GI-Graphs and Their Groups

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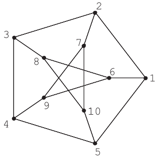
Generalized Petersen graphs



In 1950 the class of generalized Petersen graphs was introduced by Coxeter and around 1970 popularized by Frucht, Graver and Watkins.

Let $n \geq 3$ and k be such that $1 \leq k < n$ and $k \neq n/2$.

$$
\begin{aligned}
V(G(n,k)) &= \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\} \\
E(G(n,k)) &= \{u_i u_{i+1},\ \ u_i v_i,\ \ v_i v_{i+k};\ i = 0, \ldots, n-1\},
\end{aligned}
$$

where the subscripts are to be read modulo n. Since $G(n,k) = G(n, n-k)$ we usually take $1 \leq k < n/2$.

Some properties of *GP*-graphs

- connected
- vertex-transitive if $(n, k) = (10, 2)$ or
$$k^2 \equiv \pm 1 \pmod n.$$
- edge-transitive if
$$(n, k) \in \{(4, 1), (5, 2), (8, 3), (10, 2), (10, 3), (12, 5), (24, 5)\}.$$
- each edge-transitive is also vertex-transitive and hence arc-transitive.

Automorphisms of GP-graphs

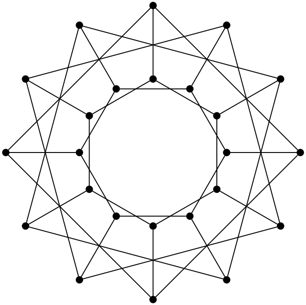
If a generalized Petersen graph is not edge-transitive, then there (may) exist only three types of automorphisms:

- rotation ρ
- reflection τ
- automorphism α that changes the outer and the inner rim if and only if $k^2 \equiv \pm 1 \pmod{n}$:

$$\alpha(u_i) = v_{ki}, \quad \alpha(v_i) = u_{ki}.$$

The automorphism group of $G(n, k)$ contains the dihedral group D_n, generated by ρ and τ.

I-graphs



I-graphs were introduced in the Foster census in 1988 by Bouwer *et al.*

They represent a slight further albeit important generalization of the renowned Petersen graph.

Let $n \geq 3$ and j, k be such that $1 \leq j, k < n$ and $j, k \neq n/2$.

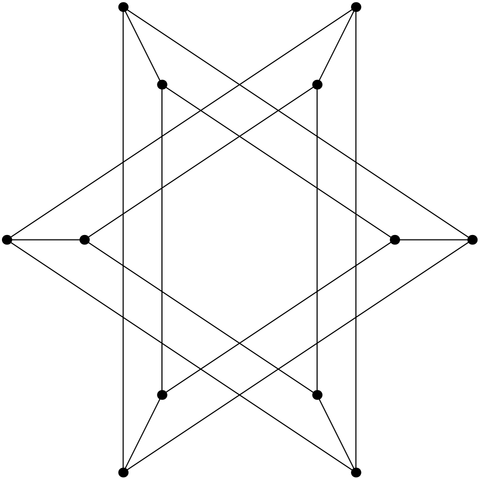
$$\begin{aligned} V(I(n,j,k)) &= \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\} \\ E(I(n,j,k)) &= \{u_i u_{i+j}, \quad u_i v_i, \quad v_i v_{i+k}; \ i = 0, \ldots, n-1\}, \end{aligned}$$

where the subscripts are to be read modulo n.

Some properties of I-graphs

- Not all connected! Let $\gcd(n, j, k) = d$ and let $n = n_0 d, j = j_0 d, k = k_0 d$. Then $I(n, j, k)$ consists of d isomorphic copies of $I(n_0, j_0, k_0)$.$I(n, j, k)$ is connected if and only if
$$\gcd(n, j, k) = 1.$$
- $I(n, j, k)$ is a GP-graph if and only if
$$\gcd(n, j) = 1 \quad \text{or} \quad \gcd(n, k) = 1.$$
- vertex- or edge-transitive only if GP-graphs
- I-graphs less popular than GP-graphs.
- Recently Žitnik, Horvat and Pisanski used I-graphs to prove that all GP-graphs are unit-distance graphs (JKMS 2012).

I-graph $I(6, 2, 2)$ is not connected



Standard form of an I-graph.

- $I(n, j, k) = I(n, k, j)$
- $I(n, j, k) = I(n, n - j, k)$
- Using these facts we may always assume that in $I(n, j, k)$ we have $1 \leq j \leq k < n/2$. In this case the I-graph is in a *standard form*.

