Abstract Polytopes: Regular, Semiregular and Chiral

Barry Monson (UNB)

(from projects with Egon Schulte,
Daniel Pellicer and Gordon Williams)

SODO – Queenstown, February, 2012

supported in part by the NSERC of Canada

What are abstract polytopes?

An **abstract** ***n*-polytope** $\mathcal{Q}$ is a poset having some of the key structural properties of the face lattice of a convex n-polytope, although $\mathcal{Q}$

- need not be a lattice
- need not be finite
- need not have a familiar geometric realization

The abstract 3-polytopes include all convex polyhedra, face-to-face tessellations and many less familiar structures. But

you can safely think of a finite 3-polytope as a *map on a compact surface*.

Do we want details?

What are abstract polytopes?

An **abstract** *n*-**polytope** $\mathcal{Q}$ is a poset having some of the key structural properties of the face lattice of a convex *n*-polytope, although $\mathcal{Q}$

- need not be a lattice
- need not be finite
- need not have a familiar geometric realization

The abstract 3-polytopes include all convex polyhedra, face-to-face tessellations and many less familiar structures. But

you can safely think of a finite 3-polytope as a *map on a compact surface*.

Do we want details?

UNB

What are abstract polytopes?

An **abstract** ***n*****-polytope** $\mathcal{Q}$ is a poset having some of the key structural properties of the face lattice of a convex *n*-polytope, although $\mathcal{Q}$

- need not be a lattice
- need not be finite
- need not have a familiar geometric realization

The abstract 3-polytopes include all convex polyhedra, face-to-face tessellations and many less familiar structures. But

you can safely think of a finite 3-polytope as a *map on a compact surface*.

Do we want details?

UNB

What are abstract polytopes?

An **abstract** n**-polytope** $\mathcal{Q}$ is a poset having some of the key structural properties of the face lattice of a convex n-polytope, although $\mathcal{Q}$

- need not be a lattice
- need not be finite
- need not have a familiar geometric realization

The abstract 3-polytopes include all convex polyhedra, face-to-face tessellations and many less familiar structures. But

you can safely think of a finite 3-polytope as a *map on a compact surface*.

Do we want details?

What are abstract polytopes?

An **abstract** n**-polytope** $\mathcal{Q}$ is a poset having some of the key structural properties of the face lattice of a convex n-polytope, although $\mathcal{Q}$

- need not be a lattice
- need not be finite
- need not have a familiar geometric realization

The abstract 3-polytopes include all convex polyhedra, face-to-face tessellations and many less familiar structures. But

you can safely think of a finite 3-polytope as a *map on a compact surface*.

Do we want details?

The symmetry of $\mathcal{Q}$

is encoded in the group $\Gamma = \Gamma(\mathcal{Q})$ of all order-preserving bijections (= automorphisms) of $\mathcal{Q}$.

Each automorphism is det'd by its action on any one *flag* Φ; for a polyhedron, a flag

$$\Phi = \text{incident [vertex, edge, facet] triple}$$

Def. $\mathcal{Q}$ is *regular* if Γ is transitive on flags.

Examples:

- any polygon ($n = 2$) is (abstractly, i.e. combinatorially) regular
- the usual tiling of $\mathbb{E}^3$ by unit cubes is an infinite regular 4-polytope
- the Platonic solids ($n = 3$). Look, for example, at $\Rightarrow$

UNB

The symmetry of $\mathcal{Q}$

is encoded in the group $\Gamma = \Gamma(\mathcal{Q})$ of all order-preserving bijections (= automorphisms) of $\mathcal{Q}$.

Each automorphism is det'd by its action on any one *flag* Φ; for a polyhedron, a flag

$$\Phi = \text{incident [vertex, edge, facet] triple}$$

Def. $\mathcal{Q}$ is *regular* if Γ is transitive on flags.

Examples:

- any polygon ($n = 2$) is (abstractly, i.e. combinatorially) regular
- the usual tiling of $\mathbb{E}^3$ by unit cubes is an infinite regular 4-polytope
- the Platonic solids ($n = 3$). Look, for example, at $\Rightarrow$

The symmetry of $\mathcal{Q}$

is encoded in the group $\Gamma = \Gamma(\mathcal{Q})$ of all order-preserving bijections (= automorphisms) of $\mathcal{Q}$.

Each automorphism is det'd by its action on any one *flag* Φ; for a polyhedron, a flag

$$\Phi = \text{incident [vertex, edge, facet] triple}$$

Def. $\mathcal{Q}$ is *regular* if Γ is transitive on flags.

Examples:

- any polygon ($n = 2$) is (abstractly, i.e. combinatorially) regular
- the usual tiling of $\mathbb{E}^3$ by unit cubes is an infinite regular 4-polytope
- the Platonic solids ($n = 3$). Look, for example, at $\Rightarrow$

The symmetry of $\mathcal{Q}$

is encoded in the group $\Gamma = \Gamma(\mathcal{Q})$ of all order-preserving bijections (= automorphisms) of $\mathcal{Q}$.

Each automorphism is det'd by its action on any one *flag* Φ; for a polyhedron, a flag

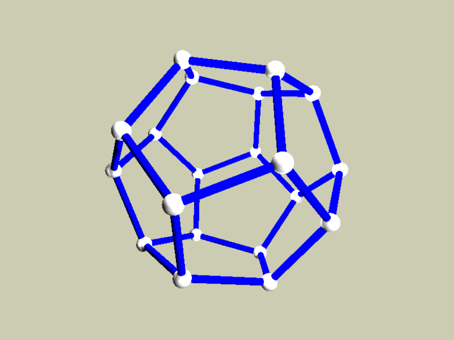
$$\Phi = \text{incident [vertex, edge, facet] triple}$$

Def. $\mathcal{Q}$ is *regular* if Γ is transitive on flags.

Examples:

- any polygon ($n = 2$) is (abstractly, i.e. combinatorially) regular
- the usual tiling of $\mathbb{E}^3$ by unit cubes is an infinite regular 4-polytope
- the Platonic solids ($n = 3$). Look, for example, at $\Rightarrow$

Example 1. The regular dodecahedron $\mathcal{D}$ (facets removed)



Here $\Gamma(\mathcal{D})$ is the Coxeter group

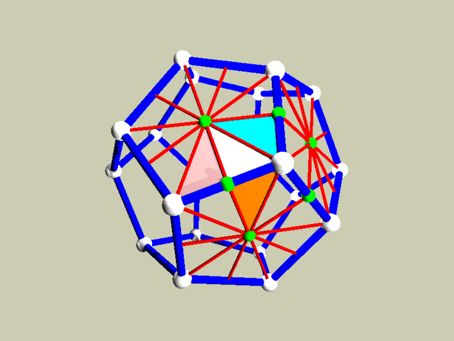
$$H_3 = \bullet \overset{5}{\text{———}} \bullet \overset{3}{\text{———}} \bullet$$

of order 120.

The flags correspond exactly to the triangles in a barycentric subdivision of the surface of $\mathcal{D}$.

Here is part of that $\Rightarrow$

A base flag for $\mathcal{D}$, adjacent flags and generators



By transitivity, pick any base flag $= \Phi$ [white]
Then
0-adjacent flag $=: \Phi^0$ [pink]
1-adjacent flag $=: \Phi^1$ [cyan]
2-adjacent flag $=: \Phi^2$ [orange]
For $i = 0, 1, 2$, there is a unique automorphism

$$\rho_i : \Phi \mapsto \Phi^i \ .$$

Then $\Gamma(\mathcal{D}) = \langle \rho_0, \rho_1, \rho_2 \rangle$.
Think reflections!

The convex regular polyhedra (=Platonic solids) and the Kepler-Poinsot star-polyhedra $\mathcal{P}$

Local data for both polyhedron $\mathcal{P}$ and its group $\Gamma(\mathcal{P})$ reside in the Schläfli symbol or type $\{p, q\}$.

Platonic solids: $\{3,3\}$ (tetrahedron), $\{3,4\}$ (octahedron), $\{4,3\}$ (cube), $\{3,5\}$ (icosahedron),$\{5,3\}$ (dodecahedron)

Kepler (ca. 1619) $\{\frac{5}{2},5\}$ (small stellated dodecahedron), $\{\frac{5}{2},3\}$ (great stellated dodecahedron)

Poinsot (ca. 1809) $\{5,\frac{5}{2}\}$ (great dodecahedron), $\{3,\frac{5}{2}\}$ (great isosahedron)

Want to see some?

