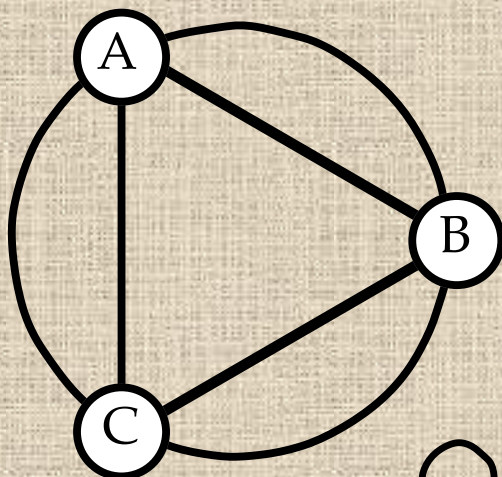


If a {4, 4} map has a *glide* that is a tricirculant:

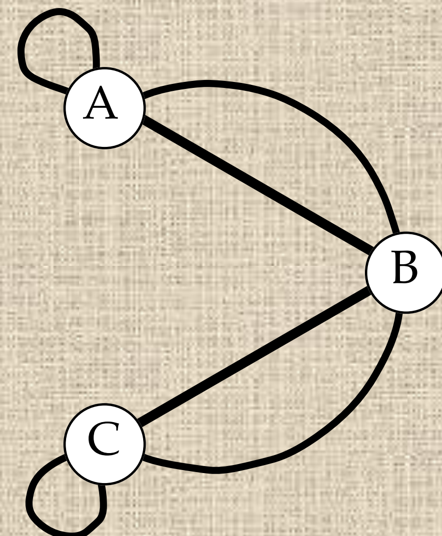


A tricirculant symmetry of a {4, 4} map has one of these diagrams:

