The quantitative characterization of all finite simple groups

Wujie Shi (施武杰）

Chongqing University of

Arts and Sciences, P.R. China

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Let G be a finite group, $|G|$ the order of G and $\pi_e(G) = \{\ o(g) \mid g \in G\ \}$ the set of all element orders for G (also say it the spectrum of G).

In 1987, we put forward the following conjecture:

Conjecture. **All finite simple groups G can determined uniformly using their orders $|G|$ and their element orders $\pi_e(G)$.**

That is, for all finite simple groups we may characterize them using only their orders and the sets of their element orders.

In 1987, after I wrote a letter to Prof. John G. Thompson and reported the above conjecture.

Thompson pointed that, “Good luck with your conjecture about simple groups. I hope you continue to work on it”, …, “This would certainly be a nice theorem”…, in his reply letters.

The warmly encouragement prompt us to finish this work.

With Prof. J.G.Thompson（Oct.1989 in Canberra）

Classification Theorem of F.S.G.

If G is a finite simple groups, then G is isomorphic to one of the following groups:

1. Abelian simple groups $\boldsymbol{Z_p}$, p is prime;

2. non-abelian simple groups

 (a) alternating groups $\boldsymbol{A_n}$, $n \geq 5$,

 (b) 16 infinite series groups of Lie type, ${}^2F_4(2)'$,

 (c) 26 sporadic groups.

For $\boldsymbol{Z_p}$, it is obvious and only need "the order of G"

For 26 sporadic groups, we need checked using ATLAS, see

1. Wujie Shi, A new characterization of the sporadic simple groups, Group Theory - Porc. Singapore Group Theory Conf. 1987, Walter de Gruyter Berlin-New York, 1989.

For $L_n(q)$ and A_n , see

2. Wujie Shi and Jianxing Bi, A characteristic property for each finite projective special linear group, Lecture Notes in Math., Springer-Verlag, 1456(1990).

3. Wujie Shi and Jianxing Bi, A new characterization of the alternating groups, Southeast Asian Bull. Math., 16 (1992).

For Suzuki–Ree groups, we have

4. Wujie Shi and Jianxing Bi, A characterization of Suzuki–Ree groups, Sci. in China, Ser. A, 34 (1991).

For Chevalley groups G_2, F_4, E_6, E_7, E_8 and the twisted Chevalley groups, and the Tits group, see

5. Wujie Shi, The pure quantitative characterization of finite simple groups(I), Prog. Nat. Sci., 4 (1994).

For $U_n(q)$, we have

6. Hongping Cao and Wujie Shi, Pure quantitative characterization of finite projective special unitary groups, Sci. China, Ser. A, 45 (2002).

For ${}^2D_n(q)$ and $D_l(q)$ (l odd), see

7. Mingchun Xu and W. Shi, Pure quantitative characterization of finite simple groups ${}^2D_n(q)$ and $D_l(q)$ (l odd), Alg. Coll., 10 (2003).

Since we did not prove the conjecture before 2009, if there exist a counter example for this conjecture, most occurred in $B_n(q)$ and $C_n(q)$, q odd. Thus we proved

8. Wujie Shi, Pure quantitative characterization of finite simple groups, Front. Math. China, 2(2007).

9. M. A. Grechkoseeva, On difference between the spectra of the simple groups $B_n(q)$ and $C_n(q)$, Siberian Mathematical Journal, 48 (2007).

The above papers proved that $\pi_e(B_n(q)) \neq \pi_e(C_n(q))$ for all odd q, thus the counter example is not exist.

The following lemma play a key in proofs of refs. [1-7].

Lemma (Shi; 1992) Let G be a finite non-abelian simple group. If $p^k \parallel |G|$, where p is an odd prime, and $|G| < p^{3k}$, then G is isomorphic to one of the following groups:

(1) A simple group of Lie type in characteristic p;

(2) A_5 (p=5); A_6 (p=3); A_9 (p=3);

(3) $L_2(p-1)$ (p is Fermat prime); $L_2(8)$ (p=3); $U_5(2)$(p=3).

For $p = 2$, we have a similar result.

This lemma also answered the following **E. Artin's problem:** Find out all simple groups of order |G| which contain a Sylow subgroup of order greater than $|G|^{1/3}$.

Using the classification of finite simple groups (CFSG) we get the above lemma.

From 1987 to 2003, in refs. [1-7] the authors proved that The Conjecture is correct for all finite simple groups except B_n, C_n and D_n (n even).

In the end of 2009, A. V. Vasilev, M. A. Grechkoseeva, and V. D. Mazurov proved that Conjecture is correct for B_n, C_n and D_n (n even). See

10. A. V. Vasilev, M. A. Grechkoseeva, and V. D. Mazurov, On finite groups isospectral to simple symplectic and orthogonal groups, Siberian Math. J., 50(2009).

11. A. V. Vasilev, M. A. Grechkoseeva, and V. D. Mazurov, Characterization of the finite simple groups by spectrum and order, Algebra and Logic, 48(2009).

They analyze the problem of formerly proof (depend on the lemma) and established several lemmas.

Thus this conjecture is proved, that is, all finite simple groups can determined by their “two orders”.

However, it does not true for some very small groups.
Examples:

1. $\mathbf{D_8}$(dihedral group of order 8) and $\mathbf{Q_8}$(quaternion group)
$|D_8| = |Q_8| = 8$ and $\pi_e(D_8) = \pi_e(Q_8) = \{1, 2, 4\}$,
but $D_8 \not\cong Q_8$, they are different groups.
2. $\mathbf{G_1} = Z_3 \times Z_3 \times Z_3$ and $\mathbf{G_2} = (Z_3 \times Z_3) \rtimes Z_3$ (semi-direct product), $|\mathbf{G_1}| = |\mathbf{G_2}|$ and $\pi_e(\mathbf{G_1}) = \pi_e(\mathbf{G_2}) = \{1, 3\}$,
but $\mathbf{G_1}$ and $\mathbf{G_2}$ are not isomorphism.
So, simple groups are simple.

