On the Split Structure of Lifted groups, I

Aleksander Malnič
University of Ljubljana and University of Primorska

Joint work with Rok Požar

Symmetries of Discrete Objects
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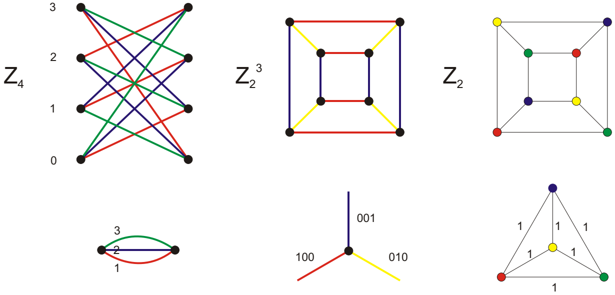

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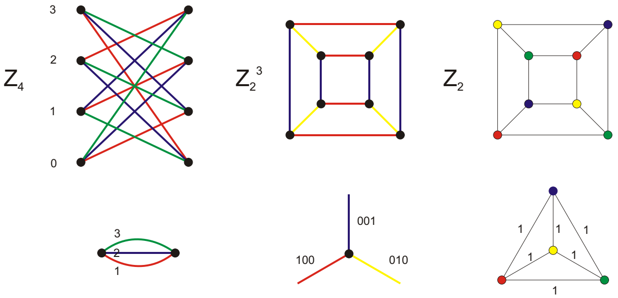

Construction/reconstruction
by a **voltage assignment** $\zeta\colon X \to \Gamma \cong \mathrm{CT}_p$

Motivation in AGT: Studying symmetries of graphs

Lifting automorphisms along regular covering projections

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Applications

Construction of infinite families, compiling lists,
and classification of graphs with interesting symmetry properties.

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- Efficient algorithms? Computer implementation?

Basic lifting lemma

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  - $g^{\sharp}$ can be 'represented' by a matrix over $\mathbb{Z}$ (not uniquelly)

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- **Abelian covers**: there is an algorithm using group presentations **without explicit construction of** $\tilde{G}$.

Split extensions via presentations (abelian covers)

Lemma 2. Let $G = \langle g_1, g_2, \ldots, g_n \mid R_1, R_2, \ldots, R_m \rangle$. Then the extension $\mathrm{id} \to \mathrm{CT}_p \to \tilde{G} \to G \to \mathrm{id}$ splits $\Leftrightarrow$ some set of lifts $\{\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n\}$ satisfies the above relations $R_1, \ldots R_m$.

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- Can be used to test for the existence of **normal complements**.

Split extensions – special cases wrt the action of $\bar{G}$

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Some $\bar{G}$ has an invariant section

over a G-invariant subset $\Omega \subset V$

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Strongly split extension (over Ω)

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- However, for **abelian** covers there is an efficient algorithm.

Abelian covers: Finding a sectional complement

Adapting the algorithm for finding an orbit

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Note.

- Computations can be carried out over $\mathbb{Z}$.

Finding all covers of X s.t. $\tilde{G}$ strongly splits over Ω

Define

$\mathrm{Cone}_X(\Omega) = X + *$, where $*$ adjacent to Ω

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Note.

- We can explicitly find all elementary abelian covers along which G lifts as a strongly split extension.
- The problem reduces to finding invariant subspaces of matrix group linearly representing the action of G on $H_1(X, \mathbb{Z}_p)$.

Example – finding all elementary abelian G-split covers

Find all connceted regular $\mathbb{Z}_p^k$-covers of K_4 such that $\mathbb{Z}_4$ lifts as a split extension with an invariant section.

Line	Condition	Dim	Voltage array
1.	$p \equiv -1\,(4)$	1	$[1],[1],[1],[1],[0],[0]$
2.		2	$\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}-1\\-1\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}$
3.		3	$\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\-1\end{bmatrix},\begin{bmatrix}1\\-1\\-1\end{bmatrix},\begin{bmatrix}1\\-1\\1\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix}$
4.	$p \equiv 1\,(4), \lambda_0^2 = -1$	1	$[1],[1],[1],[1],[0],[0]$
5.		1	$[1],[\lambda_0],[-1],[-\lambda_0],[0],[0]$
6.		2	$\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-\lambda_0\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}1\\\lambda_0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}$
7.		2	$\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}\lambda_0\\-\lambda_0\end{bmatrix},\begin{bmatrix}-1\\-1\end{bmatrix},\begin{bmatrix}-\lambda_0\\\lambda_0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}$
8.		3	$\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\\lambda_0\\-\lambda_0\end{bmatrix},\begin{bmatrix}1\\-1\\-1\end{bmatrix},\begin{bmatrix}1\\-\lambda_0\\\lambda_0\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix}$
9.	$p = 2$	1	$[1],[1],[1],[1],[1],[1]$
10.		2	$\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}$

Thank you!