UNB

The classical convex regular polytopes, their Schläfli symbols and finite Coxeter groups with string diagrams

name	symbol	# facets	(Coxeter) group	order
$n = 4$:				
simplex	$\{3, 3, 3\}$	5	$A_4 \simeq S_5$	$5!$
cross-polytope	$\{3, 3, 4\}$	16	B_4	384
cube	$\{4, 3, 3\}$	8	B_4	384
24-cell	$\{3, 4, 3\}$	24	F_4	1152
600-cell	$\{3, 3, 5\}$	600	H_4	14400
120-cell	$\{5, 3, 3\}$	120	H_4	14400
$n > 4$:				
simplex	$\{3, 3, \ldots, 3\}$	$n + 1$	$A_n \simeq S_{n+1}$	$(n + 1)!$
cross-polytope	$\{3, \ldots, 3, 4\}$	2^n	B_n	$2^n \cdot n!$
cube	$\{4, 3, \ldots, 3\}$	$2n$	B_n	$2^n \cdot n!$

Regular polytopes and string C-groups

Schulte (1982) showed that the abstract regular n-polytopes $\mathcal{P}$ correspond exactly to the *string C-groups of rank n* (which we often study in their place).

The Correspondence Theorem.

Part 1. If $\mathcal{P}$ is a regular n-polytope, then $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a *string C-group*.

Part 2. Conversely, if $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a string C-group, then we can reconstruct an n-polytope $\mathcal{P}(\Gamma)$ (in a natural way as a coset geometry on Γ).

Furthermore, $\Gamma(\mathcal{P}(\Gamma)) \simeq \Gamma$ and $\mathcal{P}(\Gamma(\mathcal{P})) \simeq \mathcal{P}$.

UNB

Recap: what is a string C-group?

Means: having fixed a base flag Φ in $\mathcal{P}$, for $0 \leq j \leq n-1$ there is a unique automorphism $\rho_j \in \Gamma(\mathcal{P})$ mapping Φ to the j-adjacent flag Φ^j. These involutions generate $\Gamma(\mathcal{P})$ and satisfy the relations implicit in some string (Coxeter) diagram, like

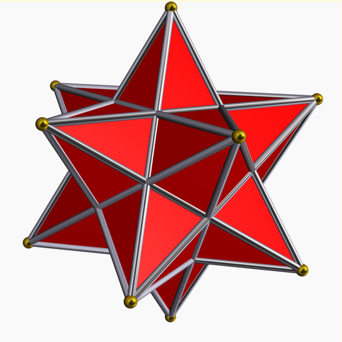
$$\bullet \overset{p_1}{\rule[0.5ex]{2em}{0.4pt}} \bullet \overset{p_2}{\rule[0.5ex]{2em}{0.4pt}} \bullet \rule[0.5ex]{1em}{0.4pt} \ldots \rule[0.5ex]{1em}{0.4pt} \bullet \overset{p_{n-1}}{\rule[0.5ex]{2em}{0.4pt}} \bullet\ ,$$

and perhaps other relations, so long as this *intersection condition* continues to hold:

$$\langle \rho_k : k \in I \rangle \cap \langle \rho_k : k \in J \rangle = \langle \rho_k : k \in I \cap J \rangle$$

(for all $I, J \subseteq \{0, \ldots, n-1\}$).
Notice that $\mathcal{P}$ then has Schläfli type $\{p_1, \ldots, p_{n-1}\}$.

Example 2. A modern look at a a classical object



(from Wikipedia)

The small stellated dodecahedron $\{\frac{5}{2}, 5\}$ is a Euclidean realization of the map $\mathcal{M} = \{5, 5 \,|\, 3\}$. This quotient of the infinite tessellation $\{5, 5\}$ of $\mathbb{H}^2$ is determined by specifying that '1st holes' be triangular, i.e. $\Gamma(\mathcal{M}) = \langle \rho_0, \rho_1, \rho_2 \rangle$, where

$$\rho_0^2 = \rho_0^2 = \rho_0^2 = (\rho_0\rho_1)^5 = (\rho_1\rho_2)^5$$
$$= (\rho_0\rho_2)^2 = (\rho_0\rho_1\rho_2\rho_1)^3 = 1.$$

(The 'extra' hole relation is red. You can see the 'hole', which is really an anti-hole.)

In contrast, the great icosahedron $\{3, \frac{5}{2}\}$ is isomorphic to the convex regular icosahedron $\{3, 5\}$.

Abstract Semiregular Polytopes

There are already too many regular polytopes, so let's relax the constraints and get even more!

Abstract Semiregular Polytopes

There are already too many regular polytopes, so let's relax the constraints and get even more!

Definition. A polytope $\mathcal{Q}$ is said to be *semiregular* if its facets are regular and $\Gamma(\mathcal{Q})$ is transitive on vertices.

Abstract Semiregular Polytopes

There are already too many regular polytopes, so let's relax the constraints and get even more!

Definition. A polytope $\mathcal{Q}$ is said to be *semiregular* if its facets are regular and $\Gamma(\mathcal{Q})$ is transitive on vertices.

- the classical Archimedean solids, prisms, antiprisms

Abstract Semiregular Polytopes

There are already too many regular polytopes, so let's relax the constraints and get even more!

Definition. A polytope $\mathcal{Q}$ is said to be *semiregular* if its facets are regular and $\Gamma(\mathcal{Q})$ is transitive on vertices.

- the classical Archimedean solids, prisms, antiprisms
- any regular polytope

Abstract Semiregular Polytopes

There are already too many regular polytopes, so let's relax the constraints and get even more!

<u>Definition</u>. A polytope $\mathcal{Q}$ is said to be *semiregular* if its facets are regular and $\Gamma(\mathcal{Q})$ is transitive on vertices.

- the classical Archimedean solids, prisms, antiprisms
- any regular polytope

So this is a broad generalization of 'regular'. One can generalize even more: uniform polytopes inductively have uniform facets, again with $\Gamma(\mathcal{Q})$ vertex-transitive, taking polygons to be uniform to start.

Abstract Semiregular Polytopes

There are already too many regular polytopes, so let's relax the constraints and get even more!

Definition. A polytope $\mathcal{Q}$ is said to be *semiregular* if its facets are regular and $\Gamma(\mathcal{Q})$ is transitive on vertices.

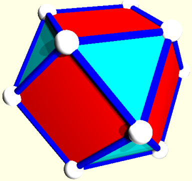
- the classical Archimedean solids, prisms, antiprisms
- any regular polytope

So this is a broad generalization of 'regular'. One can generalize even more: uniform polytopes inductively have uniform facets, again with $\Gamma(\mathcal{Q})$ vertex-transitive, taking polygons to be uniform to start.

Instead let's be at least a little more restrictive ...

Alternating Semiregular Polytopes

Definition An abstract semiregular n-polytope $\mathcal{S}$ is alternating if it has two (necessarily compatible) types of regular facets, say $\mathcal{P}$ and $\mathcal{Q}$, appearing in alternating fashion around each $(n-3)$-face.



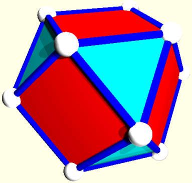
The cuboctahedron is an alternating semiregular 3-polytope. Here $k = 2$ each of triangles $\mathcal{P} = \{3\}$ and squares $\mathcal{Q} = \{4\}$ alternate around each 0-face = vertex. Each rectangular vertex-figure is a 'geometrically alternating' polygon, but is abstractly regular of course.

A truncated tetrahedron, for example, is semiregular but not alternating.

Example 4.

Alternating Semiregular Polytopes

Definition An abstract semiregular n-polytope $\mathcal{S}$ is alternating if it has two (necessarily compatible) types of regular facets, say $\mathcal{P}$ and $\mathcal{Q}$, appearing in alternating fashion around each $(n-3)$-face.



The cuboctahedron is an alternating semiregular 3-polytope. Here $k = 2$ each of triangles $\mathcal{P} = \{3\}$ and squares $\mathcal{Q} = \{4\}$ alternate around each 0-face = vertex.

