On the Split Structure of Lifted groups, II

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SODO, Queenstown, New Zeleand

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Stating Problem I

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Setting

$p\colon X \times_\zeta \Gamma \to X$ a regular cover of connected graphs given by voltages

Suppose $G \leq \mathrm{Aut}\, X$ lifts to $\tilde{G}$

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Further assumptions

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However

For complexity reasons we would like to avoid the construction of the derived graph and the lifted group

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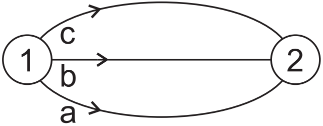
Since CT_p is abelian

voltages can be viewed as 'vectors' over $\mathbb{Z}$
and evaluation of $g_i^{\#}(c)$ as $M_i \underline{c}$

Example



Voltage group: $\Gamma =< x \mid 7x = 0 >\cong \mathbb{Z}_7$
voltages: $\zeta(a) = 4x, \quad \zeta(b) = 2x, \quad \zeta(c) = x$

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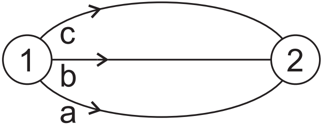

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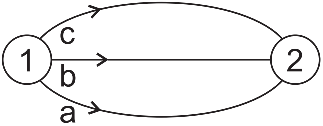

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$\tilde{G}$ **has a sectional complement** $\Leftrightarrow$ **system has a solution**

Package for computing with covers

Main functions

- construction of the voltage space,
- construction of (T,u)-reduced voltage assignment,
- construction of the derived cover,
- testing whether an automorphism lifts,
- testing weather two projections are equivalent,
- computing a lift,
- construction of the group of covering transformations,
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Additional feature

generalised graphs: multiple links, loops, and **semiedges**

Thank you!