Towards a geometry of operads: Dendroidal polytopes

Ittay Weiss

The University of the South Pacific

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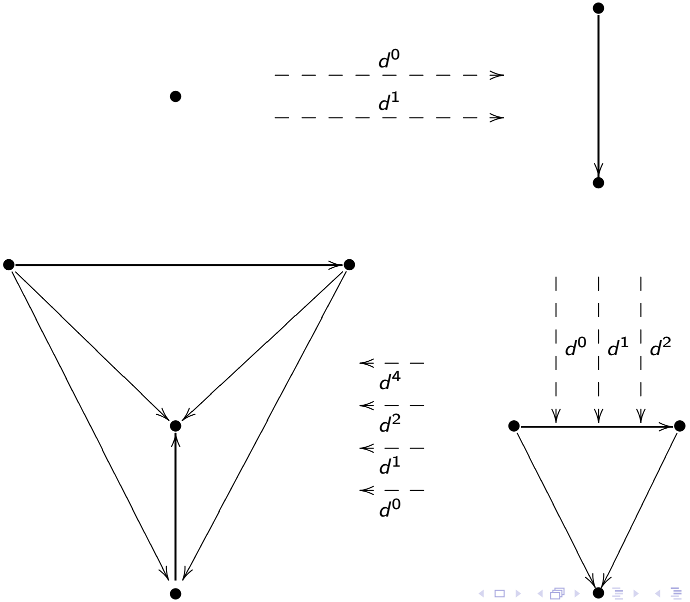
Definition

A *simplicial set* is a collection $\{X_n\}_{n=0}^{\infty}$ of sets together with, for every $n \geq 1$, $n+1$ *face maps* $d_i : X_n \to X_{n-1}$ and for every $n \geq 1$, n *degeneracy maps* $s_j : X_{n-1} \to X_n$ satisfying the *simplicial identities:*

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && \text{if } i < j \\ d_i s_j &= s_{j-1} d_i && \text{if } i < j \\ d_i s_j &= id && \text{if } i = j \text{ or } i = j+1 \\ d_i s_j &= s_j d_{i-1} && \text{if } i > j+1 \\ s_i s_j &= s_j s_{i-1} && \text{if } i > j+1 \end{aligned}$$

Note that a simplicial set is a **discrete combinatorial object**. There are two ways to understand the simplicial identities. Geometrically as the duals of face embeddings and order theoretically as duals of maximal linear embeddings.

Simplicial identities - geometrically



Simplicial identities - order theoretically

- Denote by L_n a linearly ordered set with $n+1$ elements (for purposes of following slides, think of these as linear trees).
- For each $n \geq 1$ there are precisely $n+1$ maximal embedding $d^i : L_{n-1} \to L_n$ characterized by $\mathrm{Im}(d^i)$ missing the i-th element. These are called *coface* maps.
- For each $n \geq 1$ there are precisely n minimal surjections $s^j : L_n \to L_{n-1}$ charachterized by the j-th element being the only one to be repeated (twice). These are called *codegeneracy* maps.
- Every order preserving map $L_n \to L_m$ is a composition of codegeneracies followed by cofaces.
- The relations satisfied by the cofaces and codegeneracies are dual to the simplicial identities.

The Homotopy Hypothesis

- For a topological space A, $\mathrm{Sing}(A)$ is the simplicial set where $\mathrm{Sing}(A)_n = \mathrm{Map}(\Delta^n, A)$, called the *singular complex* of A.
- For a simplicial set X associate a *geometric realization* $|X|$ by taking a disjoint union of simplices, one of dimension n for every element $x \in X_n$, and gluing them together according to the face maps $d_i : X_n \to X_{n-1}$.

Theorem

(Quillen, 1967) Topological spaces and simplicial sets each support a model structure such that the singular complex and geometric realization constructions form an equivalence.

- Informally, the theorem states that as far as homotopy theory is concerned there is no difference between topological spaces and simplicial sets.
- It allows the use of combinatorial techniques in topology.
- The theorem is rather hard to prove.
- Slogan: the geometry of linear trees is homotopy theory.

Polytopes

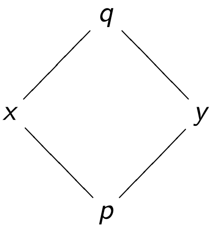
Recall that an *abstract polytope* is a poset P (whose elements are called faces) such that:

- P has a bottom element $\bot$ and a top element $\top$
- All flags (e.g., maximal chains) in P have the same length (which then implies that every $p \in P$ has a well-defined dimension)
- P is strongly connected
- P satisfies the diamond condition

where

- Strongly connected means that every two flags are connected by a sequence of adjacent flags
- The diamond condition states that if $p, q \in P$ are faces of dimension $n-1$ and $n+1$ respectively then there are precisely two faces $x, y \in P$ of dimension n such that

Polytopes



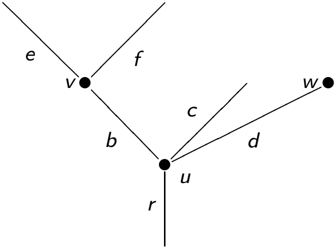
Loosely stated: a face of codimension 2 is a face of a face in precisely two ways.