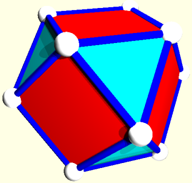
Each rectangular vertex-figure is a 'geometrically alternating' polygon, but is abstractly regular of course.

A truncated tetrahedron, for example, is semiregular but not alternating.

Example 4.

Alternating Semiregular Polytopes

Definition An abstract semiregular n-polytope $\mathcal{S}$ is alternating if it has two (necessarily compatible) types of regular facets, say $\mathcal{P}$ and $\mathcal{Q}$, appearing in alternating fashion around each $(n-3)$-face.

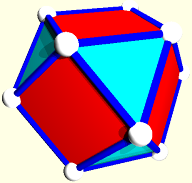


The cuboctahedron is an alternating semiregular 3-polytope. Here $k = 2$ each of triangles $\mathcal{P} = \{3\}$ and squares $\mathcal{Q} = \{4\}$ alternate around each 0-face = vertex. Each rectangular vertex-figure is a 'geometrically alternating' polygon, but is abstractly regular of course.

A truncated tetrahedron, for example, is semiregular but not alternating.

Example 4.

Definition An abstract semiregular n-polytope $\mathcal{S}$ is alternating if it has two (necessarily compatible) types of regular facets, say $\mathcal{P}$ and $\mathcal{Q}$, appearing in alternating fashion around each $(n-3)$-face.



Example 4.

The cuboctahedron is an alternating semiregular 3-polytope. Here $k = 2$ each of triangles $\mathcal{P} = \{3\}$ and squares $\mathcal{Q} = \{4\}$ alternate around each 0-face = vertex.

Each rectangular vertex-figure is a 'geometrically alternating' polygon, but is abstractly regular of course.

A truncated tetrahedron, for example, is semiregular but not alternating.

Thing 1 and Thing 2 – build an alternating polytope

You are given – unlimited copies of regular polyhedra $\mathcal{P}$ and $\mathcal{Q}$ having matching facets

Your task – Start with a single $\mathcal{P}$. Attach a copy of $\mathcal{Q}$ to each $\mathcal{P}$-facet, then a copy of $\mathcal{P}$ to each remaining 'exposed' facet of a $\mathcal{Q}$, and so on in alternating fashion with the

Edge Rule k – close up around each edge after k $\mathcal{P}$'s and k $\mathcal{Q}$'s.

- Can this be done?
- Is the resulting 'complex' $\mathcal{S}$ a 4–polytope?
- If so, what is the symmetry group $\mathrm{Aut}(\mathcal{S})$?

Thing 1 and Thing 2 – build an alternating polytope

You are given – unlimited copies of regular polyhedra $\mathcal{P}$ and $\mathcal{Q}$ having matching facets

Your task – Start with a single $\mathcal{P}$. Attach a copy of $\mathcal{Q}$ to each $\mathcal{P}$-facet, then a copy of $\mathcal{P}$ to each remaining 'exposed' facet of a $\mathcal{Q}$, and so on in alternating fashion with the

Edge Rule k – close up around each edge after k $\mathcal{P}$'s and k $\mathcal{Q}$'s.

- Can this be done?
- Is the resulting 'complex' $\mathcal{S}$ a 4–polytope?
- If so, what is the symmetry group $\mathrm{Aut}(\mathcal{S})$?

Thing 1 and Thing 2 – build an alternating polytope

You are given – unlimited copies of regular polyhedra $\mathcal{P}$ and $\mathcal{Q}$ having matching facets

Your task – Start with a single $\mathcal{P}$. Attach a copy of $\mathcal{Q}$ to each $\mathcal{P}$-facet, then a copy of $\mathcal{P}$ to each remaining 'exposed' facet of a $\mathcal{Q}$, and so on in alternating fashion with the

Edge Rule k – close up around each edge after k $\mathcal{P}$'s and k $\mathcal{Q}$'s.

- Can this be done?
- Is the resulting 'complex' $\mathcal{S}$ a 4–polytope?
- If so, what is the symmetry group $\mathrm{Aut}(\mathcal{S})$?

Thing 1 and Thing 2 – build an alternating polytope

You are given – unlimited copies of regular polyhedra $\mathcal{P}$ and $\mathcal{Q}$ having matching facets

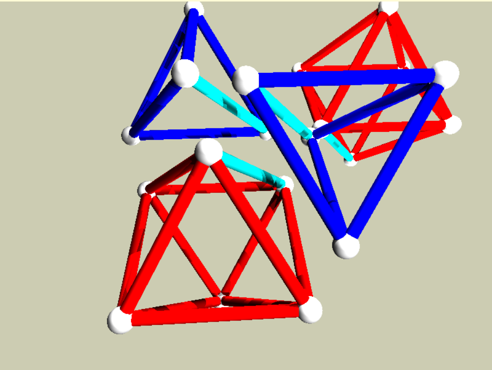
Your task – Start with a single $\mathcal{P}$. Attach a copy of $\mathcal{Q}$ to each $\mathcal{P}$-facet, then a copy of $\mathcal{P}$ to each remaining 'exposed' facet of a $\mathcal{Q}$, and so on in alternating fashion with the

Edge Rule k – close up around each edge after k $\mathcal{P}$'s and k $\mathcal{Q}$'s.

- Can this be done?
- Is the resulting 'complex' $\mathcal{S}$ a 4–polytope?
- If so, what is the symmetry group $\mathrm{Aut}(\mathcal{S})$?

Example 5 (rich). Take

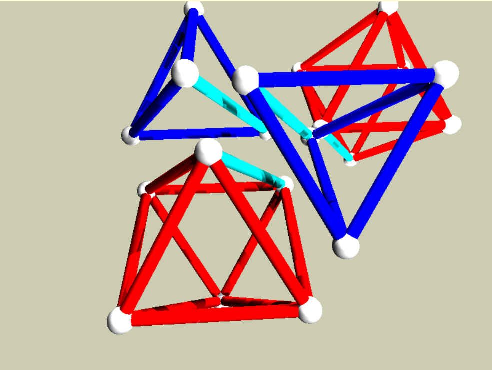
the regular octahedron $\mathcal{P} = \{3,4\}$ and
the regular tetrahedron $\mathcal{Q} = \{3,3\}$



$\Leftarrow$ Assemble $k = 2$ of each around each edge.
We get a familiar tessellation $\mathcal{S}$ of $\mathbb{E}^3$.
This *abstract semiregular 4-polytope* therefore has a Euclidean realization.

Example 5 (rich). Take

the regular octahedron $\mathcal{P} = \{3, 4\}$ and
the regular tetrahedron $\mathcal{Q} = \{3, 3\}$



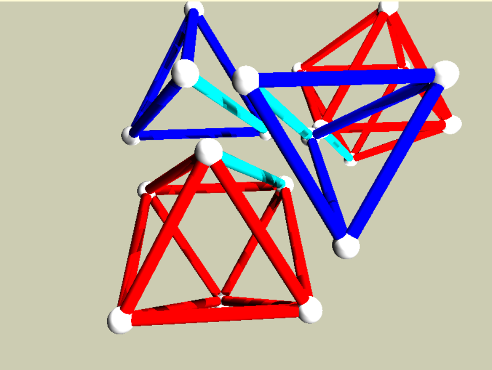
$\Leftarrow$ Assemble $k = 2$ of each around each edge.

We get a familiar tessellation $\mathcal{S}$ of $\mathbb{E}^3$.

This *abstract semiregular 4-polytope* therefore has a Euclidean realization.

Example 5 (rich). Take

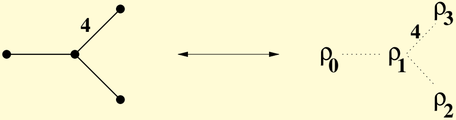
the regular octahedron $\mathcal{P} = \{3,4\}$ and
the regular tetrahedron $\mathcal{Q} = \{3,3\}$



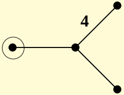
$\Leftarrow$ Assemble $k = 2$ of each around each edge.
We get a familiar tessellation $\mathcal{S}$ of $\mathbb{E}^3$.
This *abstract semiregular 4-polytope* therefore has a Euclidean realization.