Question 1. Find out the application for "all finite simple groups can determined by their "two orders".

Question 2. Proving the Conjecture, can or not independent on C.F.S.G., that is, can or not judge the simplicity of finite groups using only the "two orders"?

Question 3. Weaken the condition of "two orders", characterize all finite simple groups.

Question 4. Let G_1, G_2,…, G_k be finite groups. If $\pi_e(G_1) = \pi_e(G_2) = \ldots = \pi_e(G_k)$, whether or not the order of simple group G_i (if exsit) is the smallest?

For the set $\{|G|\}$, there are many famous and interesting results. For example, Sylow theorm. But we do not know more information for the set $\boldsymbol{\pi_e(G)}$.

Obviously, $\boldsymbol{\pi_e(G)}$ is a subset of the set $\boldsymbol{Z^+}$ of positive integers, and a difficult problem is:

“ Which subset of Z^+ constitute a set of orders of element of a group? ”

If $m \in \pi_e(G)$ and $n \mid m$, we have $n \in \pi_e(G)$, that is, it has a closure property for division.

Some interesting results for $\pi_e(G)$

If we divide the set $\pi_e(G)$ into {1}, the set $\pi_e'(G)$ consisting of primes and the set $\pi_e''(G)$ consisting of composite numbers, then we have

Theorem 1(Deng, Shi; 1997). Let G be any finite group. Then $|\pi(G)| = |\pi_e'(G)| \leq |\pi_e''(G)| + 3$, and if the equality holds, then G is simple. Moreover, these simple groups are all determined only by the set $\pi_e(G)$.

Theorem 2(Brandl, Shi; 1991). Let G be a finite group whose element orders are consecutive integers. That is, $\pi_e(G) = \{1, 2, 3, \ldots, n\}$. Then $n \leq 8$.

On Thompson's two problems

From 1987 to 1991, after I wrote some letters to Thompson and reported the above conjecture. Thompson posed the following two problems.

For each finite group G and each integer $d \geq 1$, let $G(d) = \{x \in G\ ;\ x^d = 1\}$.

Definition. G_1 and G_2 are of **the same order type** if and only if $|G_1(d)|=|G_2(d)|$, $d=1, 2, \ldots\ldots$.

Thompson problem(1987). **Suppose G_1 and G_2 are groups of the same order type. Suppose also that G_1 is solvable. Is it true that G_2 is also necessarily solvable?**

In Thompson’s Private letter he pointed out that

“The problem arose initially in the study of algebraic number fields, and is of considerable interest”.

In 1987, J.G. Thompson gave an example as follows:

$$G_1 = 2^4 : A_7 \text{ and } G_2 = L_3(4) : 2_2$$

both are maximal subgroups of M_{23}. From this example we get G_1 and G_2 have the same "order equation" :

$$|G_1| = |G_2| = n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8 + n_{14}$$
$$= 1 + 435 + 2240 + 6300 + 8064 + 6720 + 5760$$
$$+ \ 5040 + 5760.$$

$$\pi_e(G_1) = \pi_e(G_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 14\}.$$

It showed that the " same order type " can not decide the simple section of G.

But if Thompson's problem is affirmative, then the "same order type " may determine the solvability of all finite groups.

Another Thompson's conjecture, which is also aim at characterizing all finite simple groups by a quantity set, is posed in 1988, which is appeared in another communication letter:

If G is a finite group, set $N(G) = \{n \in \mathbf{Z}^+ ;\ G$ has a conjugacy class with $|C| = n\}$

Thompson conjecture (1988). If G and M are finite groups and $N(G) = N(M)$, and if in addition, M is a non-Abelian simple group while the center of G is 1, then G and M are isomorphic.

For Thompson two problems, we (Chen, Shen and Shi and others)

have solved the case in which the prime graph is not connected.

How deal with the case of connected prime graph？

References (only some papers for Conjecture)

1. Wujie Shi, A new characterization of the sporadic simple groups, Group Theory - Porc. Singapore Group Theory Conf. 1987, Walter de Gruyter Berlin-New York, 1989.
2. Wujie Shi and Jianxing Bi, A characteristic property for each finite projective special linear group (with J.X. Bi), Lecture Notes in Math., Springer-Verlag, 1456 (1990).
3. Wujie Shi and Jianxing Bi, A characterization of Suzuki–Ree groups, Sci. in China, Ser. A, 34 (1991).
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6. Hongping. Cao and Wujie Shi, Pure quantitative characterization of finite projective special unitary groups, Sci. China, Ser. A, 45 (2002).
7. Mingchun Xu and W. Shi, Pure quantitative characterization of finite simple groups ${}^2D_n(q)$ and $D_l(q)$ (l odd), Alg. Coll., 10 (2003).
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9. M. A. Grechkoseeva, On difference between the spectra of the simple groups $B_n(q)$ and $C_n(q)$, Siberian Mathematical Journal, 48 (2007).

10. A. V. Vasilev, M. A. Grechkoseeva, and V. D. Mazurov, On finite groups isospectral to simple symplectic and orthogonal groups, Siberian Math. J., 50 (2009).
11. A. V. Vasilev, M. A. Grechkoseeva, and V. D. Mazurov, Characterization of the finite simple groups by spectrum and order, Algebra and Logic, 48(2009).

The End Thank You ! 谢谢！