- Every Euclidean polytope P gives rise to an abstract polytope $\mathrm{Face}(P)$ by considering the poset of faces of P.
- Not every abstract polytope arises in this way.
- The two ways of understanding the simplicial identities are embodied in the isomorphism

$$\mathrm{Sub}(L_n) \cong \mathrm{Face}(\Delta^n).$$

Operads

An operad is an algebraic structure that is an abstraction of the algebra of compositions of multivariable functions. The following is a picture of (part of) an operad:

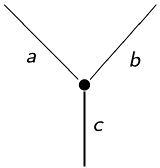


- Trees are fundamental objects in the study of operads.
- The simplicial relations extend to dendroidal relations giving rise to dendroidal sets.
- Is there a tree formalism that correctly captures the combinatorics of operads?

Tree formalisms

- For every definition of a tree T there is a notion of the poset of subobjects $\mathrm{Sub}(T)$
- For different definitions of tree these posets can be different
- Is there a definition of tree such that the accompanying poset $\mathrm{Sub}(T)$ has a relevant operadic interpretation?
- Motivated by the isomorphism $\mathrm{Sub}(L_n) \cong \mathrm{Face}(\Delta^n)$, can every tree be realized as a polytope?
- Are there other geometric realizations of trees?

More specifically, for the tree T given by



we expect $\mathrm{Sub}(\mathrm{T}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b, c\}\}$. Just this requirement already excludes most (all?) familiar formalisms.

New tree formalism

There is an extension of order theory that permits for an axiomatization of trees that captures the operadic combinatorics. Within that formalism

Theorem

Given any tree T its poset of subobject $P = \mathrm{Sub}(T)$ satisfies

- P has a bottom element $\bot$ and a top element $\top$
- All flags in P have the same length
- P is strongly connected
- P satisfies the restricted diamond condition: If $p, q \in P$ are faces of dimensions $n-1$ and $n+1$ respectively, and $p \neq \bot$, then there exist precisely two faces x, y of dimension n such that $p < x, y < q$.

Dendroidal polytopes

Definition

A poset P satisfying the conditions above is called a *dendroidal abstract polytope.*

Work in progress:

- There exists a generalization of Euclidean spaces in which dendroidal abstract polytopes can (sometimes) be realized.
- All dendroidal abstract polytopes of the form $\mathrm{Sub}(T)$ can be so realized.
- A geometry of operads emerges!

Broad posets

- Given a set A the free commutative monoid on it is $A^+ = \{(a_1, \cdots, a_n) \mid n \in \mathbb{N}_{\geq 0}, a_i \in A\}$ with addition.
- A *broad relation* R on a set A is a subset $R \subseteq A^+ \times A$.
- We denote, as usual, $a_1 + \cdots + a_n Ra$.
- R is reflexive if aRa holds for all $a \in A$.
- R is anti-symmetric if aRb and bRa imply $a = b$.
- R is transitive if $a_1 + \cdots + a_n Ra$ and for $b_i \in A^+$ holds $b_i Ra_i$ then $b_1 + \cdots + b_n Ra$.
- R is a broad poset if R is reflexive, anti-symmetric, and transitive. We then write $R = \leq$.

Induced posets

A broad poset $(A, \leq)$ induces two relations, one on A and one on A^+, as follows.

- For $a, b \in A$ declare that $a \leq_d b$ if there exist some $x \in A^+$ such that $a + x \leq b$.
- For $a_1 + \cdots + a_n, b_1 + \cdots + b_k \in A^+$ declare $a_1 + \cdots + a_n \leq b_1 + \cdots + b_k$ if there exist k elements $x_1, \cdots, x_k \in A^+$ such that $a_1 + \cdots + a_n = x_1 + \cdots + x_k$ and $x_i \leq b_i$ hold.

Note that

- $(A, \leq_d)$ need not be anti-symmetric. If $(A, \leq)$ is finite then $(A, \leq_d)$ is a poset.
- $(A^+, \leq)$ is always a poset.

Trees

Given a broad poset $(A, \leq)$ and an element $a \in A$ let $a_{\downarrow} = \{b \in A^+ \mid b < a\}$. If $a_{\downarrow} = \emptyset$ then a is called a *leaf*.

Definition

A *tree* is a broad poset $(A, \leq)$ which is dendroidally ordered in the sense that:

- $(A, \leq_d)$ has a maximum element r called the *root*.
- $(A, \leq)$ is simple (meaning that if $a_1 + \cdots + a_n \leq a$ holds then $a_i = a_j$ implies $i = j$).
- for every $a \in A$ either a is a leaf or $a_{\downarrow}$ has a maximum in $(A^+, \leq)$.

Thank You!