Building $\mathcal{S}$ from Wythoff's Construction

The (infinite) Coxeter group $\Gamma = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$ of type $\tilde{B}_3$ has diagram



and acts discretely on Euclidean space $\mathbb{E}^3$. We get $\mathcal{S}$ from Wythoff's construction, as encoded in the modified diagram



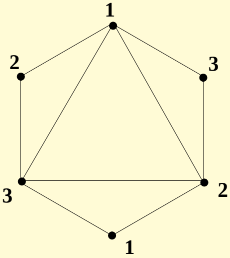
Begin with vertex set = Γ-orbit of the unique point fixed by ρ_1, ρ_2, ρ_3; etc.

see it

UNB

Change an ingredient . . .

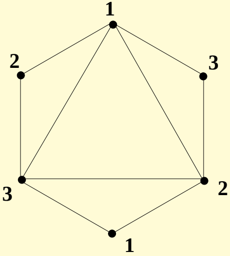
Keep the regular tetrahedron $\mathcal{Q} = \{3, 3\}$ but switch to the regular hemioctahedron $\mathcal{P} = \{3, 4\}_3$:



This projective map $\mathcal{P}$ has 3 vertices, 6 edges and 4 triangular facets.

We still try to put $k = 2$ of each around each edge. But now our construction is best done using an

Keep the regular tetrahedron $\mathcal{Q} = \{3, 3\}$ but switch to the regular hemioctahedron $\mathcal{P} = \{3, 4\}_3$:



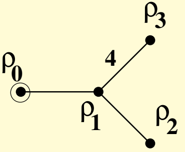
This projective map $\mathcal{P}$ has 3 vertices, 6 edges and 4 triangular facets.

We still try to put $k = 2$ of each around each edge. But now our construction is best done using an

Abstract Version of Wythoff's Construction (with E. Schulte) in which

the vertices, edges, 2-faces (= polygons) and 3-faces (=facets) of the new $\mathcal{S}$ are (identified with) right cosets of certain standard subgroups of the <u>new</u> group Γ generated by $\rho_0, \rho_1, \rho_2, \rho_3$ and having defining relations $\Downarrow$

Γ :



$$\rho_0^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = 1$$

$$(\rho_0\rho_1)^3 = (\rho_1\rho_2)^3 = (\rho_1\rho_3)^4 = 1$$

$$(\rho_0\rho_2)^2 = (\rho_0\rho_3)^2 = (\rho_2\rho_3)^2 = 1$$

<u>and</u> the new projectifying relation

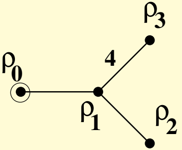
$$(\rho_0\rho_1\rho_3)^3 = 1\ .$$

Remark. It's not clear that $\mathcal{S}$ 'survives intact', since the new relation could destroy polytopality.

Abstract Version of Wythoff's Construction (with E. Schulte) in which

the vertices, edges, 2-faces (= polygons) and 3-faces (=facets) of the new $\mathcal{S}$ are (identified with) right cosets of certain standard subgroups of the <u>new</u> group Γ generated by $\rho_0, \rho_1, \rho_2, \rho_3$ and having defining relations $\Downarrow$

Γ :



$$\rho_0^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = 1$$

$$(\rho_0\rho_1)^3 = (\rho_1\rho_2)^3 = (\rho_1\rho_3)^4 = 1$$

$$(\rho_0\rho_2)^2 = (\rho_0\rho_3)^2 = (\rho_2\rho_3)^2 = 1$$

<u>and</u> the new projectifying relation

$$(\rho_0\rho_1\rho_3)^3 = 1 \; .$$

Remark. It's not clear that $\mathcal{S}$ 'survives intact', since the new relation could destroy polytopality.

But all is well - and we get a finite 4-polytope $\mathcal{S}$! (Example 6)

It's true, but not obvious, that Γ is now finite. In fact,

- the ρ_j's survive as involutions
- still $\langle \rho_0, \rho_1, \rho_2 \rangle \simeq \Gamma(\mathcal{Q})$ ($\simeq \mathbb{S}_4$) [group for tetrahedra]
- still $\langle \rho_0, \rho_1, \rho_3 \rangle \simeq \Gamma(\mathcal{P})$ (also $\simeq \mathbb{S}_4$)[group for hemioctahedra]
- $|\Gamma| = 192$, so there are $8 = 192/24$ facets of each type; two tetrahedra and two hemioctahedra occur alternately around each edge.
- $\mathcal{S}$ can't be regular since $\mathcal{P} \not\simeq \mathcal{Q}$. But it is alternating semiregular – all facets are regular and $\Gamma(\mathcal{S})$ is vertex-transitive.

But all is well - and we get a finite 4-polytope $\mathcal{S}$! (Example 6)

It's true, but not obvious, that Γ is now finite. In fact,

- the ρ_j's survive as involutions
- still $\langle \rho_0, \rho_1, \rho_2 \rangle \simeq \Gamma(\mathcal{Q})$ ($\simeq \mathbb{S}_4$) [group for tetrahedra]
- still $\langle \rho_0, \rho_1, \rho_3 \rangle \simeq \Gamma(\mathcal{P})$ (also $\simeq \mathbb{S}_4$)[group for hemioctahedra]
- $|\Gamma| = 192$, so there are $8 = 192/24$ facets of each type; two tetrahedra and two hemioctahedra occur alternately around each edge.
- $\mathcal{S}$ can't be regular since $\mathcal{P} \not\simeq \mathcal{Q}$. But it is alternating semiregular – all facets are regular and $\Gamma(\mathcal{S})$ is vertex-transitive.

But all is well - and we get a finite 4-polytope $\mathcal{S}$! (Example 6)

It's true, but not obvious, that Γ is now finite. In fact,

- the ρ_j's survive as involutions
- still $\langle \rho_0, \rho_1, \rho_2 \rangle \simeq \Gamma(\mathcal{Q})$ ($\simeq \mathbb{S}_4$) [group for tetrahedra]
- still $\langle \rho_0, \rho_1, \rho_3 \rangle \simeq \Gamma(\mathcal{P})$ (also $\simeq \mathbb{S}_4$)[group for hemioctahedra]
- $|\Gamma| = 192$, so there are $8 = 192/24$ facets of each type; two tetrahedra and two hemioctahedra occur alternately around each edge.
- $\mathcal{S}$ can't be regular since $\mathcal{P} \not\simeq \mathcal{Q}$. But it is alternating semiregular – all facets are regular and $\Gamma(\mathcal{S})$ is vertex-transitive.

But all is well - and we get a finite 4-polytope $\mathcal{S}$! (Example 6)

It's true, but not obvious, that Γ is now finite. In fact,

- the ρ_j's survive as involutions
- still $\langle \rho_0, \rho_1, \rho_2 \rangle \simeq \Gamma(\mathcal{Q})$ ($\simeq \mathbb{S}_4$) [group for tetrahedra]
- still $\langle \rho_0, \rho_1, \rho_3 \rangle \simeq \Gamma(\mathcal{P})$ (also $\simeq \mathbb{S}_4$)[group for hemioctahedra]
- $|\Gamma| = 192$, so there are $8 = 192/24$ facets of each type; two tetrahedra and two hemioctahedra occur alternately around each edge.
- $\mathcal{S}$ can't be regular since $\mathcal{P} \not\simeq \mathcal{Q}$. But it is alternating semiregular – all facets are regular and $\Gamma(\mathcal{S})$ is vertex-transitive.

UNB

But all is well - and we get a finite 4-polytope $\mathcal{S}$! (Example 6)

It's true, but not obvious, that Γ is now finite. In fact,

- the ρ_j's survive as involutions
- still $\langle \rho_0, \rho_1, \rho_2 \rangle \simeq \Gamma(\mathcal{Q})$ ($\simeq \mathbb{S}_4$) [group for tetrahedra]
- still $\langle \rho_0, \rho_1, \rho_3 \rangle \simeq \Gamma(\mathcal{P})$ (also $\simeq \mathbb{S}_4$)[group for hemioctahedra]
- $|\Gamma| = 192$, so there are $8 = 192/24$ facets of each type; two tetrahedra and two hemioctahedra occur alternately around each edge.
- $\mathcal{S}$ can't be regular since $\mathcal{P} \not\simeq \mathcal{Q}$. But it is alternating semiregular – all facets are regular and $\Gamma(\mathcal{S})$ is vertex-transitive.

