Distinguishing Maps: Angles, Cliques, and Transpositions

Thomas W. Tucker

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The orientation-preserving case

Let A be a subgroup of $Aut(G)$. Call an edge uv a τ-edge for A if some non-identity $a \in A$ fixes u and v (e.g a reflection across uv for a map). If there is no such edge, A is no-τ.

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Theorem (TWT *Elec J. Comb. 2011*). If $A \subset Aut(G)$ is no-τ and $D(A, V) > 2$, then G is $C_3, C_4, C_5, K_4, K_5, K_7, O_6, O_8$. For all except O_8 there is a unique (well-known) orientable map M with $Aut^+(M) = A$. There is no such map for O_8.

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So $S_2(O_n) = O_{n+2}$ and $S_1(K_n) = K_{n+1}$.

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Examples: Intransitive $d < 3$

Type	map	G	surface	V-orb	E-orb
bipart		$K_{1,n}, K_{2,n}$	sphere	2	1 or 2
1 rad	C_n	$S_1(C_n)$	sphere	2	2
2 rad	C_n	$S_2(C_n)$	sphere	3	3
1 rad	T1	$S_1(K_4)$	torus	2	3
1 rad	T2	$S_1(K_5)$	$g = 2$	2	3
3 rad	K_4^P	$S_3(K_4)$	proj	2	2
2 rad	T2	$S_2(K_5)$	$g = 2$	3	4
bipart	$B(m, n)$	$K_{m,n}$	$g = 3, 4, 6$	2	1

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bipart	$B(m, n)$	$K_{m,n}$	$g = 3, 4, 6$	2	1

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Here $B(m, n)$ is map from $CM(Z_m \times Z_n, (a, a^{-1}b, b^{-1})$, where we place a vertex in each a^m and b^n face, join through vertices of original graph (and throw out old vertices and edges). Can be viewed as hypermap.

Examples: Transitive, not $D^+(M) = 3$

Name	G	d	$Stab$	g	τ
$CM(Z_4, (1, -1, 2))$	K_4	3	D_1	1	mix
$CM(Z_5, (1, -1, 2, -2))$	K_5	4	D_1	2	no
$CM(Z_2^3, (x, y, z))$	cube	3	D_3	1	reg
$CM(Z_2^3, (x, x+y, y, y+z, z, z+x))$	O_8	6	D_3	7	all
$CM(Q, (i, j, k)^b)$	O_8	6	D_3	5	no
$CM(Z_3^2, (x, y)^b)$	$C_3 \times C_3$	4	D_4	1	reg
$CM(Z_3^2, (x, y, -x+y)^b)$	$K_{3,3,3}$	6	D_6	1	reg
$CM(Z_3^2, (x, x+y, y, y-x)^b)$	K_9	8	D_4	10	all
$B(3, 3)$	$K_{3,3}$	3	D_n	1	reg
$B(4, 4)$	$K_{4,4}$	4	D_4	3	reg
$B(5, 5)$	$K_{5,5}$	5	D_5	6	reg

Also K_6 in projective plane.

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Given $A \subset Aut(M)$ with an edge orbit E_A, we can "twist" all the edges in E_A, to get a new map (probably nonorientable) M' with $Aut(M) = Aut(M')$ and action is the same on the vertex set (the underlying graph stays the same). Usual Petrie duality is to twist all edges.

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Example For standard map M for double pyramid $S_2(C_5)$ in the sphere, there are three edge-orbit using the D_5 action fixing the apexes. This map has $D(M) = 3$ under that action, so we can get 8 maps in all by applying various partial Petrie duals.

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So the above tables can lead to many different maps with same action of $Aut(M)$ on V.

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We have talked and resolved the differences

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Theorem 1 Suppose $A \subset Aut(G)$, where G is connected and edge stabilizers in A are abelian (e.g for maps $Z_2 \times Z_2$). Then $D(A, V) > 3$ implies $G = K_4$.

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Finally, for $c = 4$ each K_4 has the symmetry of the tetrahedron and for $c = 6$ each K_6 has the symmetry of K_6 in the projective plane (antipodal quotient of the icosahedron).

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Remark Holds for graph G and $A \subset Aut(G)$ with natural dihedral vertex-stabilizers (i.e. A_v acts on d neighbors as D_d in the natural way).

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Since an angle reflector reserves orientation, this shows why in the orientation-preserving case there are no bent open angles, so you get K_n, O_{2n}, C_n.