# New variational principle for discrete integrable systems

Frank Nijhoff
University of Leeds
*Royal Society/Leverhulme Trust Senior Research Fellow (2011-12)*

(joint work with Sarah Lobb and Pavlos Xenitidis)

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# Discrete Integrable Systems

- Subject on the interface of pure & applied mathematics and mathematical physics.
- Discrete Objects: ordinary and partial difference equations, and corresponding algebraic and geometric structures.
- Research areas related to the subject:

- Discrete differential geometry (*Differenzengeometrie*);
- Integrable dynamics and dynamical maps (QRT maps in elliptic surfaces);
- Random matrix models;
- Discrete holomophic/conformal maps and discrete analytic function theory;
- Nevanlinna theory and the Vojta dictionary in Number Theory;
- Difference equations over finite fields and tropical geometry;
- Combinatorics: octahedral recurrences and cluster algebras;
- Discrete special functions ($q$- and elliptic hypergeometric series);
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# Outline

- Integrable partial difference equations – quadrilateral lattice equations
- Multidimensional consistency (cubic consistency) and integrability
- ABS list of quadrilateral lattice equations
- Lagrangian structures and closure relation
- Variational principle for (discrete) Lagrangian multi-forms

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# Notation

We consider partial difference equations on a regular lattice described by the following variables:

- two <u>independent</u> discrete variables $n_1, n_2$ ($n_i \in \mathbb{Z}$) corresponding to two lattice directions;
- lattice parameters $\alpha_1$, $\alpha_2$ ($\alpha_i \in \mathbb{C}$) associated with the grid width;
- a scalar dependent variable, i.e. function of the lattice $u(n_1, n_2)$ taking values in $\mathbb{C}$ (or $\mathbb{R}$).

Elementary lattice shifts are denoted as follows:

$$u_1 := u(n_1 + 1, n_2)\ , \quad u_2 := u(n_1, n_2 + 1)\ , \quad u_{12} := u(n_1 + 1, n_2 + 1)\ ,$$




We may also consider additional lattice directions, where the dependent variable $u$ depends on any number of discrete variables: $u(n_1, n_2, n_3, \cdots)$.

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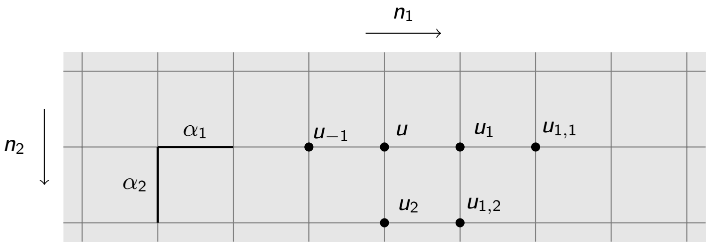


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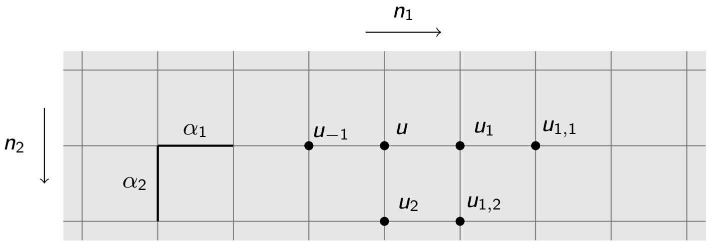


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# Quadrilateral lattice equations

We will be considering 2-dimensional lattice equations of the form

$$Q(u, u_1, u_2, u_{1,2}; \alpha_1, \alpha_2) = 0$$

where $Q$ is affine-linear in all four arguments. Multilinearity ensures we may solve the equation uniquely for any argument.




**Remark:** Such equations can also be considered on arbitrary quadgraphs, (and by duality can be mapped on a regular lattice) [Adler, 1996]. Well-posedness of initial-value problems can be considered both for the regular lattice [Papageorgiou, FWN, Capel, 1990] or for quadgraphs [Adler, Veselov, 2001].

