Computation of normalizers of intransitive permutation groups in symmetric groups

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1. Normalizers and Coherent Configurations
2. An algorithm applicable to transitive groups
3. Subdirect Product
4. Methods and Examples

Normalizers and Coherent Configurations

Let G and H be permutation groups on a set Ω.

- The normalizer of H in G is defined by
$$\mathrm{Norm}(G,H) = \{g \in G | g^{-1}Hg = H\}.$$
- The orbit of H containing a point $i \in \Omega$ is defined by $i^H = \{i^h | h \in H\}$.
- H acts naturally in Ω^2 by $(i,j)^h = (i^h, j^h)$ for $i,j \in \Omega$.
- Let $R_0, R_1, \cdots$ and R_d be the orbits of H on Ω^2.
- The coherent configuration C formed by H is a graph on Ω with colored edge sets $R_0, R_1, \cdots$ and R_d.

  Let $A = Aut(C)$ the automorphism group of the configuration.

【Fact】$\mathrm{Norm}(G,H) \subseteq A$

Definition of Coherent Configuration

Let subsets $R_0, R_1, \cdots, R_d \subseteq \Omega^2$.
Definition. $C = (\Omega, \{R_k\}_{k=0,1,\cdots,d})$ is a coherent configuration if it satisfies the following:

1. $R_0 \cup R_1 \cup \cdots \cup R_d = \Omega^2$ and $R_i \cap R_j = \emptyset$ if $i \neq j$;
2. for some $r < d$
$$R_0 \cup R_1 \cup \cdots \cup R_r = \{(x,x) | x \in \Omega\};$$
3. for every k there exists k^* such that
$$R_{k^*} = {}^t R_k = \{(y,x) | (x,y) \in R_k\};$$
4. there exist constant numbers $p_{i,j,k}$ such that for any $(x,z) \in R_k$
$$p_{i,j,k} = \#\{y \in \Omega | (x,y) \in R_i \text{ and } (y,z) \in R_j\}.$$

An Example of Coherent Configuration and its Automorphism

$\Omega = \{1, 2, 3, 4, 5, 6\}$,
$H = \text{sub} < Sym(6) | (1, 5, 4)(2, 6, 3), (1, 6, 3, 2, 5, 4) >$

H has 4 orbits R_0, R_1, R_2 and $R_3 \subset \Omega^2$.

$$C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 2 & 2 & 3 & 3 \\ 1 & 0 & 2 & 2 & 3 & 3 \\ 3 & 3 & 0 & 1 & 2 & 2 \\ 3 & 3 & 1 & 0 & 2 & 2 \\ 2 & 2 & 3 & 3 & 0 & 1 \\ 2 & 2 & 3 & 3 & 1 & 0 \end{pmatrix} \end{array} \xrightarrow{(3,5,4,6) \in A} \begin{pmatrix} 0 & 1 & 3 & 3 & 2 & 2 \\ 1 & 0 & 3 & 3 & 2 & 2 \\ 2 & 2 & 0 & 1 & 3 & 3 \\ 2 & 2 & 1 & 0 & 3 & 3 \\ 3 & 3 & 2 & 2 & 0 & 1 \\ 3 & 3 & 2 & 2 & 1 & 0 \end{pmatrix}$$

$A = \text{sub} < Sym(6) | (3, 5, 4, 6), (1, 6, 2, 5)(3, 4) > \supsetneq H$

An Example of Coherent Configuration and its Automorphism

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$$A = \text{sub} < Sym(6)|(3,5,4,6),(1,6,2,5)(3,4) > \supsetneq H$$

An algorithm applicable to transitive groups

Sometimes it takes a long time for the computation of normalizers.

【Assumption】($N =$Norm(G, H), $H \subseteq G$)

$\exists O = i^H = i^N$, a common orbit for some $i \in \Omega \Longrightarrow N =< H, N_i >$

Especially, if H is transitive, Ω is the common orbit.