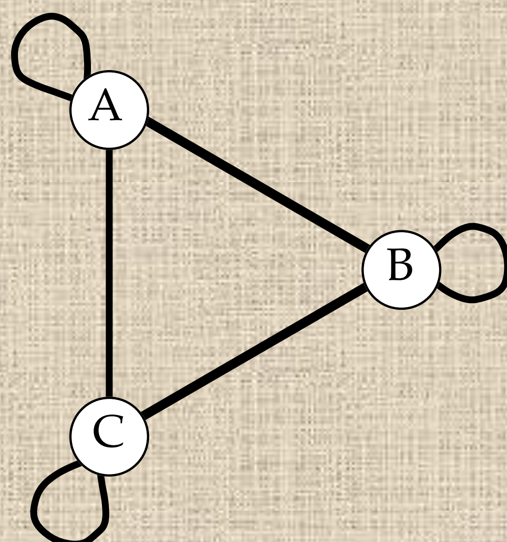
TC1



TC6

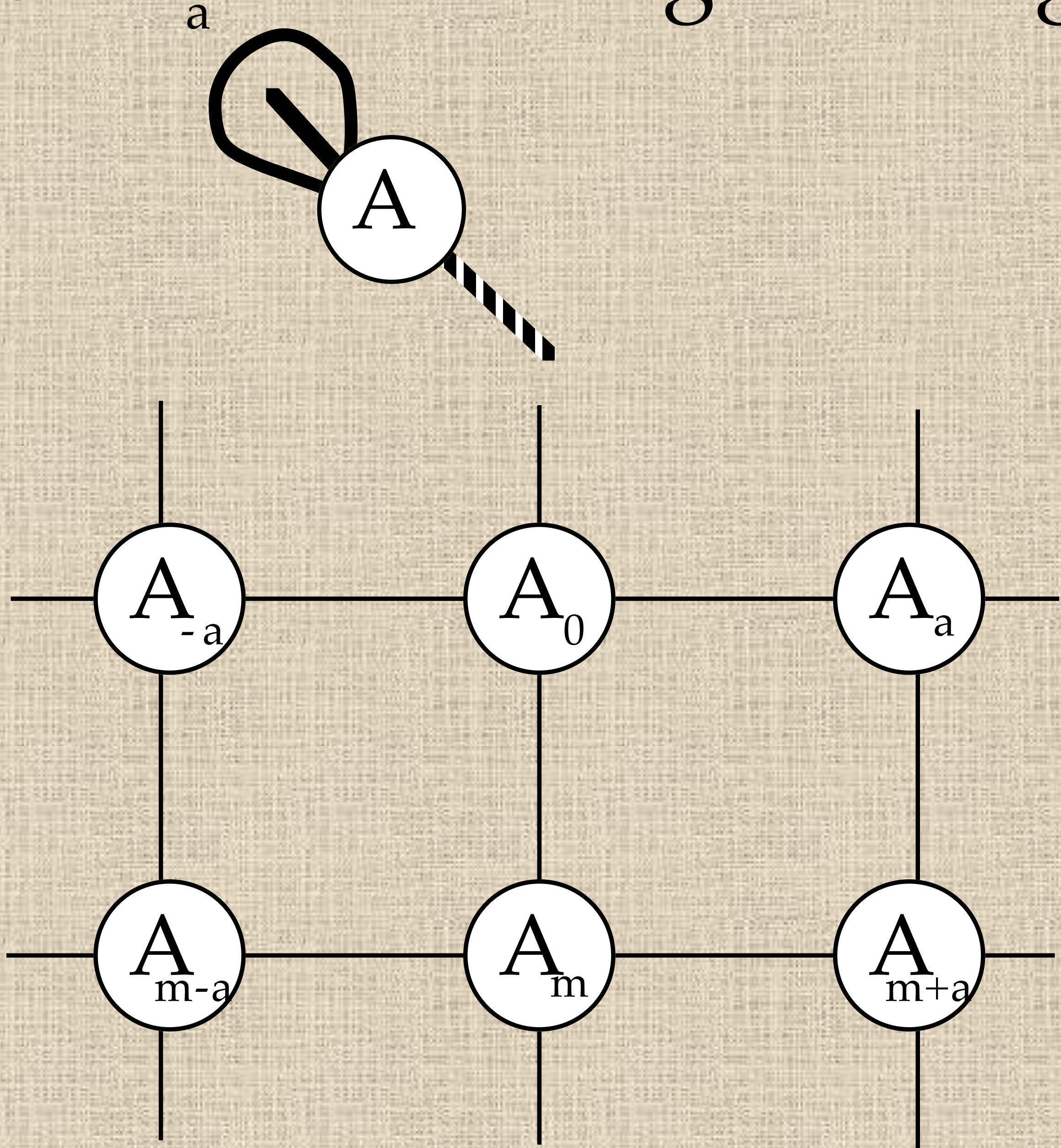


TC8

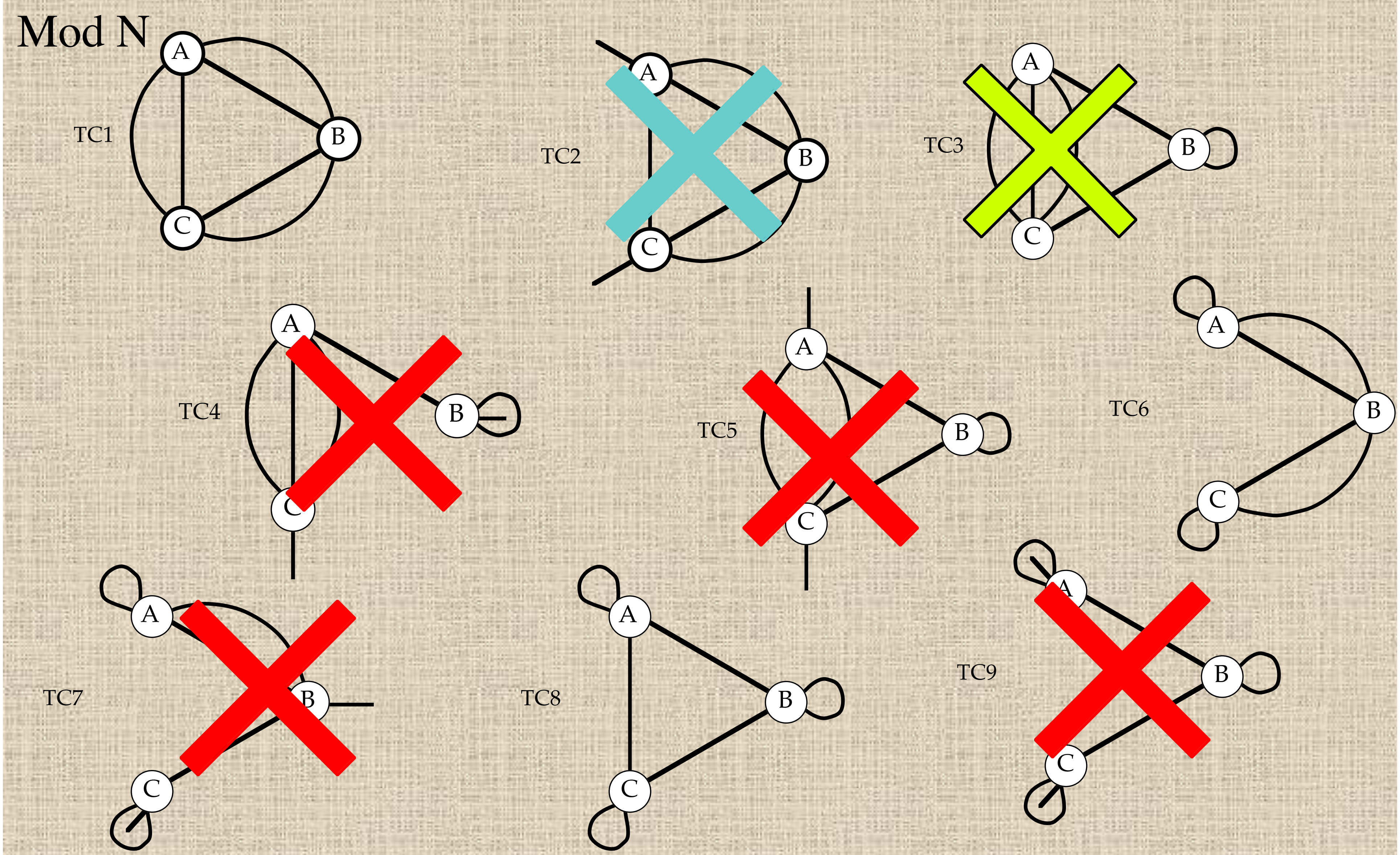


Theorem(Wilson & Potocnik): If a tetravalent edge-transitive graph has an edge which belongs to at least two 4-cycles, then (with one exception having 14 vertices) it is the skeleton of a toroidal map.

Consider this diagram fragment:

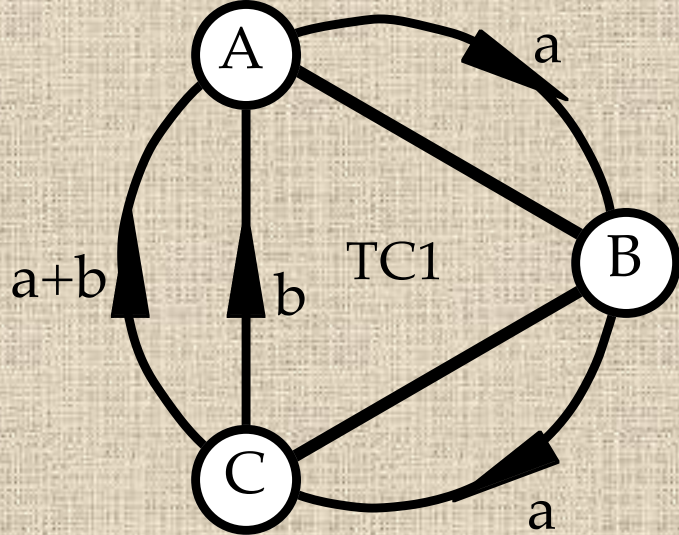


Tricirculant tetravalent graphs

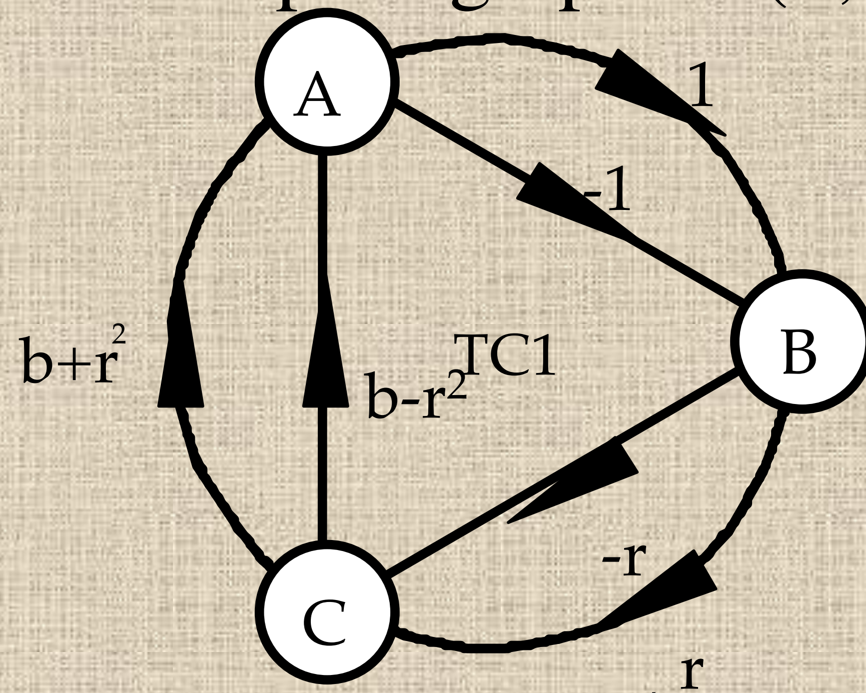


TC1

Toroidal



Spidergraph PS(3, n; r)