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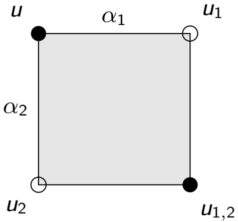


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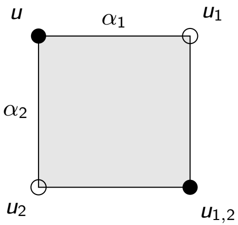


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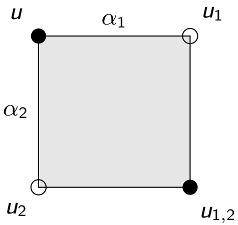


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# Consistency around a cube

Copies of the equation in each pair of lattice directions:

$$Q(u, u_1, u_2, u_{1,2}; \alpha_1, \alpha_2) = 0$$
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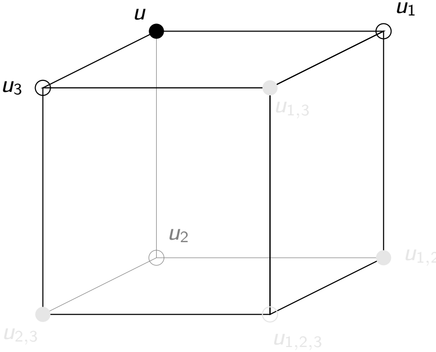

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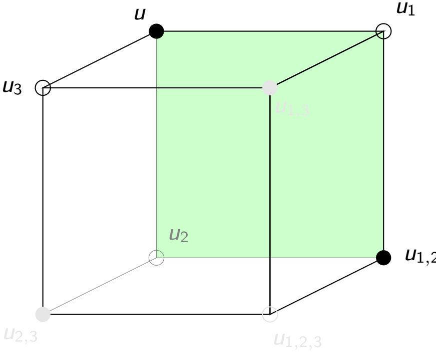

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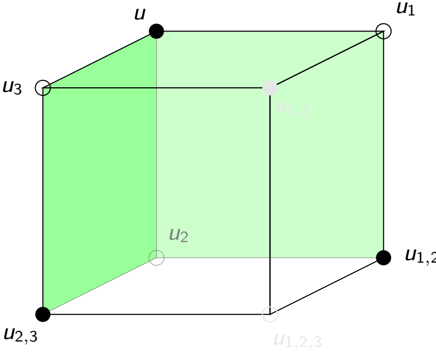

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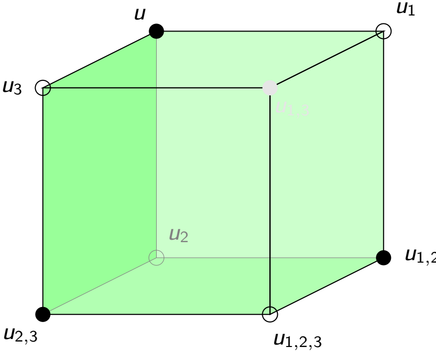

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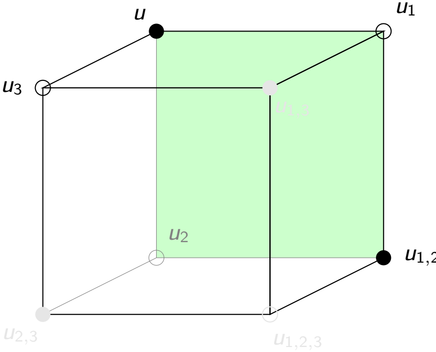

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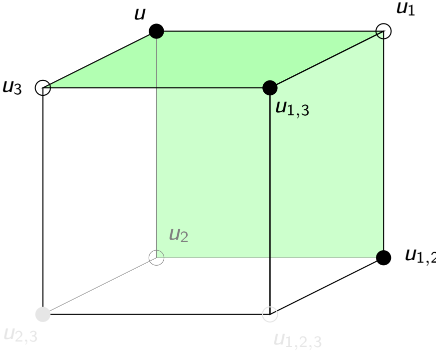