【Example】 $\Omega = \{1, 2, \cdots, 64\}$

$H =$PermutationGroup< 64 | (1,45,62,26,2,42,52,25,16,36,56,28,15,39, 58,27)(3,44,55,19,13,34,51,18,14, 37,61,17, 4,47,57,20)(5,35,49,32,11, 41,53,29,12,46,59,30, 6,40,63,31) (7,38,60,21, 8,33,54,22,10,43,50,23, 9,48,64,24), (1,63,11,51, 2,58,12,54)(3,56,15,62, 4,49,16,59)(5,50,9, 60,6,55,10,61) (7,57,13,53,8,64,14,52)(17,40,27,39)(18,47,22,35)(19,38, 29,41)(20,45) (21,44,31,43)(23,42,25,37)(24,33)(26,46)(28,48,32,36) (30,34) >;

- H is transitive $\Longrightarrow$ $N =< H, N_1 >$

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【Example continued】

Moreover repeatedly,

- $\{2^h | h \in H_1\}$ is the only one orbit of H_1 of size 15 (a common orbit of H_1 and N_1) $\Longrightarrow N_1 =< H_1, N_{1,2} >$ and so $N =< H, N_{1,2} >$
- $\{3^h | h \in H_{1,2}\}$ is the only one orbit of $H_{1,2}$ of size 14 $\Longrightarrow N =< H, N_{1,2,3} > \subseteq < H, \text{Norm}(Aut(C'''), H_{1,2,3}) >$, where C''' is the coherent configuration formed by $H_{1,2,3}$.
- $\text{Index}(Aut(C'''), H_{1,2,3}) \approx 5 \cdot 10^{21}$ $(\text{Index}(Aut(C), H) \approx 3 \cdot 10^{44})$
- $N''' = \text{Norm}(Aut(C'''), H_{1,2,3})$ and $\text{Index}(N''', H_{1,2,3}) = 1152$.
- $\#[u : u \textit{ in Transversal}(N''', H_{1,2,3}) | H^u \textit{ eq } H] = 2 \Longrightarrow \text{Index}(N, H) = 2$

Total computing time 7.89 seconds

SymmetricNormalizer(H) >60 seconds

【Remark】This is replaced by Norm(N''', H). $\cdots (N''' \not\supseteq H)$

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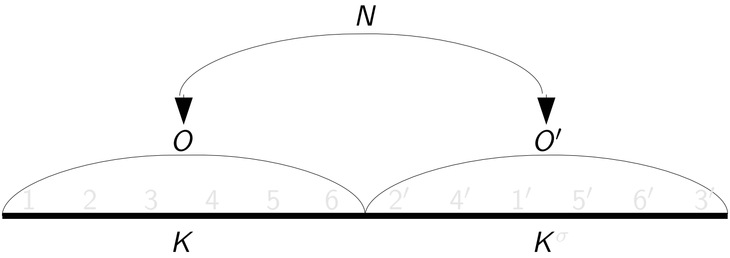
Subdirect Product

Case : H and N =Norm(G, H) have no common orbits.
(An easy example)

- H has two orbit O and O'.
- N =Norm(G, H) interchanges O and O'.

$\Longrightarrow$ OrbitImage(H, O) $\cong$OrbitImage(H, O') $\cong K$, so $H \subseteq K \times K^{\sigma}$



$$\sigma = \begin{pmatrix} 1' & 2' & 3' & 4' & 5' & 6' \\ 2' & 4' & 1' & 5' & 6' & 3' \end{pmatrix} = (1', 2', 4', 5', 6', 3')$$

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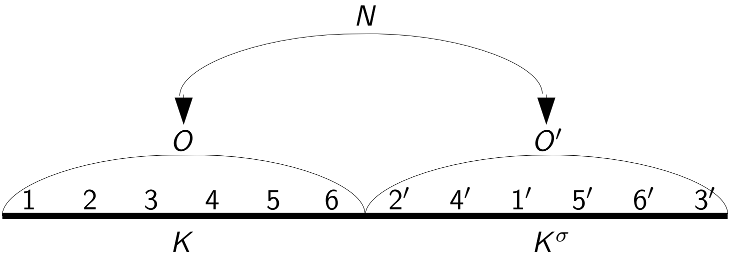

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Methods and Examples

Ω consists of 4 isomorphic orbits $O_1, \cdots O_4$ of H

- OrbitImage(H, O_1) $\cong$PrimitiveGroup(16, 20) $= K$
- There are 31 isomorphism classes of groups satisfying this condition.