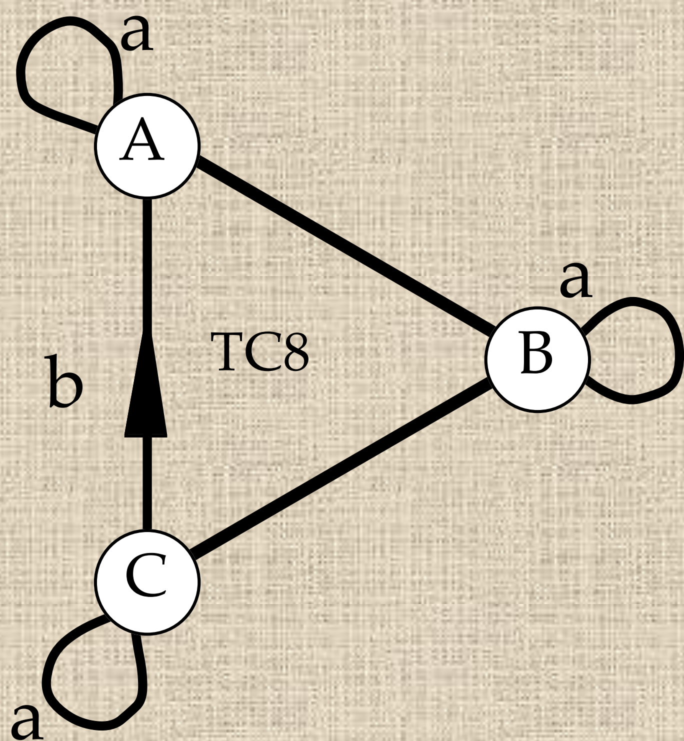
b = 0 or n/2

$r^3 = \pm 1 \pmod{n}$

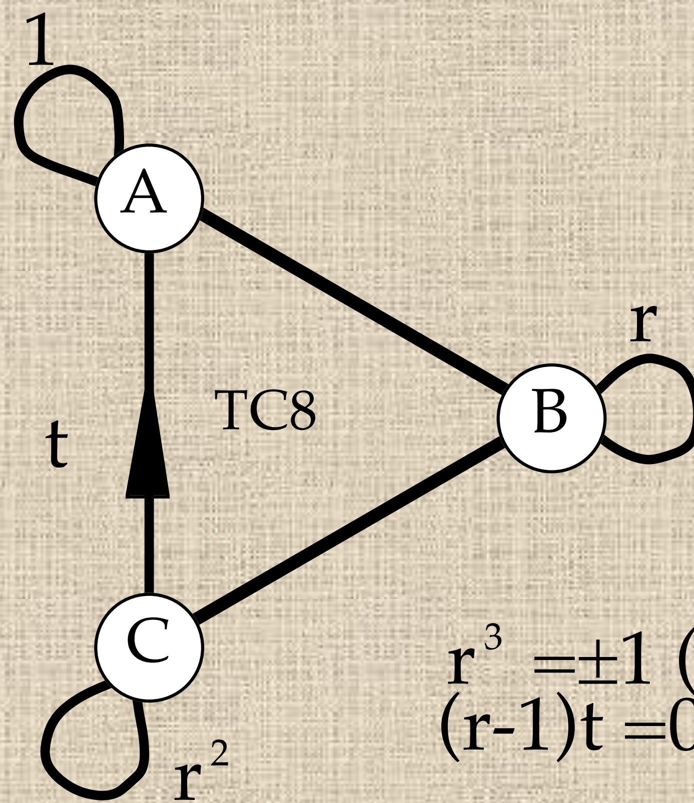
Plus sporadic examples at n = 4, 8, 8

TC8

Toroidal



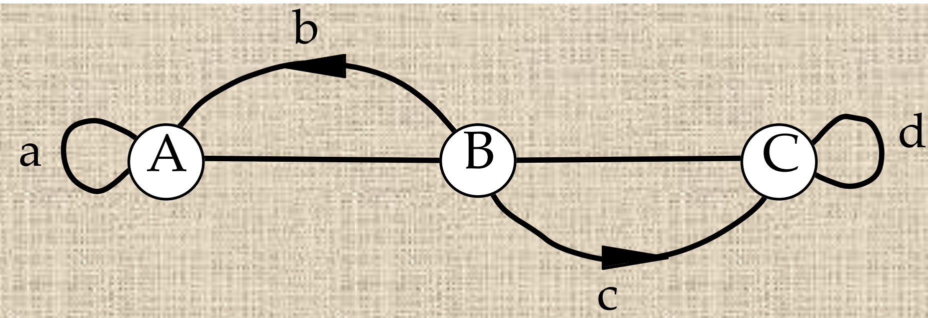
MSY(3, n; r, t)



$r^3 = \pm 1 \pmod{n}$
$(r-1)t = 0 \pmod{n}$

Marusic & Sparl, JACo 2008

TC6



Propellor graphs

$Pr_N(a, b, c, d)$

Matthew Sterns

2-weaving

Tip->ABABA . .

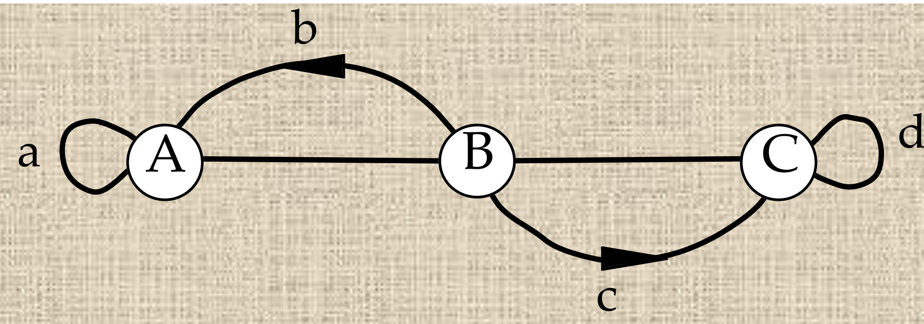
$Pr_N(1, 2d, 2, d)$ for N even and $d^2 = \pm 1 \pmod N$

4-weaving

Tip -> ABCBABCBA . .