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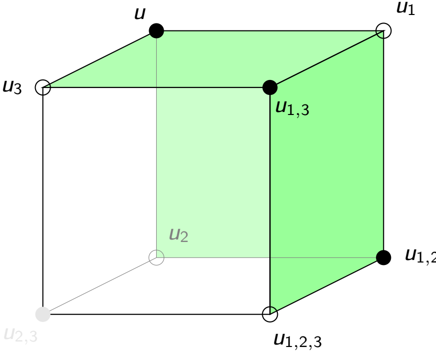

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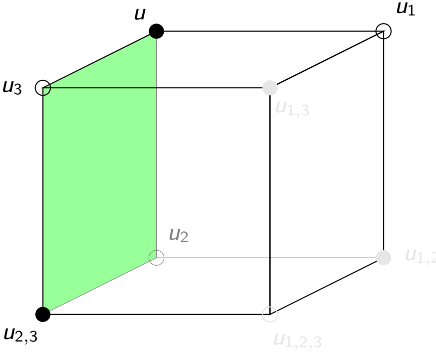

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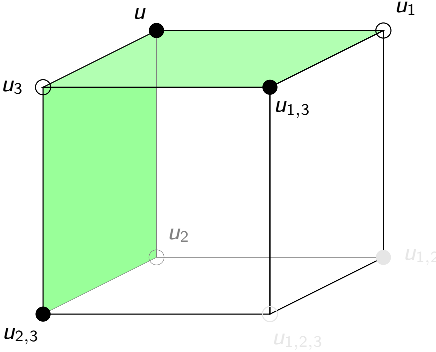

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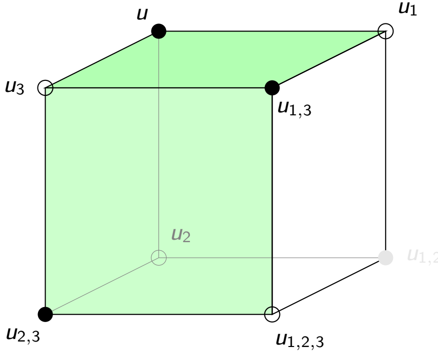


If these three values coincide, the equation is called consistent-around-the-cube.

# Example: lattice potential Korteweg-de Vries equation

The equation is

$$Q(u, u_1, u_2, u_{1,2}; \alpha_1, \alpha_2) = (u - u_{1,2})(u_1 - u_2) - \alpha_1 + \alpha_2 = 0$$

or equivalently

$$u_{1,2} = \frac{u(u_1 - u_2) - \alpha_1 + \alpha_2}{u_1 - u_2}$$

which when shifted in a 3rd lattice direction is

$$\begin{aligned} u_{1,2,3} &= \frac{u_3(u_{1,3} - u_{2,3}) - \alpha_1 + \alpha_2}{u_{1,3} - u_{2,3}} \\ &= -\frac{(\alpha_1 - \alpha_2)u_1 u_2 + (\alpha_2 - \alpha_3)u_2 u_3 + (\alpha_3 - \alpha_1)u_3 u_1}{(\alpha_1 - \alpha_2)u_3 + (\alpha_2 - \alpha_3)u_1 + (\alpha_3 - \alpha_1)u_2} \end{aligned}$$

Clearly the result is independent of the way in which the value at the outer vertex is calulated!
Note, furthermore, that the value $u_{1,2,3}$ no longer depends on the value $u$ at the "origin". This is called the *tetrahedron property*.