【Method W】

Following the above argument, we may assume

- $H \subseteq$DirectProduct([K, K, K, K])
- SymmetricNormalizer(H) $\subseteq$WreathProduct(M, Sym(4)) $= W$, where $M =$SymmetricNormalizer(K)

$\Longrightarrow$ SymmetricNormalizer(H) $=$Normalizer(W, H)

【Method D】

For each τ of $4! = 24$ elements of W permuting $O_1, \cdots, O_4$,

- $\exists \sigma \in$DirectProduct([M, M, M, M]) $= D$ such that $H^{\tau\sigma} = H$?

$\Longrightarrow$SymmetricNormalizer(H) $=<$Normalizer(D, H)$, \tau\sigma, \cdots >$

By Magma calculator (Time is in seconds)

No	Ind	#$[O_i]$	SymmetricNorm(H)			Norm(W,H)			Norm(D,H)···	
1	24	1	0.01	0.00	0.00	0.01	0.01	0.01	0.27	0.26
2	6	2	0.01	0.01	0.01	0.01	0.02	0.02	0.30	0.28
3	6	2	0.01	0.00	0.00	0.01	0.01	0.02	0.28	0.27
4	6	2	0.01	0.00	0.00	0.01	0.01	0.01	0.23	0.24
5	2	3	0.02	0.02	0.00	0.01	0.01	0.02	0.35	0.28
6	2	3	0.84	7.77	0.04	7.25	0.16	0.03	0.27	0.23
7	4	2	0.05	32.28	0.04	0.05	14.71	0.02	0.26	0.25
8	8	1	0.01	0.02	0.00	0.01	0.01	0.02	0.24	0.23
9	2	3	0.02	0.01	0.01	0.01	0.02	0.02	0.33	0.30
10	4	2	0.06	0.24	0.18	0.04	0.02	0.13	0.27	0.26
11	8	1	0.01	0.26	0.00	0.01	0.01	0.01	0.25	0.24
12	4	2	0.01	0.00	0.00	0.01	0.01	0.02	0.26	0.22
13	6	2	0.01	0.02	0.01	0.01	0.01	0.02	0.30	0.34
14	6	2	6.38	0.03	> 60	7.18	3.64	11.53	0.21	0.24
15	6	2	1.79	21.78	10.17	1.78	2.33	0.02	0.24	0.21

No	Ind	$\#[O_i]$	SymmetricNorm(H)			Norm(W,H)			Norm(D,H)···	
16	6	2	0.09	0.45	0.01	>60	0.02	0.02	0.35	0.31
17	6	2	24.80	>60	>60	>60	>60	12.78	0.27	0.22
18	24	1	29.644	>60	91.972	>60	>60	29.34	0.27	0.26
19	24	1	4.16	2.69	0.02	0.28	0.06	1.10	0.29	0.24
20	2	3	0.13	0.47	38.91	13.27	0.04	0.15	0.38	0.36
21	8	1	>60	>60	>60	>60	>60	>60	0.27	0.24
22	8	1	0.02	0.01	0.01	0.01	0.01	0.03	0.38	0.33
23	4	2	0.02	0.02	0.02	0.02	0.03	0.03	0.53	0.46
24	4	2	0.01	0.01	0.01	0.02	0.02	0.02	0.36	0.32
25	4	2	0.02	0.02	0.01	0.01	0.01	0.03	0.35	0.30
26	4	2	0.04	0.05	0.15	0.03	0.05	0.04	0.59	0.50
27	4	2	6.35	0.54	3.57	0.02	36.92	3.91	0.37	0.38
28	8	1	0.03	0.05	6.80	0.07	0.03	14.55	0.36	0.34
29	4	2	0.03	0.04	0.03	0.04	0.04	0.04	0.55	0.45
30	8	1	1.40	12.12	0.05	0.63	10.37	0.08	0.35	0.31
31	24	1	0.00	0.03	0.04	0.00	0.01	0.01	0.04	0.03
Total time $\geq$			136.0	258.9	332.0	270.8	248.6	134.0	9.8	8.9