$Pr_N(1, b, b+4, 2b+3)$ for $4|N$ and $8b+16 = 0 \pmod N$ and $b = 1 \pmod 4$

TC6



Propellor graphs

$Pr_N(a, b, c, d)$

$Pr_5(1, 1, 2, 2)$

$Pr_{10}(1, 1, 2, 2)$

$Pr_{10}(1, 4, 3, 2)$

$Pr_{10}(1, 1, 3, 3)$

$Pr_{10}(2, 3, 1, 4)$

Appendix:
Other circulancies

Circulant trivalent graphs

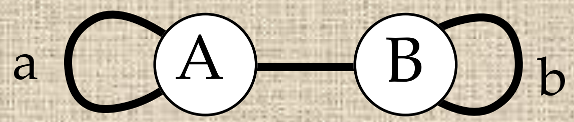
Mod N

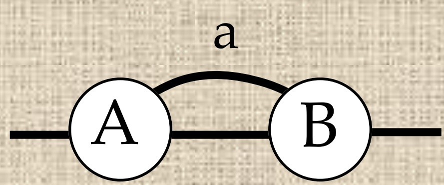
N=4, a=1, tetrahedron

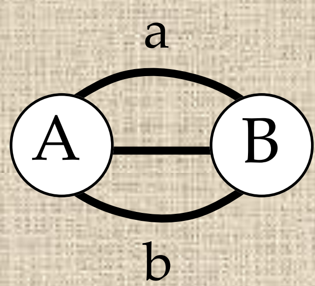
N=6, a=1, $K_{3,3}$

Bicirculant trivalent graphs

Mod N

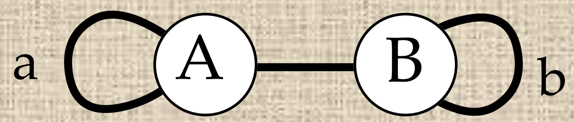






Bicirculant trivalent graphs (1)

Mod N



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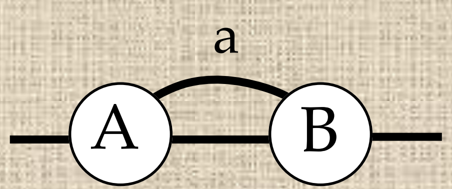
Bicirculant trivalent graphs (2)

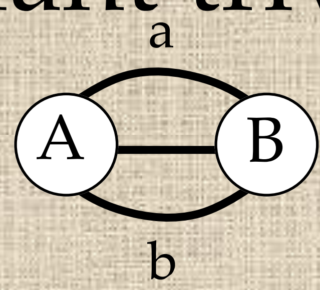
Mod N



N=2, a = 1, K_4

Bicirculant trivalent graphs (3)

Mod N



N=any, a = 1, b =r, $\{6,3\}_{B, C}$

for (B, C) = 1.

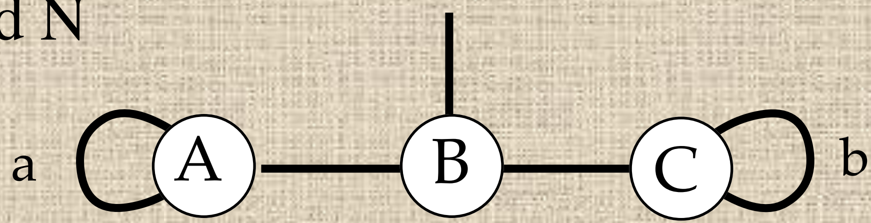
Homework 1:

a. Given N and r, find B, C.

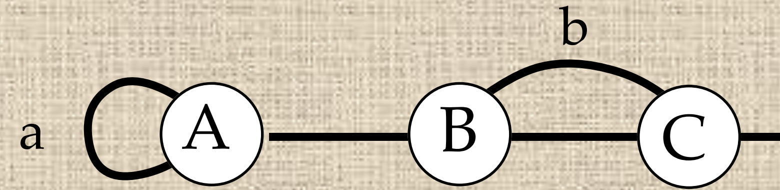
b. Given (B, C) = 1, find N, r.