More on this new $\mathcal{S}$

- The polytope $\mathcal{S}$ is universal for assembling tetrahedra and hemioctahedra face-to-face, with two each alternately surrounding any edge.
- $\mathcal{S}$ has a unique, minimal regular cover of Schläfli type $\{3, 12, 4\}$ and group order $2^{13} \cdot 3^2 = 73728$.
- But further collapse is possible. Each vertex-figure is a centrally-symmetric cuboctahedron. So let's collapse these to semiregular hemicuboctahedra by adjoining the relation

$$(\rho_1\rho_2\rho_3)^3 = 1$$

Polytopality survives this further collapse and we get the

More on this new $\mathcal{S}$

- The polytope $\mathcal{S}$ is universal for assembling tetrahedra and hemioctahedra face-to-face, with two each alternately surrounding any edge.
- $\mathcal{S}$ has a unique, minimal regular cover of Schläfli type $\{3, 12, 4\}$ and group order $2^{13} \cdot 3^2 = 73728$.
- But further collapse is possible. Each vertex-figure is a centrally-symmetric cuboctahedron. So let's collapse these to semiregular hemicuboctahedra by adjoining the relation

$$(\rho_1\rho_2\rho_3)^3 = 1$$

Polytopality survives this further collapse and we get the

More on this new $\mathcal{S}$

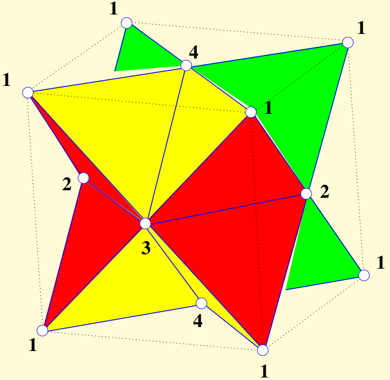
- The polytope $\mathcal{S}$ is universal for assembling tetrahedra and hemioctahedra face-to-face, with two each alternately surrounding any edge.
- $\mathcal{S}$ has a unique, minimal regular cover of Schläfli type $\{3, 12, 4\}$ and group order $2^{13} \cdot 3^2 = 73728$.
- But further collapse is possible. Each vertex-figure is a centrally-symmetric cuboctahedron. So let's collapse these to semiregular hemicuboctahedra by adjoining the relation

$$(\rho_1 \rho_2 \rho_3)^3 = 1$$

Polytopality survives this further collapse and we get the

Example 7. The Tomotope $\mathcal{T}$ (w. D. Pellicer, G. Williams)

To visualize $\mathcal{T}$ slice out a $2 \times 2 \times 2$ cube containing eight tetrahedra, a core octahedron and three other octahedra, each split into four identical but non-regular tetrahedra. The latter pieces fit into the twelve 'dimples' on the surface of the *stella octangula*:



More on visualizing $\mathcal{T}$

Next identify opposite square faces of the $2 \times 2 \times 2$ cube in toroidal fashion, so that the eight original vertices of the cube become one.

Finally reflect in the centre of the core octahedron and so identify antipodal faces of all ranks.

see it again

You see the 4 vertices, $4 = 8/2$ tetrahedra and 1 hemioctahedron (hidden in the core). The other three hemioctahedra are red, yellow and green, and 'run around' the belts of those colours.

There are 12 edges, on which $\Gamma(\mathcal{T})$ acts faithfully, and 16 triangular 2-faces.

The permutations

Aside: $\Gamma(\mathcal{T}) \simeq \mathbb{Z}_2^4 \rtimes \mathbb{S}_3$ has order 96 an can be obtained from the *crystallographic* group $\tilde{B}_3$ by reduction mod 2.

More on visualizing $\mathcal{T}$

Next identify opposite square faces of the $2 \times 2 \times 2$ cube in toroidal fashion, so that the eight original vertices of the cube become one.

Finally reflect in the centre of the core octahedron and so identify antipodal faces of all ranks.

▸ see it again

You see the 4 vertices, $4 = 8/2$ tetrahedra and 1 hemioctahedron (hidden in the core). The other three hemioctahedra are red, yellow and green, and 'run around' the belts of those colours.

There are 12 edges, on which $\Gamma(\mathcal{T})$ acts faithfully, and 16 triangular 2-faces.

▸ The permutations

Aside: $\Gamma(\mathcal{T}) \simeq \mathbb{Z}_2^4 \rtimes \mathbb{S}_3$ has order 96 an can be obtained from the *crystallographic* group $\tilde{B}_3$ by reduction mod 2.

UNB

More on visualizing $\mathcal{T}$

Next identify opposite square faces of the $2 \times 2 \times 2$ cube in toroidal fashion, so that the eight original vertices of the cube become one.

Finally reflect in the centre of the core octahedron and so identify antipodal faces of all ranks.

see it again

You see the 4 vertices, $4 = 8/2$ tetrahedra and 1 hemioctahedron (hidden in the core). The other three hemioctahedra are red, yellow and green, and 'run around' the belts of those colours.

There are 12 edges, on which $\Gamma(\mathcal{T})$ acts faithfully, and 16 triangular 2-faces.

The permutations

Aside: $\Gamma(\mathcal{T}) \simeq \mathbb{Z}_2^4 \rtimes \mathbb{S}_3$ has order 96 an can be obtained from the *crystallographic* group $\tilde{B}_3$ by reduction mod 2.

More on visualizing $\mathcal{T}$

Next identify opposite square faces of the $2 \times 2 \times 2$ cube in toroidal fashion, so that the eight original vertices of the cube become one.

Finally reflect in the centre of the core octahedron and so identify antipodal faces of all ranks.

see it again

You see the 4 vertices, $4 = 8/2$ tetrahedra and 1 hemioctahedron (hidden in the core). The other three hemioctahedra are red, yellow and green, and 'run around' the belts of those colours.

There are 12 edges, on which $\Gamma(\mathcal{T})$ acts faithfully, and 16 triangular 2-faces.

The permutations

Aside: $\Gamma(\mathcal{T}) \simeq \mathbb{Z}_2^4 \rtimes \mathbb{S}_3$ has order 96 an can be obtained from the *crystallographic* group $\tilde{B}_3$ by reduction mod 2.

UNB

The tomotope has $\mathcal{T}$ has a strange property

that cannot hold for maps (rank 3 polytopes). It has infinitely many mutually non–isomorphic minimal regular covers.
There is such a finite, minimal regular cover $\mathcal{P}_p$ for each prime p. Each of these 4-polytopes has type $\{3, 12, 4\}$.

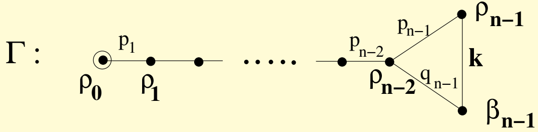
Intuitively: infinitely many distinct sets of minimal assembly instructions for $\mathcal{T}$ using standardized regular parts.

See The Tomotope, B. Monson, D. Dellicer, G. Williams, to appear in Ars Mathematica Contemporanea.

Enough with the examples - on with the Theorems

Egon Schulte and I have proved

Theorem 1. A combinatorial Wythoff's construction works for any group generated by involutions satisfying at least the relations suggested by



and also satisfying an *intersection condition* (akin to that for string C-groups).

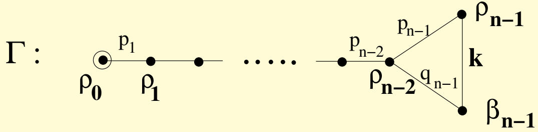
Note: k or various p_j's could equal 2: no branch then.

Definition Call Γ a *tail-triangle group*.

Enough with the examples - on with the Theorems

Egon Schulte and I have proved

Theorem 1. A combinatorial Wythoff's construction works for any group generated by involutions satisfying at least the relations suggested by



and also satisfying an *intersection condition* (akin to that for string C-groups).

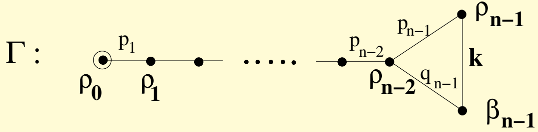
Note: k or various p_j's could equal 2: no branch then.