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The equation is

$$Q(u, u_1, u_2, u_{1,2}; \alpha_1, \alpha_2) = (u - u_{1,2})(u_1 - u_2) - \alpha_1 + \alpha_2 = 0$$

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$$u_{1,2} = \frac{u(u_1 - u_2) - \alpha_1 + \alpha_2}{u_1 - u_2}$$

which when shifted in a 3rd lattice direction is

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# ABS Classification

In 2002 Adler, Bobenko and Suris (ABS)[1] classified all scalar quadrilateral PΔEs of the form

$$Q(u, u_i, u_j, u_{i,j}; \alpha_i, \alpha_j) = 0 \ ,$$

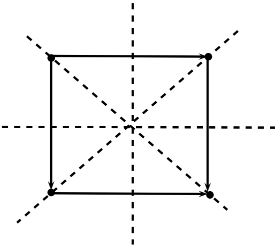
up to Möbius equivalence, exhibiting multidimensional consistency and subject to the additional conditions:

**a)** Affine-linearity: $Q$ is affine-linear in each argument, i.e., in each vertex variable $u$,$u_i$, $u_j$,$u_{i,j}$;

**b)** Tetrahedron property: in the cubic consistency, the evaluation of the point on the cube given by $u_{i,j,k}$ is *independent of* $u$.

**c)** Symmetry: $Q$ is invariant under the group $D_4$ of symmetries of the square, i.e.

$$\begin{aligned} &Q(u, u_i, u_j, u_{i,j}; \alpha_i, \alpha_j) = 0 \\ \Leftrightarrow\ &Q(u, u_j, u_i, u_{i,j}; \alpha_j, \alpha_i) = 0 \\ \Leftrightarrow\ &Q(u_i, u, u_{i,j}, u_j; \alpha_i, \alpha_j) = 0 \end{aligned}$$



[1] Adler, V.E., A.I. Bobenko and Yu.B. Suris. Classification of Integrable Equations on Quad-Graphs, the Consistency Approach. *Communications in Mathematical Physics*, 2003: **233**, pp.513-543.

# The ABS list of quadrilateral P$\Delta$Es

$$(u - u_{ij})(u_i - u_j) - \alpha_i + \alpha_j = 0 \qquad \text{(H1)}$$

$$(u - u_{ij})(u_i - u_j) + (\alpha_j - \alpha_i)(u + u_i + u_j + u_{ij}) - \alpha_i^2 + \alpha_j^2 = 0 \qquad \text{(H2)}$$

$$\mathrm{e}^{-\alpha_i/a}(uu_i + u_j u_{ij}) - \mathrm{e}^{-\alpha_j/2}(uu_j + u_i u_{ij}) + \delta(\mathrm{e}^{-\alpha_i} - \mathrm{e}^{-\alpha_j}) = 0 \qquad \text{(H3)}$$

$$\alpha_i(u - u_j)(u_i - u_{ij}) - \alpha_j(u - u_i)(u_j - u_{ij}) + \delta^2 \alpha_i \alpha_j(\alpha_i - \alpha_j) = 0 \qquad \text{(Q1)}$$

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$$\begin{aligned} &\mathrm{sn}(\alpha_i)(uu_i + u_j u_{ij}) - \mathrm{sn}(\alpha_j)(uu_j + u_i u_{ij}) - \mathrm{sn}(\alpha_i - \alpha_j)(uu_{ij} + u_i u_j) + \\ &k\, \mathrm{sn}(\alpha_i)\, \mathrm{sn}(\alpha_j)\, \mathrm{sn}(\alpha_i - \alpha_j)\, (1 + uu_i u_j u_{ij}) = 0 \end{aligned} \qquad \text{(Q4)}$$

In (Q4), $\mathrm{sn}$ is the Jacobi elliptic function $\mathrm{sn}(x|k)$ with modulus $k$.

Two further equations A1 and A2 in the list are related by point transformation to Q1 and $(\mathrm{Q3})_{\delta=0}$.

