A census of cubic vertex-transitive graphs

Gabriel Verret (Primorska), P. Potočnik and P. Spiga

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WLOG, we may assume connectedness.

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Using ad hoc methods, McKay and Royle (1996) obtained a list which is complete up to 94 vertices.

Using some new theoretical results and a few tricks, we construced all cubic vertex-transitive graphs of order at most 1280.

Three natural cases

Let Γ be a cubic G-vertex-transitive graph and let m be the number of orbits of $G_v^{\Gamma(v)}$ (the permutation group induced by the action of a vertex-stabiliser G_v in its action on the neighbourhood $\Gamma(v)$).

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We deal with each of these separately.

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The census of cubic arc-transitive graphs is now complete up to 10000 vertices (Conder).

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These simple tricks are enough to make the $m = 3$ case computationally feasible, except when G has order 512 or 1024 (too many groups).

2-groups

For n a power of 2, let R_n be the class of groups which have order n and admit a generating set consisting of 3 involutions or of 2 elements, one of which is an involution.

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Once we have constructed R_{512} and R_{1024}, we apply to the groups in these classes the same procedure which we used for other orders.

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The most difficult case. The main problem in this case is that a vertex-stabiliser can be arbitrarily large. (In fact, very large with respect to $|\mathrm{V}(\Gamma)|$).

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By a paper of Miklavic, Potočnik and Wilson, arc-transitive cycle decompositions of 4-valent graphs are well-understood, so it suffices to find all 4-valent arc-transitive graphs of order at most 640.

4-valent arc-transitive graphs

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We characterised the graphs for which $|G_v|$ is "very large" with respect to the the order of the graph.

4-valent arc-transitive graphs

Theorem (PSV)

Let (Γ, G) *be locally-*D_4*. Then one of the following holds:*

- $\Gamma \cong \mathrm{C}(r, s)$,
- (Γ, G) *is one of 18 exceptions,*
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This allows us to use the amalgam method (construct the amalgams and find all normal subgroups up to a certain index).

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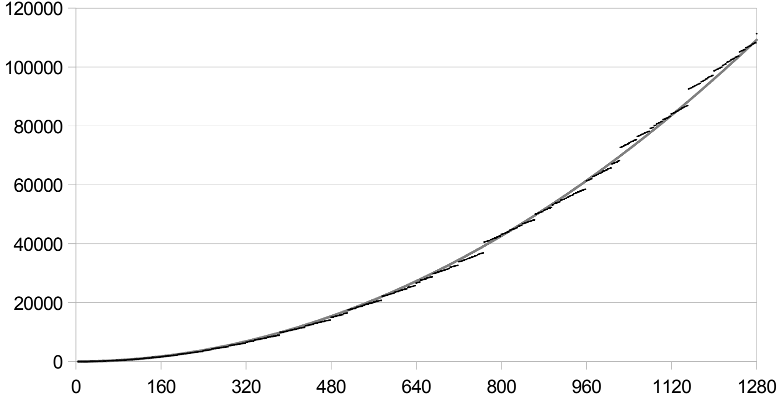
Side note : by combining our data with the census of small 2-arc-transitive 4-valent graphs (Potočnik), we get all 4-valent arc-transitive graphs of order at most 640.

Number of graphs of order up to *n*



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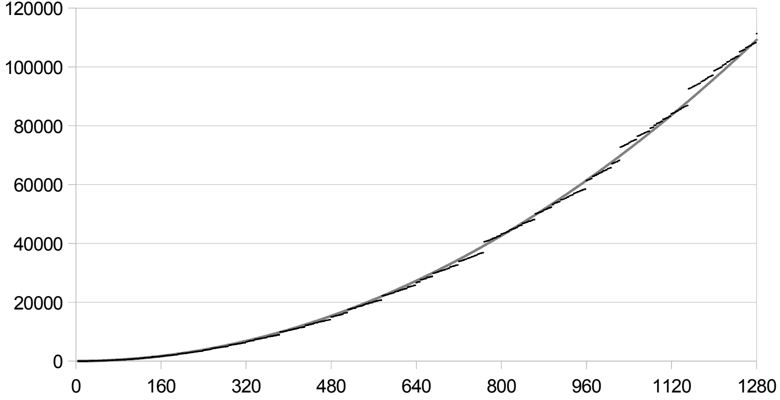

In gray is the graph of the function $n \mapsto n^2/15$. In an upcoming paper, we prove that $\log(f(n)) \in \Theta((\log n)^2)$.