Definition Call Γ a *tail-triangle group*.

Enough with the examples - on with the Theorems

Egon Schulte and I have proved

Theorem 1. A combinatorial Wythoff's construction works for any group generated by involutions satisfying at least the relations suggested by



and also satisfying an *intersection condition* (akin to that for string C-groups).

Note: k or various p_j's could equal 2: no branch then.

Definition Call Γ a *tail-triangle group*.

And

Theorem 2. For any two regular n-polytopes $\mathcal{P}$ and $\mathcal{Q}$ with matching (i.e. isomorphic) facets, there exists a semiregular $(n+1)$-polytope $\mathcal{S}$ with infinitely many facets of type $\mathcal{P}$ and $\mathcal{Q}$, occurring alternately around each face of corank 2. (In this case, $\Gamma(\mathcal{S})$ contains a certain free product with amalgamation.)

(See *Semiregular polytopes and Amalgamated C-groups*, Adv. in Math., to appear.)

Some almost final words on monodromy ...

The structure of regular covers $\mathcal{R}$ of a general polytope $\mathcal{Q}$ has a lot to do with the *monodromy group* $\mathrm{Mon}(\mathcal{Q})$.

$\mathrm{Mon}(\mathcal{Q})$ is a natural permutation group on the flag set of $\mathcal{Q}$; this action commutes with the action of automorphisms on flags.

$\mathrm{Mon}(\mathcal{Q})$ is a string group generated by involutions, nearly a string C-group. But for the tomotope these generators fail the intersection condition.

The fall out: covering and likely other combinatorial questions are complicated.

See *Mixing and Monodromy of Abstract Polytopes*, B.M., D. Pellicer, G. Williams, in purgatory.

UNB

Some almost final words on monodromy ...

The structure of regular covers $\mathcal{R}$ of a general polytope $\mathcal{Q}$ has a lot to do with the *monodromy group* $\mathrm{Mon}(\mathcal{Q})$.

$\mathrm{Mon}(\mathcal{Q})$ is a natural permutation group on the flag set of $\mathcal{Q}$; this action commutes with the action of automorphisms on flags.

$\mathrm{Mon}(\mathcal{Q})$ is a string group generated by involutions, nearly a string C-group. But for the tomotope these generators fail the intersection condition.

The fall out: covering and likely other combinatorial questions are complicated.

See *Mixing and Monodromy of Abstract Polytopes*, B.M., D. Pellicer, G. Williams, in purgatory.

Some almost final words on monodromy ...

The structure of regular covers $\mathcal{R}$ of a general polytope $\mathcal{Q}$ has a lot to do with the *monodromy group* $\mathrm{Mon}(\mathcal{Q})$.

$\mathrm{Mon}(\mathcal{Q})$ is a natural permutation group on the flag set of $\mathcal{Q}$; this action commutes with the action of automorphisms on flags.
$\mathrm{Mon}(\mathcal{Q})$ is a string group generated by involutions, nearly a string C-group. But for the tomotope these generators fail the intersection condition.

The fall out: covering and likely other combinatorial questions are complicated.

See *Mixing and Monodromy of Abstract Polytopes*, B.M., D. Pellicer, G. Williams, in purgatory.

UNB

Some almost final words on monodromy ...

The structure of regular covers $\mathcal{R}$ of a general polytope $\mathcal{Q}$ has a lot to do with the *monodromy group* $\mathrm{Mon}(\mathcal{Q})$.

$\mathrm{Mon}(\mathcal{Q})$ is a natural permutation group on the flag set of $\mathcal{Q}$; this action commutes with the action of automorphisms on flags.

$\mathrm{Mon}(\mathcal{Q})$ is a string group generated by involutions, nearly a string C-group. But for the tomotope these generators fail the intersection condition.

The fall out: covering and likely other combinatorial questions are complicated.

See *Mixing and Monodromy of Abstract Polytopes*, B.M., D. Pellicer, G. Williams, in purgatory.

Some final words on chirality ... Example 8

In 2008, M. Conder, I. Hubard, T. Pisanski ended the long search for any *chiral* polytopes $\mathcal{P}$ of higher rank (this meant > 4 for $\mathcal{P}$ finite).

We recently noticed that $\mathrm{Mon}(\mathcal{P})$ fails the intersection condition for one of their examples. So take $\Gamma = S_6$ with these generators:

$$\sigma_1 = (1,2,3), \sigma_2 = (1,3,2,4), \sigma_3 = (1,5,4,3), \sigma_4 = (1,2,3)(4,6,5).$$

Then Γ is the automorphism group of a chiral 5-polytope $\mathcal{P}$, evidently with type $\{3,4,4,3\}$

But $\mathrm{Mon}(\mathcal{P})$, which has order 518400 and the same type, fails the intersection condition. We don't understand where this leads ...

Some final words on chirality ... Example 8

In 2008, M. Conder, I. Hubard, T. Pisanski ended the long search for any *chiral* polytopes $\mathcal{P}$ of higher rank (this meant > 4 for $\mathcal{P}$ finite).

We recently noticed that $\mathrm{Mon}(\mathcal{P})$ fails the intersection condition for one of their examples. So take $\Gamma = S_6$ with these generators:

$\sigma_1 = (1,2,3), \sigma_2 = (1,3,2,4), \sigma_3 = (1,5,4,3), \sigma_4 = (1,2,3)(4,6,5).$

Then Γ is the automorphism group of a chiral 5-polytope $\mathcal{P}$, evidently with type $\{3,4,4,3\}$

But $\mathrm{Mon}(\mathcal{P})$, which has order 518400 and the same type, fails the intersection condition. We don't understand where this leads ...

Some final words on chirality ... Example 8

In 2008, M. Conder, I. Hubard, T. Pisanski ended the long search for any *chiral* polytopes $\mathcal{P}$ of higher rank (this meant > 4 for $\mathcal{P}$ finite).

We recently noticed that $\mathrm{Mon}(\mathcal{P})$ fails the intersection condition for one of their examples. So take $\Gamma = S_6$ with these generators:

$\sigma_1 = (1, 2, 3), \sigma_2 = (1, 3, 2, 4), \sigma_3 = (1, 5, 4, 3), \sigma_4 = (1, 2, 3)(4, 6, 5)$.

Then Γ is the automorphism group of a chiral 5-polytope $\mathcal{P}$, evidently with type $\{3, 4, 4, 3\}$

But $\mathrm{Mon}(\mathcal{P})$, which has order 518400 and the same type, fails the intersection condition. We don't understand where this leads ...

Some final words on chirality ... Example 8

In 2008, M. Conder, I. Hubard, T. Pisanski ended the long search for any *chiral* polytopes $\mathcal{P}$ of higher rank (this meant > 4 for $\mathcal{P}$ finite).

We recently noticed that $\mathrm{Mon}(\mathcal{P})$ fails the intersection condition for one of their examples. So take $\Gamma = S_6$ with these generators:

$\sigma_1 = (1,2,3), \sigma_2 = (1,3,2,4), \sigma_3 = (1,5,4,3), \sigma_4 = (1,2,3)(4,6,5).$

Then Γ is the automorphism group of a chiral 5-polytope $\mathcal{P}$, evidently with type $\{3,4,4,3\}$

But $\mathrm{Mon}(\mathcal{P})$, which has order 518400 and the same type, fails the intersection condition. We don't understand where this leads ...

UNB

Some final words on chirality ... Example 8

In 2008, M. Conder, I. Hubard, T. Pisanski ended the long search for any *chiral* polytopes $\mathcal{P}$ of higher rank (this meant > 4 for $\mathcal{P}$ finite).

We recently noticed that $\mathrm{Mon}(\mathcal{P})$ fails the intersection condition for one of their examples. So take $\Gamma = S_6$ with these generators:

$\sigma_1 = (1,2,3), \sigma_2 = (1,3,2,4), \sigma_3 = (1,5,4,3), \sigma_4 = (1,2,3)(4,6,5).$

Then Γ is the automorphism group of a chiral 5-polytope $\mathcal{P}$, evidently with type $\{3,4,4,3\}$

But $\mathrm{Mon}(\mathcal{P})$, which has order 518400 and the same type, fails the intersection condition. We don't understand where this leads ...