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## 3-leg forms & Lagrangians

All ABS equations can be written in either of two forms:

$$\underline{\text{additive :}} \qquad \psi(u,u_1;\alpha_1)-\psi(u,u_2;\alpha_2)=\phi(u,u_{1,2};\alpha_1,\alpha_2)$$

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for some functions $\psi$ and $\phi$.

$$H1: \qquad (u+u_1)-(u+u_2)=\frac{\alpha_1-\alpha_2}{u-u_{12}}$$

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where in the latter: $u=\sqrt{k}\,\operatorname{sn}(x)$ and $\alpha_i=\sqrt{k}\,\operatorname{sn}(a_i)$

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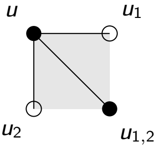


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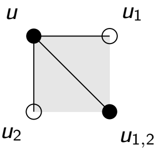


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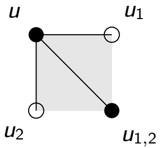


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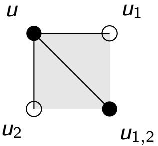


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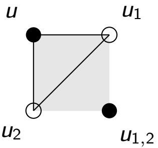

# Variational formalism: Discrete Euler-Lagrange equations

Starting from an action

$$S[u(n_1, n_2)] = \sum_{n_1, n_2 \in \mathbb{Z}} \mathscr{L}(u, u_1, u_2; \alpha_1, \alpha_2)$$

following the usual *least-action principle*, the lattice equations are those for which $S$ attains a minimum under local variations $\delta u$ of the dependent variable. Thus,

$$\begin{aligned}\delta S &= \sum_{n_1, n_2 \in \mathbb{Z}} \left\{ \frac{\partial}{\partial u} \mathscr{L}(u, u_1, u_2; \alpha_1, \alpha_2) \delta u + \frac{\partial}{\partial u_1} \mathscr{L}(u, u_1, u_2; \alpha_1, \alpha_2) \delta u_1 \right. \\ &\qquad \left. + \frac{\partial}{\partial u_2} \mathscr{L}(u, u_1, u_2; \alpha_1, \alpha_2) \delta u_2 \right\} = 0\end{aligned}$$

Resumming each of the terms we get:

$$\begin{aligned}0 &= \sum_{n_1, n_2 \in \mathbb{Z}} \left\{ \frac{\partial}{\partial u} \mathscr{L}(u, u_1, u_2; \alpha_1, \alpha_2) + \frac{\partial}{\partial u} \mathscr{L}(u_{-1}, u, u_{-1,2}; \alpha_1, \alpha_2) \right. \\ &\qquad \left. + \frac{\partial}{\partial u} \mathscr{L}(u_{-2}, u_{1,-2}, u; \alpha_1, \alpha_2) \right\} \delta u\end{aligned}$$

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$$\frac{\partial}{\partial u} \Big( \mathscr{L}(u, u_1, u_2; \alpha_1, \alpha_2) + \mathscr{L}(u_{-1}, u, u_{-1,2}; \alpha_1, \alpha_2) + \mathscr{L}(u_{-2}, u_{1,-2}, u; \alpha_1, \alpha_2) \Big) = 0$$

# Variational formalism: Discrete Euler-Lagrange equations

Starting from an action

$$S[u(n_1, n_2)] = \sum_{n_1, n_2 \in \mathbb{Z}} \mathscr{L}(u, u_1, u_2; \alpha_1, \alpha_2)$$

following the usual *least-action principle*, the lattice equations are those for which $S$ attains a minimum under local variations $\delta u$ of the dependent variable. Thus,

$$\begin{aligned} \delta S &= \sum_{n_1, n_2 \in \mathbb{Z}} \left\{ \frac{\partial}{\partial u} \mathscr{L}(u, u_1, u_2; \alpha_1, \alpha_2) \delta u + \frac{\partial}{\partial u_1} \mathscr{L}(u, u_1, u_2; \alpha_1, \alpha_2) \delta u_1 \right. \\ &\qquad \left. + \frac{\partial}{\partial u_2} \mathscr{L}(u, u_1, u_2; \alpha_1, \alpha_2) \delta u_2 \right\} = 0 \end{aligned}$$

