Imprimitivity of locally finite 1-ended planar graphs

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with Mark E. Watkins

Introduction

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Want: *Let* Γ *be an infinite, locally finite, 1-ended planar graph, and let* $Aut(\Gamma)$ *be transitive on the vertex set of* Γ*. Then* $Aut(\Gamma)$ *is imprimitive.*

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