Some Conjectures and a Question

- **Conjecture**. The monodromy group for any convex polytope is a string C-group.

  Is this known? (I have some doubts it is true ...)

Some Conjectures and a Question

- **Conjecture**. The monodromy group for any convex polytope is a string C-group.

  Is this known? (I have some doubts it is true ...)

- **Conjecture**. The monodromy group for any convex polytope *with combinatorially regular facets of any type* is a string C-group. (This seems likely.)

UNB

Some Conjectures and a Question

- **Conjecture**. The monodromy group for any convex polytope is a string C-group.

  Is this known? (I have some doubts it is true ...)

- **Conjecture**. The monodromy group for any convex polytope *with combinatorially regular facets of any type* is a string C-group. (This seems likely.)

- **Conjecture**. There is a finite abstract (convex?, chiral?) polytope $\mathcal{Q}$ whose minimal regular covers are all infinite. (Certainly $\mathrm{Mon}(\mathcal{Q})$ could not then be a string C-group.)

Some Conjectures and a Question

- **Conjecture**. The monodromy group for any convex polytope is a string C-group.

  Is this known? (I have some doubts it is true ...)
- **Conjecture**. The monodromy group for any convex polytope *with combinatorially regular facets of any type* is a string C-group. (This seems likely.)
- **Conjecture**. There is a finite abstract (convex?, chiral?) polytope $\mathcal{Q}$ whose minimal regular covers are all infinite. (Certainly $\mathrm{Mon}(\mathcal{Q})$ could not then be a string C-group.)
- **Question** Can one always assemble thing 1 and thing 2 for any finite integer $k \geq 2$?

Many thanks to our organizers!

References

[1] M. Conder, I. Hubard and T. Pisanski, Constructions for chiral polytopes, J. London Math. Soc., **77** (2008), 115–129.

[2] M. I. Hartley, All Polytopes are Quotients, and Isomorphic Polytopes Are Quotients by Conjugate Subgroups, Discrete Comput. Geom., **21** (1999), 289–298.

[3] M. I. Hartley, More on quotient polytopes, Aequationes Math., **57** (1999), 108–120.

[4] I. Hubard, A. Orbanić, A. I. Weiss, Monodromy groups and self-invariance, Canadian Journal of Math, **61** (2009), 1300–1324.

References, continued

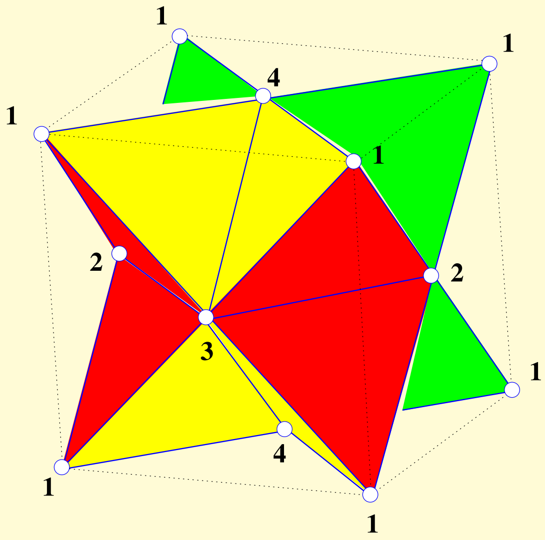
[5] P. McMullen and E. Schulte, Abstract Regular Polytopes, Encyclopedia of Mathematics and its Applications, **92**, Cambridge University Press, Cambridge, 2002.

[6] B. Monson and E. Schulte, Semiregular polytopes and amalgamated C-groups, to appear in Advances in Mathematics.

[7] B.Monson, D. Pellicer and G. Williams, The Tomotope, to appear in Ars Mathematica Contemporanea.

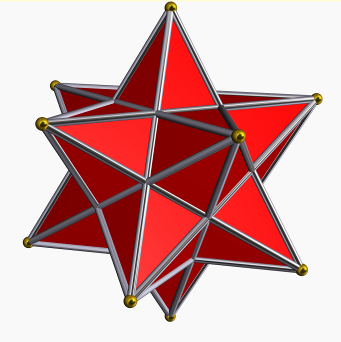
[8] B.Monson, D. Pellicer and G. Williams, Mixing and Monodromy of Abstract Polytopes, in prepartion but nearly done.

Picture of the *Tomotope* $\mathcal{T}$:

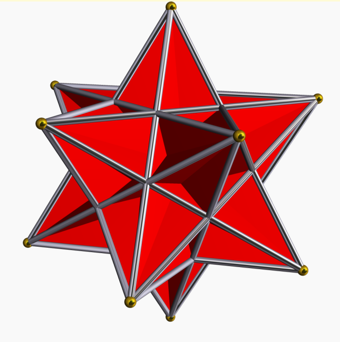


▸ back to description

Example 3. Two regular star-polyhedra (courtesy Wikipedia)



Small stellated dodec. $\{\frac{5}{2}, 5\}$

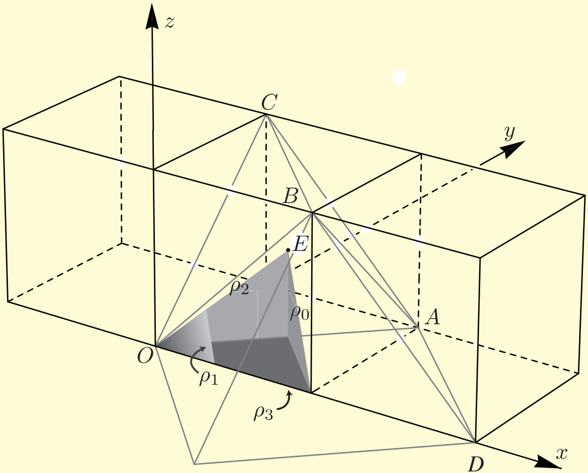


Great icos. $\{3, \frac{5}{2}\}$
$\simeq$ convex reg. icos.

Go Back

UNB

The semiregular tessellation $\mathcal{S}$ of $\mathbb{R}^3$



go back

The group $\Gamma(\mathcal{T})$

acts faithfully on edges of $\mathcal{T}$, so we have this permutation representation:

$$\rho_0 = (5,10)(6,9)(7,12)(8,11)$$

$$\rho_1 = (1,6)(2,5)(3,8)(4,7)$$

$$\rho_2 = (5,9)(6,10)(7,11)(8,12)$$

$$\rho_3 = (5,8)(6,7)(9,12)(10,11)$$

Back

UNB

The n-polytope $\mathcal{Q}$

is a poset whose elements ($=$ *faces*) satisfy:

get back

The n-polytope $\mathcal{Q}$

is a poset whose elements ($=$ *faces*) satisfy:

- $\mathcal{Q}$ has a unique minimal face F_{-1} and maximal face F_n

get back

The n-polytope $\mathcal{Q}$

is a poset whose elements (= *faces*) satisfy:

- $\mathcal{Q}$ has a unique minimal face F_{-1} and maximal face F_n
- Every maximal chain or *flag* has $n+2$ faces

get back

The n-polytope $\mathcal{Q}$

is a poset whose elements (= *faces*) satisfy:

- $\mathcal{Q}$ has a unique minimal face F_{-1} and maximal face F_n
- Every maximal chain or *flag* has $n+2$ faces

  so $\mathcal{Q}$ has a strictly monotone rank function onto $\{-1, 0, \ldots, n\}$

get back

UNB

The n-polytope $\mathcal{Q}$

is a poset whose elements (= *faces*) satisfy:

- $\mathcal{Q}$ has a unique minimal face F_{-1} and maximal face F_n
- Every maximal chain or *flag* has $n+2$ faces

  so $\mathcal{Q}$ has a strictly monotone rank function onto $\{-1, 0, \ldots, n\}$
- $\mathcal{Q}$ is strongly flag connected

get back

The n-polytope $\mathcal{Q}$

is a poset whose elements ($=$ *faces*) satisfy:

- $\mathcal{Q}$ has a unique minimal face F_{-1} and maximal face F_n
- Every maximal chain or *flag* has $n+2$ faces

  so $\mathcal{Q}$ has a strictly monotone rank function onto $\{-1, 0, \ldots, n\}$
- $\mathcal{Q}$ is strongly flag connected
- $\mathcal{Q}$ satisfies the 'diamond' condition:

get back

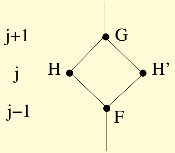
The n-polytope $\mathcal{Q}$

is a poset whose elements (= *faces*) satisfy:

- $\mathcal{Q}$ has a unique minimal face F_{-1} and maximal face F_n
- Every maximal chain or *flag* has $n+2$ faces

  so $\mathcal{Q}$ has a strictly monotone rank function onto $\{-1, 0, \ldots, n\}$
- $\mathcal{Q}$ is strongly flag connected
- $\mathcal{Q}$ satisfies the 'diamond' condition:

  whenever $F < G$ with $\text{rank}(F) = j-1$ and $\text{rank}(G) = j+1$ there exist exactly two j-faces H with $F < H < G$



get back

The n-polytope $\mathcal{Q}$

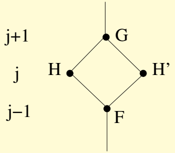
is a poset whose elements (= *faces*) satisfy:

- $\mathcal{Q}$ has a unique minimal face F_{-1} and maximal face F_n
- Every maximal chain or *flag* has $n+2$ faces

  so $\mathcal{Q}$ has a strictly monotone rank function onto $\{-1, 0, \ldots, n\}$
- $\mathcal{Q}$ is strongly flag connected

  via adjacency in the flag graph; this rules out, for example, the disjoint union of two polyhedra
- $\mathcal{Q}$ satisfies the 'diamond' condition:

  whenever $F < G$ with $\text{rank}(F) = j-1$ and $\text{rank}(G) = j+1$ there exist exactly two j-faces H with $F < H < G$



get back

So when is $\mathrm{Mon}(Q)$ a string C-group?

Some results below may be well-known, others new:

$\mathrm{Mon}(\mathcal{Q})$ is a string C-group if

- $\mathcal{Q}$ is any polyhedron ($d = 3$, regardless of symmetry); or
- $\mathcal{Q}$ is regular of any rank (in which case $\Gamma(\mathcal{Q}) \simeq \mathrm{Mon}(\mathcal{Q})$); or
- all facets of $\mathcal{Q}$ are regular quotients of one particular regular facet (or dually); or
- $\mathcal{Q}$ has any mixture of regular facets together with flag-transitive vertex-figures (or dually).
- **Thus**: if $\mathcal{Q}$ is any simplicial (or simple) convex polytope, then $\mathrm{Mon}(\mathcal{Q})$ is a string C-group.

  Example. The cyclic convex 4-polytope $\mathcal{Q}$ on 6 vertices thereby has a regular cover of Schläfli type $\{3, 3, 12\}$; $\mathrm{Mon}(\mathcal{Q})$ is a string C-group of order $2^6 \cdot 3^7$.

So when is $\mathrm{Mon}(Q)$ a string C-group?

Some results below may be well-known, others new:

$\mathrm{Mon}(\mathcal{Q})$ is a string C-group if

- $\mathcal{Q}$ is any polyhedron ($d = 3$, regardless of symmetry); or
- $\mathcal{Q}$ is regular of any rank (in which case $\Gamma(\mathcal{Q}) \simeq \mathrm{Mon}(\mathcal{Q})$); or
- all facets of $\mathcal{Q}$ are regular quotients of one particular regular facet (or dually); or
- $\mathcal{Q}$ has any mixture of regular facets together with flag-transitive vertex-figures (or dually).
- **Thus**: if $\mathcal{Q}$ is any simplicial (or simple) convex polytope, then $\mathrm{Mon}(\mathcal{Q})$ is a string C-group.

  Example. The cyclic convex 4-polytope $\mathcal{Q}$ on 6 vertices thereby has a regular cover of Schläfli type $\{3, 3, 12\}$; $\mathrm{Mon}(\mathcal{Q})$ is a string C-group of order $2^6 \cdot 3^7$.

So when is $\mathrm{Mon}(Q)$ a string C-group?

Some results below may be well-known, others new:

$\mathrm{Mon}(\mathcal{Q})$ is a string C-group if

- $\mathcal{Q}$ is any polyhedron ($d = 3$, regardless of symmetry); or
- $\mathcal{Q}$ is regular of any rank (in which case $\Gamma(\mathcal{Q}) \simeq \mathrm{Mon}(\mathcal{Q})$); or
- all facets of $\mathcal{Q}$ are regular quotients of one particular regular facet (or dually); or
- $\mathcal{Q}$ has any mixture of regular facets together with flag-transitive vertex-figures (or dually).
- **Thus**: if $\mathcal{Q}$ is any simplicial (or simple) convex polytope, then $\mathrm{Mon}(\mathcal{Q})$ is a string C-group.

  Example. The cyclic convex 4-polytope $\mathcal{Q}$ on 6 vertices thereby has a regular cover of Schläfli type $\{3, 3, 12\}$; $\mathrm{Mon}(\mathcal{Q})$ is a string C-group of order $2^6 \cdot 3^7$.

So when is $\mathrm{Mon}(Q)$ a string C-group?

Some results below may be well-known, others new:

$\mathrm{Mon}(\mathcal{Q})$ is a string C-group if

- $\mathcal{Q}$ is any polyhedron ($d = 3$, regardless of symmetry); or
- $\mathcal{Q}$ is regular of any rank (in which case $\Gamma(\mathcal{Q}) \simeq \mathrm{Mon}(\mathcal{Q})$); or
- all facets of $\mathcal{Q}$ are regular quotients of one particular regular facet (or dually); or
- $\mathcal{Q}$ has any mixture of regular facets together with flag-transitive vertex-figures (or dually).
- **Thus**: if $\mathcal{Q}$ is any simplicial (or simple) convex polytope, then $\mathrm{Mon}(\mathcal{Q})$ is a string C-group.

  Example. The cyclic convex 4-polytope $\mathcal{Q}$ on 6 vertices thereby has a regular cover of Schläfli type $\{3, 3, 12\}$; $\mathrm{Mon}(\mathcal{Q})$ is a string C-group of order $2^6 \cdot 3^7$.

So when is $\mathrm{Mon}(Q)$ a string C-group?

Some results below may be well-known, others new:

$\mathrm{Mon}(\mathcal{Q})$ is a string C-group if

- $\mathcal{Q}$ is any polyhedron ($d = 3$, regardless of symmetry); or
- $\mathcal{Q}$ is regular of any rank (in which case $\Gamma(\mathcal{Q}) \simeq \mathrm{Mon}(\mathcal{Q})$); or
- all facets of $\mathcal{Q}$ are regular quotients of one particular regular facet (or dually); or
- $\mathcal{Q}$ has any mixture of regular facets together with flag-transitive vertex-figures (or dually).
- **Thus**: if $\mathcal{Q}$ is any simplicial (or simple) convex polytope, then $\mathrm{Mon}(\mathcal{Q})$ is a string C-group.

Example. The cyclic convex 4-polytope $\mathcal{Q}$ on 6 vertices thereby has a regular cover of Schläfli type $\{3, 3, 12\}$; $\mathrm{Mon}(\mathcal{Q})$ is a string C-group of order $2^6 \cdot 3^7$.

So when is $\mathrm{Mon}(Q)$ a string C-group?

Some results below may be well-known, others new:

$\mathrm{Mon}(\mathcal{Q})$ is a string C-group if

- $\mathcal{Q}$ is any polyhedron ($d = 3$, regardless of symmetry); or
- $\mathcal{Q}$ is regular of any rank (in which case $\Gamma(\mathcal{Q}) \simeq \mathrm{Mon}(\mathcal{Q})$); or
- all facets of $\mathcal{Q}$ are regular quotients of one particular regular facet (or dually); or
- $\mathcal{Q}$ has any mixture of regular facets together with flag-transitive vertex-figures (or dually).
- **Thus**: if $\mathcal{Q}$ is any simplicial (or simple) convex polytope, then $\mathrm{Mon}(\mathcal{Q})$ is a string C-group.

  Example. The cyclic convex 4-polytope $\mathcal{Q}$ on 6 vertices thereby has a regular cover of Schläfli type $\{3, 3, 12\}$; $\mathrm{Mon}(\mathcal{Q})$ is a string C-group of order $2^6 \cdot 3^7$.