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## Example: H1

The equation is

$$(u - u_{1,2})(u_1 - u_2) - \alpha_1 + \alpha_2 = 0$$

The equation in 3-leg form is

$$(u + u_1) - (u + u_2) = \frac{\alpha_1 - \alpha_2}{u - u_{1,2}}$$

The corresponding 3-point Lagrangian is given as[2]

$$\mathscr{L}(u, u_1, u_2; \alpha_1, \alpha_2) = (u_1 - u_2)u - (\alpha_1 - \alpha_2)\ln(u_1 - u_2)$$

The discrete Euler-Lagrange equations lead to a slightly weaker equation than H1 itself, but equivalent to a discrete derivative of the equation:

$$u_1 - u_{-2} - \frac{\alpha_1 - \alpha_2}{u - u_{1,-2}} + u_{-1} - u_2 - \frac{\alpha_1 - \alpha_2}{u - u_{-1,2}} = 0$$



[2] Capel, H.W., F.W. Nijhoff and V.G. Papageorgiou. Complete Integrability of Lagrangian Mappings and Lattices of KdV Type. *Physics Letters A*, 1991: **155**, pp.377–387.

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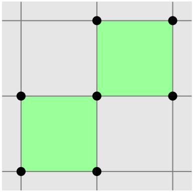

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# Closure relation and Lagrangian multiform structure

- MDC implies that on *the same dependent variable* $u = u(n_1, n_2, n_3, \ldots; \alpha_1, \alpha_2, \alpha_3, \ldots)$ one can simultaneously impose a multitude of compatible equations: P$\Delta$Es $Q_{i,j} = 0$ in any pair of independent variables $n_i, n_j$.
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- **problem:** Conventional least-action principle yields only one single equation (per component of the dependent variable $u$) as EL equation, giving the "dynamical" equation of the system;
- Thus, we need a new variational approach enabling the derivation of the multitude of equations of the MDC system.
- **Answer:** The Lagrangians will become extended objects, i.e. components of a *Lagrangian multiform.*

The existence and consistency of this approach resides on the following key observation:

**Proposition:** MDC systems of lattice equations, possess Lagrangians which obey the following closure relation:

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*on the solutions of the equations.*

Here $\Delta_i$ denotes the difference operator, i.e.. on functions $f$ of $u = u(n_1, n_2, n_3)$ we have: $\Delta_i f(u) = f(u_i) - f(u)$.

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**Example of H1**: From the explicit form of the Lagrangians we find

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&=(u_{1,2}-u_{1,3})u_1-(\alpha_2-\alpha_3)\ln(u_{1,2}-u_{1,3})-(u_2-u_3)u+(\alpha_2-\alpha_3)\ln(u_2-u_3)\\
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## Closure relation for other cases

- In a similar way the cases of H2, Q1 and A1 can be verified[3].
- The cases H1 and $(Q1)_{\delta=0}$ involve $F(u) = \ln u$, the cases H2 and $(Q1)_{\delta\neq 0}$ and A1 involve $F(u) = u \ln u$.
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$$\mathrm{Li}_2(z) = -\int_0^z z^{-1} \ln(1-z)\, dz$$

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## Universal Lagrangian

For quadrilateral *affine-linear* equations for scalar dependent variable $u = u(\mathbf{n})$

$$Q_{\mathfrak{p}_i,\mathfrak{p}_j}(u, u_i, u_j, u_{ij}) = 0 \quad , \quad u_i := u(\mathbf{n}+\mathbf{e}_i) \ , \quad u_{ij} := u(\mathbf{n}+\mathbf{e}_i+\mathbf{e}_j) \ ,$$

possessing the symmetries of the square, let us introduce the biquadratic functions:

