2-transitive finite circle geometries
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What are circle planes?

Circle planes are incidence geometries with point set P and circle set $\mathcal{C}$ that comprise Möbius, Laguerre and Minkowski planes. Circles are subsets of P with at least three points, and there are up to two different partitions of P, whose members are called generators of the plane. The most important geometric axiom is that three points no two of which are on the same generator are joined by a unique circle.

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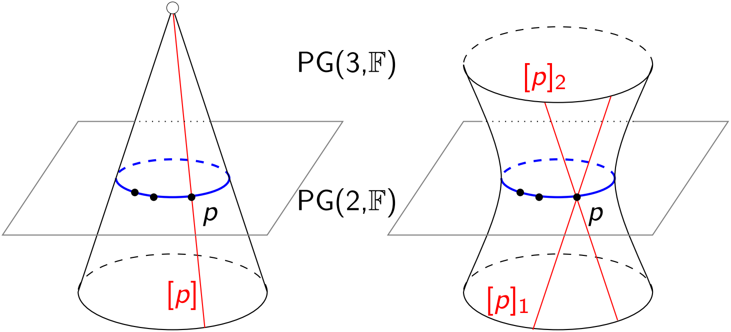
- Finite Möbius planes (or inversive planes) of order n are precisely the 3-$(n^2+1, n+1, 1)$ designs.
- Finite Laguerre planes of order n are precisely the transversal designs $\mathrm{TD}_1(3, n+1, n)$. In case of odd order, they are equivalent to anti-regular generalized quadrangles.
- Finite Minkowski planes or order n are the "doubly transversal" 3-designs with $(n+1)^2$ points. They are equivalent to sharply 3-transitive **sets** of permutations of degree $n+1$.

Models of circle planes

The *miquelian circle plane* over a field $\mathbb{F}$ is obtained as the geometry of non-trivial plane sections of a quadratic set (elliptic quadric, elliptic cone, ruled quadric) in 3-dimensional projective space over $\mathbb{F}$.



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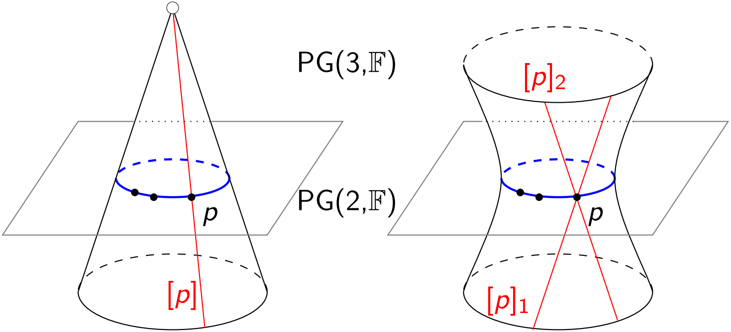

If the quadratic set is an ovoid or oval cone one obtains *embeddable* (or ovoidal) circle planes.

Finite 2-transitive Möbius or Minkowski planes

General results

- *A finite Möbius plane of even order is embeddable. (Dembowski 1964)*
- *A finite Minkowski plane of even order is miquelian. (Heise 1974)*
- *A finite circle plane of odd order with a desarguesian derivation is miquelian. (Chen, Kaerlein 1973, Thas 1994)*

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- *A finite circle plane of order at most 8 is miquelian.*
- *A Möbius or Laguerre plane of order 9 is miquellian. A Minkowski plane of order 9 is isomorphic to one corresponding to one of the two sharply 3-transitive groups of degree 10. (S. 1992)*

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Theorem

- *A finite 2-transitive Möbius plane is embeddable. (Dembowski 1964, Hering 1967)*
- *A finite 2-transitive Minkowski plane is miquelian. (Wilbrink 1982)*

So what about finite Laguerre planes?

A finite *Laguerre plane* $\mathcal{L} = (P, \mathcal{C}, \mathcal{G})$ *of order n* consists of a set P of $n(n+1)$ points, a set $\mathcal{C}$ of n^3 circles and a set $\mathcal{G}$ of $n+1$ generators (where circles and generators are both subsets of P) such that the following three axioms are satisfied:

(G) $\mathcal{G}$ partitions P and each generator contains n points.

(C) Each circle intersects each generator in precisely one point.

(J) Three points no two of which are on the same generator can be uniquely joined by a circle.

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Theorem

A finite embeddable Laguerre plane whose automorphism group is 2-transitive on the set of generators is miquelian.

Elation Laguerre planes

A *finite elation Laguerre plane* is a Laguerre plane $\mathcal{L}$ that admits a group Δ of automorphisms, called the *elation group* of $\mathcal{L}$, that acts trivially on the set of generators and regularly on the set of circles.

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There is a finite field $\mathbb{F}$ such that the point set of the elation Laguerre plane $\mathcal{L}$ can be identified with $(\mathbb{F}^m \cup \{\infty\}) \times \mathbb{F}^m$ where generators are of the form $\{x\} \times \mathbb{F}^m$, and such that the elation group Δ is isomorphic to $\mathbb{F}^{3m}$. The automorphism group of $\mathcal{L}$ is $\mathbb{F}$-linearly represented on $\mathbb{F}^{3m}$.
$m = 1$ describes the embeddable Laguerre planes.

The structure of finite 2-transitive groups

Theorem

If G is a finite 2-transitive and effective group on v points, then G contains a transitive normal subgroup H and either H is elementary abelian of prime power order v or H is simple non-abelian. (Burnside 1911)
In the latter case, H is one of the following. (Cameron 1981)

H	v	
A_n	n	$n \geq 5$
$PSL(d,q)$	$(q^d-1)/(q-1)$	$d \geq 2$, $(d,q) \neq (2,2),(2,3)$
$PSU(3,q^2)$	q^3+1	$q > 2$
$Sz(q)$	q^2+1	$q = 2^{2a+1} > 2$
${}^2G_2(q)$	q^3+1	$q = 3^{2a+1} > 3$
$PSp(2d,2)$	$2^{2d-1} \pm 2^{d-1}$	$d > 2$
M_n	n	$n = 11, 12, 22, 23, 24$

M_{11} *($v = 12$), $PSL(2,8)$ ($v = 28$), $PSL(2,11)$ ($v = 11$), A_7 ($v = 15$), $C0_3$ ($v = 276$), HS ($v = 176$).*

Finite 2-transitive elation Laguerre planes

Theorem

A finite 2-transitive elation Laguerre plane $\mathcal{L}$ is miquelian or Γ/T contains a transitive simple non-abelian normal subgroup isomorphic to

- $PSL(2, q)$, $q \neq 2, 3$,
- $PSU(3, q^2)$, $q > 2$, *or*
- $Sz(q)$, $q = 2^{2a+1} > 2$,

where T is the kernel of the action of the automorphism group Γ on $\mathcal{G}$ and q is a prime power (the order of $\mathcal{L}$).