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17	6	2	24.80	129468	> 60	> 60	160635	12.78	0.27	0.2
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19	24	1	4.16	2.69	0.02	0.28	0.06	1.10	0.29	0.2
20	2	3	0.13	0.47	38.91	13.27	0.04	0.15	0.38	0.3
21	8	1	> 60	18592	> 60	> 60	> 60	> 60	0.27	0.2
22	8	1	0.02	0.01	0.01	0.01	0.01	0.03	0.38	0.3
23	4	2	0.02	0.02	0.02	0.02	0.03	0.03	0.53	0.4
24	4	2	0.01	0.01	0.01	0.02	0.02	0.02	0.36	0.3
25	4	2	0.02	0.02	0.01	0.01	0.01	0.03	0.35	0.3
26	4	2	0.04	0.05	0.15	0.03	0.05	0.04	0.59	0.5
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28	8	1	0.03	0.05	6.80	0.07	0.03	14.55	0.36	0.3
29	4	2	0.03	0.04	0.03	0.04	0.04	0.04	0.55	0.4
30	8	1	1.40	12.12	0.05	0.63	10.37	0.08	0.35	0.3
31	24	1	0.00	0.03	0.04	0.00	0.01	0.01	0.04	0.0
Total time $\geq$			136.0	258.9	332.0	270.8	248.6	134.0	9.8	8

The group H in the Example for Transitive Group is from No 21.

H =PermutationGroup< 64 | (1,45,62,26,2,42,52,25,16,36,56,28,15,39, 58,27)(3,44,55,19,13,34,51,18,14, 37,61,17, 4,47,57,20)(5,35,49,32,11, 41,53,29,12,46,59,30, 6,40,63,31) (7,38,60,21, 8,33,54,22,10,43,50,23, 9,48,64,24), (1,63,11,51, 2,58,12,54)(3,56,15,62, 4,49,16,59)(5,50,9, 60,6,55,10,61) (7,57,13,53,8,64,14,52)(17,40,27,39)(18,47,22,35)(19,38, 29,41)(20,45) (21,44,31,43)(23,42,25,37)(24,33)(26,46)(28,48,32,36) (30,34) >;

	Ind	$[\#O_i : N_x]$	SymNorm	$\mathrm{Norm}(W,\cdot)$	$\mathrm{Norm}(D,\cdot)\cdots$
H_1	2	$[1, 15, 16, 16*2]$	>60	>60	0.07
$H_{1,2}$	8	$[1*2, 14, 16, 16*2]$	>60	0.40	0.05
$H_{1,2,3}$	1152	$[1*4, 12, 16, 16*2]$	>60	0.06	0.16

The group H in the Example for Transitive Group is from No 21.

H =PermutationGroup< 64 | (1,45,62,26,2,42,52,25,16,36,56,28,15,39, 58,27)(3,44,55,19,13,34,51,18,14, 37,61,17, 4,47,57,20)(5,35,49,32,11, 41,53,29,12,46,59,30, 6,40,63,31) (7,38,60,21, 8,33,54,22,10,43,50,23, 9,48,64,24), (1,63,11,51, 2,58,12,54)(3,56,15,62, 4,49,16,59)(5,50,9, 60,6,55,10,61) (7,57,13,53,8,64,14,52)(17,40,27,39)(18,47,22,35)(19,38, 29,41)(20,45) (21,44,31,43)(23,42,25,37)(24,33)(26,46)(28,48,32,36) (30,34) >;

	Ind	$[\#O_i : N_x]$	SymNorm	$\text{Norm}(W, \cdot)$	$\text{Norm}(D, \cdot) \cdots$
H_1	2	$[1, 15, 16, 16{*}2]$	> 60	> 60	0.07
$H_{1,2}$	8	$[1{*}2, 14, 16, 16{*}2]$	> 60	0.40	0.05
$H_{1,2,3}$	1152	$[1{*}4, 12, 16, 16{*}2]$	7014	0.06	0.16