$$\begin{aligned} Q_{u_j} Q_{u_{ij}} - Q\, Q_{u_j u_{ij}} &=: K_{\mathfrak{p},\mathfrak{q}} h_{\mathfrak{p}}(u, u_i) \\ Q_{u_i} Q_{u_{ij}} - Q\, Q_{u_i u_{ij}} &=: K_{\mathfrak{q},\mathfrak{p}} h_{\mathfrak{q}}(u, u_j) \\ Q_u Q_{u_{ij}} - Q\, Q_{u u_{ij}} &=: -K_{\mathfrak{p},\mathfrak{q}} h_{\mathfrak{r}}(u_i, u_j) \end{aligned}$$

where $K_{\mathfrak{p},\mathfrak{q}} = -K_{\mathfrak{q},\mathfrak{p}}$ is a function of the lattice parameters $\mathfrak{p}$, $\mathfrak{q}$ only. We now introduce the Lagrangian[5]

$$\begin{aligned} \mathscr{L}(u, u_i, u_j) \quad = \quad & \int_{u^0}^{u}\int_{u_i^0}^{u_i} \frac{dx\,dy}{h_{\mathfrak{p}}(x,y)} - \int_{u^0}^{u}\int_{u_j^0}^{u_j} \frac{dx\,dy}{h_{\mathfrak{q}}(x,y)} - \int_{u_i^0}^{u_i}\int_{u_j^0}^{u_j} \frac{dx\,dy}{h_{\mathfrak{r}}(x,y)} \\ & + \int_{u_i^0}^{u_i} dx \int_{u_j^0}^{Y(u^0,x,u_{ij}^0)} \frac{dy}{h_{\mathfrak{r}}(x,y)} + \int_{u_j^0}^{u_j} dy \int_{u_i^0}^{X(u^0,y,u_{ij}^0)} \frac{dx}{h_{\mathfrak{r}}(x,y)} \end{aligned}$$

where the functions $X$ and $Y$ are solutions of the equations

$$Q_{\mathfrak{p},\mathfrak{q}}(u^0, x, Y, u_{ij}^0) = 0 \quad \text{respectively} \quad Q_{\mathfrak{p},\mathfrak{q}}(u^0, X, y, u_{ij}^0) = 0 \ .$$

[5] P Xenitidis, F W Nijhoff and S Lobb, On the Lagrangian formulation of multidimensionally consistent systems, Proc. Roy. Soc. **A467** (2011) 3295.

# Q4 Equation

This equation, (originally due to V.Adler, 1998) reads:

$$\begin{aligned} Q_{\mathfrak{p}_i,\mathfrak{p}_j} &= p_i(u\,u_i + u_j u_{ij}) - p_j(u\,u_j + u_i u_{ij}) \\ &\quad - p_{ij}(u\,u_{ij} + u_i u_j) + p_i p_j p_{ij}(1 + u\,u_i u_j u_{ij}) \end{aligned}$$

where $p_i = \sqrt{k}\,\mathrm{sn}(\alpha_i;k)$ , $p_j = \sqrt{k}\,\mathrm{sn}(\alpha_j;k)$ , $p_{ij} = \sqrt{k}\,\mathrm{sn}(\alpha_{ij};k)$ with $\alpha_{ij} = \alpha_i - \alpha_j$.
For the biquadratics we have

$$h_{\mathfrak{p}}(x,y) = p(1 + x^2y^2) - \frac{1}{p}(x^2+y^2) + 2\frac{P}{p}xy \quad , \quad K_{\mathfrak{p}_i,\mathfrak{p}_j} = -p_i p_j p_{ij}$$

where $\mathfrak{p} = (p, P)$ are points on the elliptic curve given by $P^2 = p^4 - (k + 1/k)p^2 + 1$ and $k$ the modulus of the Jacobi elliptic function.
The double integral in the Lagrangian can be evaluated as:

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} \frac{dx\,dy}{h_{\mathfrak{p}}(x,y)} = -2\int_{\eta_0}^{\eta_1} d\eta\,\log\left(\frac{\mathrm{sn}(\xi_1) - \mathrm{sn}(\eta+\alpha)}{\mathrm{sn}(\xi_1) - \mathrm{sn}(\eta-\alpha)}\,\frac{\mathrm{sn}(\xi_0) - \mathrm{sn}(\eta-\alpha)}{\mathrm{sn}(\xi_0) - \mathrm{sn}(\eta+\alpha)}\right) ,$$

with $x_i = \sqrt{k}\,\mathrm{sn}(\xi_i;k)$ , $y_i = \sqrt{k}\,\mathrm{sn}(\eta_i;k)$ and $p = \sqrt{k}\,\mathrm{sn}(\alpha;k)$ .
the latter is an elliptic variant of the dilogarithm function.