Automorphisms of proper I-graphs

There (may) exist only three types of automorphisms:

- rotation ρ
- reflection τ
- automorphism φ that reflects a cycle on the inner rim and rotates or fixes cycles on the outer rim (or vice versa)

$$\varphi(u_{ij+pk}) = u_{-ij+pk}, \quad \varphi(v_{ij+pk}) = v_{-ij+pk}.$$

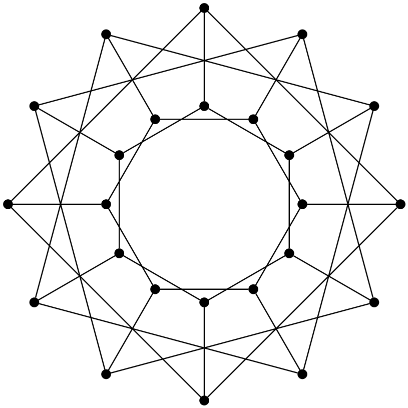
$$\psi(u_{ij+pk}) = u_{ij-pk}, \quad \psi(v_{ij+pk}) = v_{ij-pk}.$$

That happens only if

$$n = \gcd(n,j) \cdot \gcd(n,k) \text{ or } n = 2 \cdot \gcd(n,j) \cdot \gcd(n,k).$$

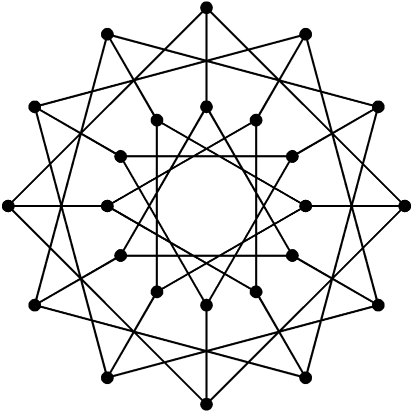
The automorphism group of $I(n,j,k)$ contains the dihedral group D_n, generated by ρ and τ.

I-graph $I(12, 2, 3)$



Here $12 = 2jk$, so each cycle of the inner rim is connected to each cycle of the outer rim with two spokes.

I-graph $I(12, 3, 4)$



Here $12 = jk$, so each cycle of the inner rim is connected to each cycle of the outer rim with one spoke.

Definition of *GI*-graphs

Let $n \geq 3$, $t \geq 1$ and $1 \leq j_k \leq n-1$, $j_k \neq n/2$ for $1 \leq k \leq t$.

A *GI-graph* $GI(n; j_1, j_2, \ldots, j_t)$ is a graph defined on the vertex set $\mathbb{Z}_t \times \mathbb{Z}_n$ with edges of two kinds:

a) **spoke edges** from (s, v) to (s', v) for all $s, s' \in \mathbb{Z}_t$, for every $v \in \mathbb{Z}_n$,
b) **layer edges** from (s, v) to $(s, v + j_s)$ and $(s, v - j_s)$ for all s and v.

The graph has nt vertices and is regular of valence $t + 1$.

Layers and spokes

For $s \in \mathbb{Z}_t$ the set

$$L_s = \{(s, v) : v \in \mathbb{Z}_n\}$$

is called a **layer** and for $v \in \mathbb{Z}_n$ the set

$$S_v = \{(s, v) : s \in \mathbb{Z}_t\}$$

is called a **spoke**.

We observe that the induced subgraph of $GI(n; j_1, j_2, \ldots, j_t)$ on every spoke is a complete graph K_t.

If $\gcd(n, j_s) = d$, the induced subgraph on the layer L_s is a union of d cycles of length n/d.

GI-graphs are

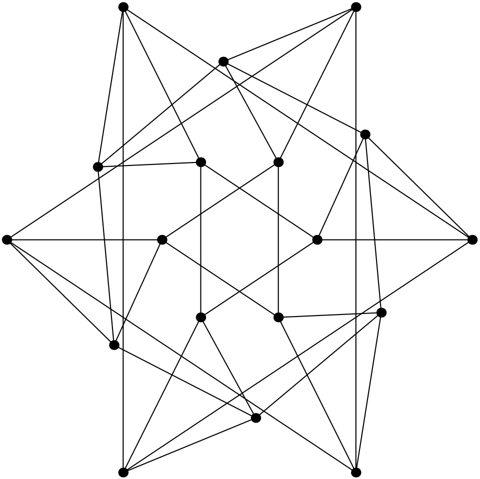
$t = 1$: unions of isomorphic cycles,

$t = 2$: *I*-graphs,
In particular, $GI(n; 1, j)$ is a generalized Petersen graph.