Graphs of order at most 1280 by type

	$m = 1$	$m = 2$	$m = 3$	Total
Cayley	386	11853	97687	109926
Non-Cayley	96	1338	0	1434
Total	482	13191	97687	111360

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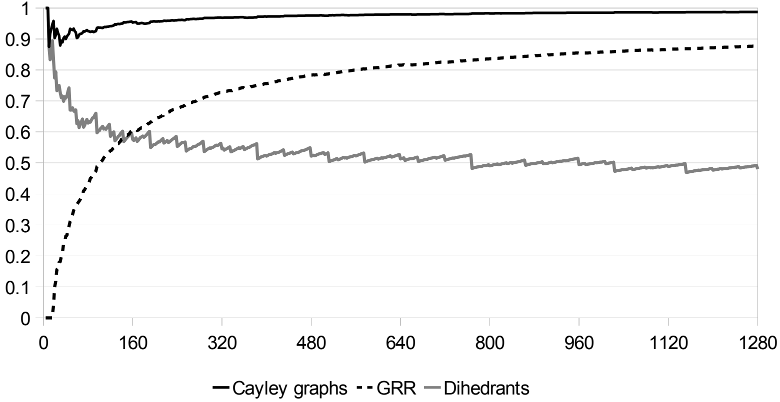
This seems to be part of a trend.

Proportion of graphs of different types



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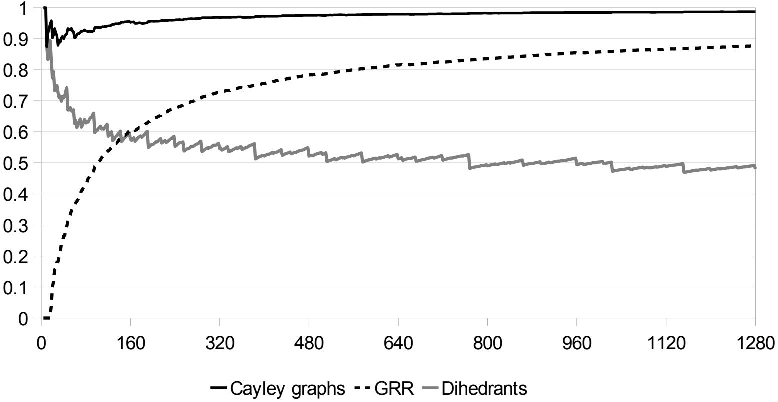

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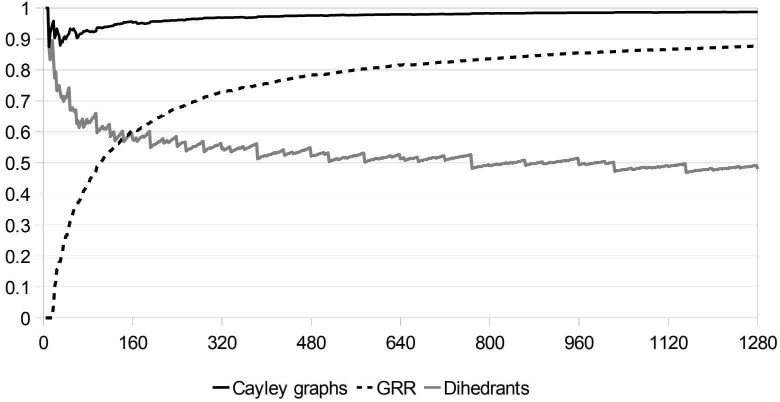

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`Magma` files containing the graphs can be found online at :
http://www.matapp.unimib.it/~spiga/