# Defining the action for discrete Lagrangian 2-forms

We now need the notion of a discrete Lagrangian 2-form.
These are oriented expressions of the form:

$$\mathscr{L}_{ij}(\mathbf{n}) = \mathscr{L}(u(\mathbf{n}), u(\mathbf{n}+\mathbf{e}_i), u(\mathbf{n}+\mathbf{e}_j); \alpha_i, \alpha_j)$$

defined on elementary plaquettes, in a multidimensional lattice, characterized by the ordered triplet $\sigma_{ij}(\mathbf{n}) = (\mathbf{n}, \mathbf{n}+\mathbf{e}_i, \mathbf{n}+\mathbf{e}_j)$




Choose a surface $\sigma$ in the multidimensional lattice consisting of a connected configuration of elementary plaquettes $\sigma_{ij}(\mathbf{n})$



Define the action to be

$$S[u(\mathbf{n});\sigma] = \sum_{\sigma_{ij}(\mathbf{n})\in\sigma} \mathscr{L}_{ij}(\mathbf{n})$$

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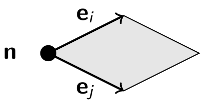


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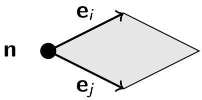


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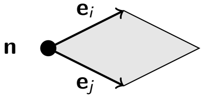


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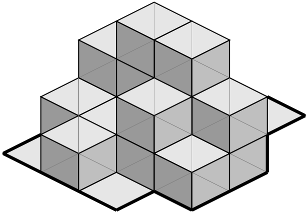

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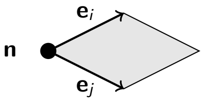


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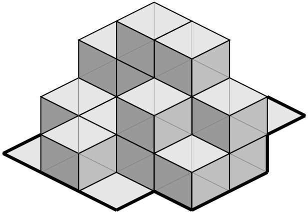

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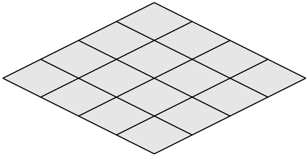
Independence of the action $S$ under local deformations of the surface is equivalent to the closure relation holding.



$$\begin{aligned} S' \quad &= \quad S - \mathscr{L}(u, u_i, u_j; \alpha_i, \alpha_j) + \mathscr{L}(u_k, u_{i,k}, u_{j,k}; \alpha_i, \alpha_j) + \mathscr{L}(u_i, u_{i,j}, u_{i,k}; \alpha_j, \alpha_k) \\ &\quad + \mathscr{L}(u_j, u_{j,k}, u_{i,j}; \alpha_k, \alpha_i) - \mathscr{L}(u, u_j, u_k; \alpha_j, \alpha_k) - \mathscr{L}(u, u_k, u_i; \alpha_k, \alpha_i) \end{aligned}$$

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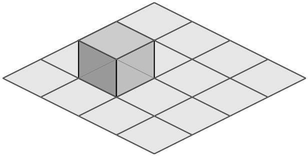

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- the main motivation is to formulate a least-action principle that produces the whole system of multidimensionally consistent equations, rather than a single equation of the motion.
- This new variational principle brings in an essential way *the geometry of the independent variables*!
- This is as much a principle which determines the *admissable Lagrangians* as well as that it selects the classical trajectories of a given system!
- Furthermore, there is a duality between (lattice) *parameters* and lattice *variables* each of which can play in turn the role of the independent variables of the system.
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