There is another generalization by Lovrečič Saražin, Pacco and Previtali, where spokes are not complete graphs but cycles. They call such graphs *generalized generalized Petersen graphs* or GGP-graphs.

$t \leq 3$: The classes of *GI*-graphs and GGP-graphs coincide.

Example: $GI(6; 1, 2, 2)$



Some properties of *GI*-graphs

For $t > 3$ the spoke edges are easy to recognize.

Proposition

Let $t > 3$. An edge of a GI-graph with t layers is a spoke-edge if and only if it belongs to some K_4.

Proposition

The graph $X = GI(n; j_1, j_2, \ldots, j_t)$ is connected if and only if

$$\gcd(n, j_1, j_2, \ldots, j_t) = 1.$$

If $\gcd(n, j_1, j_2, \ldots, j_t) = d > 1$, then X consists of d copies of $GI(n/d; j_1/d, j_2/d, \ldots, j_t/d)$.

Isomorphic GI-graphs and canoncal from.

- $GI(n; j'_1, j'_2, \ldots, j'_t)$ is the same graph as $GI(n; j_1, j_2, \ldots, j_t)$, if $j'_k \in \{j_k, -j_k\}$
- any permutation of $j_1, \ldots, j_t$ gives a GI-graph that is isomorphic to $GI(n; j_1, j_2, \ldots, j_t)$.
- Let $j_1, \ldots, j_t \notin \{0, n/2\}$ modulo n and $\gcd(n, a) = 1$. Then
$$GI(n; aj_1, \ldots aj_t) \approx GI(n; j_1, \ldots j_t).$$

Therefore we will usually assume that $j_k < n/2$ and $j_1 \leq j_2 \leq \cdots \leq j_t$.

The multi-set J is the canonical form if it is lexicographically first among all isomorphs.

Symmetry properties of *GI*-graphs

Theorem

A GI-graph is edge-transitive exactly in the following cases:

- *for* $t = 1$*,*
- *for* $t = 2$ *whenever each connected component is isomorphic to one of the 7 special generalized Petersen graphs,*
- *for* $t = 3$ *whenever each connected component is isomorphic to* $GI(3; 1, 1, 1)$*.*

In particular, there are no GI-graphs which would be edge-transitive and not arc-transitive.

Number of automorphisms of a *GI*-graph - disconnected case

We try to determine the number of automorphisms of $GI(n; J)$. Let $F(n; J)$ denote the number of automorphisms of $G = GI(n; J)$. Let $d = gcd(n, J)$. Then G is composed of d isomorphic copies of $H = GI(n, J/d)$ and

$$F(n, J) = d!F(n, J/d)$$

. This reduces the computation of F to connected GI-graphs.

Number of automorphisms of a *GI*-graph - arc-transitive case

$$F(4,1,1) = 24$$

$$F(5,1,2) = 120$$

$$F(8,1,3) = 96$$

$$F(10,1,2) = 120$$

$$F(10,1,3) = 240$$

$$F(12,1,5) = 144$$

$$F(24,1,5) = 288$$

$$F(3,1,1,1) = 72$$

Number of automorphisms of a *GI*-graph - simple *J*

Let $GI(n;J)$ be connected and not arc-transitive. Let J be a set (not a multi-set) in a standard form. Let

$$A = \{a \in Z_n^* | aJ = J\}$$

Then

$$F(n;J) = 2n|A|$$

Number of automorphisms of a *GI*-graph - multiset *J*

Let $GI(n; J)$ be connected and not arc-transitive. Let J be a multi-set in a standard form with multiplicities $m(j_i)$ and $d(j_i) = gcd(n, j_i)$. Let

$$A = \{a \in Z_n^* | aJ = J\}$$

Multiplicities must be matched by each a. Then

$$F(n; J) = 2n|A| \prod_i m(j_i)! m(j_i)^{d(j_i)-1}$$

Number of automorphisms of a *GGP*-graph?

Previous slides give an algorithm for computing $F(n; J)$ in general.

Maybe we can also compute the number of automorphisms for the related family of GGP graphs. There will be more vertex-transitive graphs, but essentially we may repeat the same line of arguments to compute the automorphisms. Instead of using full automorphims group we have to use the dihedral group.