Tricirculant trivalent graphs

Mod N

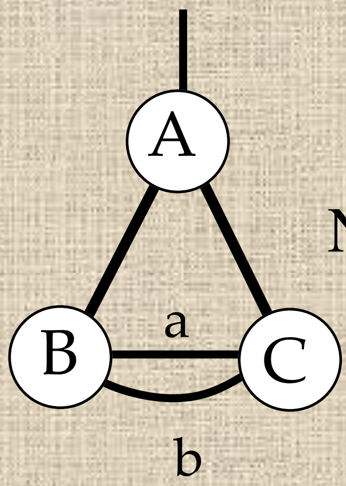


N=10, a = 1, b = 3: 8-cage

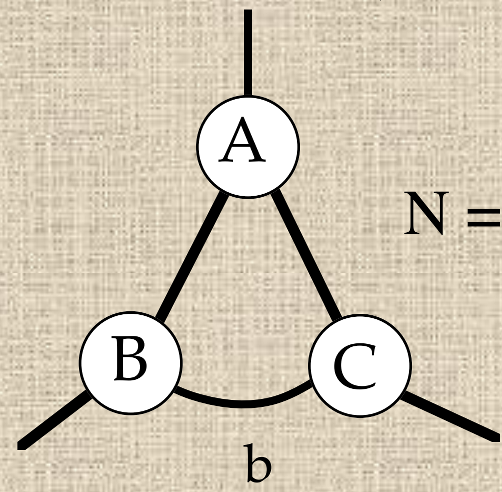


N = 6, a = 1, b = 2: Pappus

N = 18, a = 1, b = 2: $\{6, 3\}_{3,3}$



None

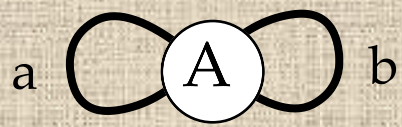


N = 2, b = 1: $K_{3,3}$

Marusic, Kutnar, Kovacs

Circulant tetravalent graphs

Mod N

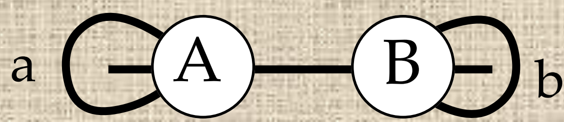


N=any, a=1, b^2 =±1 mod N

N=2m, a=1, b = m±1

Bicirculant tetravalent graphs

Mod N



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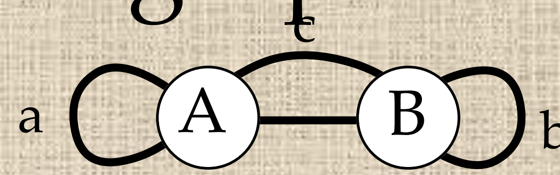
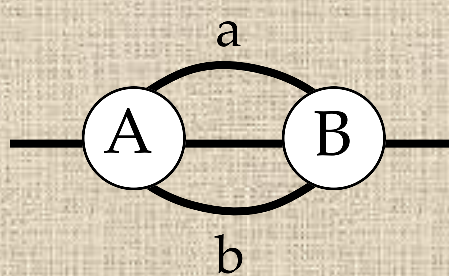
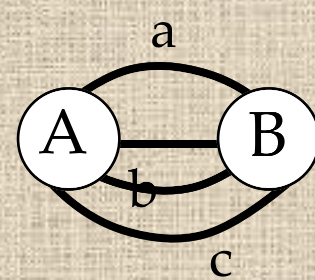


Rose window graphs, four families, all with a = 1



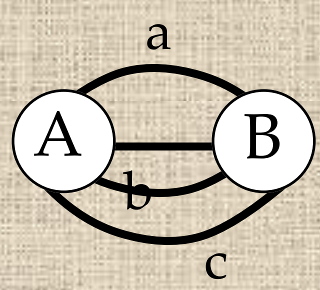
None



Kovacs, Kuzman, Malnic, Wilson

Bicirculant tetravalent graphs

Mod N



Three individual cases

N = 7, [a,b,c] = [1,2,4]

N = 13 , [a,b,c] = [1,3,9]

N = 14 , [a,b,c] = [1,4,6]

Three families:

(1) N = any, [a,b,c] = $[1, k+1, k^2+k+1]$ for $(k+1)(k^2+1) = 0$ mod N.

(2) N = any, [a,b,c] = [1,k, 1-k] for (k-1)(2k) = 0 mod N.

(3) N = product of at least 3 different primes, and none of [a,b,c] relatively prime to N.